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ON THE MODULAR PLESKEN LIE ALGEBRA

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Abstract. Let G be a finite group. The Plesken Lie algebra $\mathcal{L}[\mathrm{G}]$ is a subalgebra of the complex group algebra $\mathbf{C}[\mathrm{G}]$ and admits a direct-sum decomposition into simple Lie algebras based on the ordinary character theory of G. In this paper we review the known results on $\mathcal{L}[G]$ and related Lie algebras, as well as introduce a conjecture on a characteristic $p \neq 2$ analog $\mathcal{L}_p[\mathrm{G}]$, with a focus on when p divides the order of G.

1 Introduction

The aim of this paper is to review known results on a certain Lie algebra attached to a finite group using ordinary representation theory, as well as to suggest new directions of research that incorporate modular representations. Let us begin with some definitions.

Let k be a field, G a finite group, and $k[\mathrm{G}]$ the group algebra of G over k. The alternating bilinear map $[\ ,\ ] : k[\mathrm{G}] \times k[\mathrm{G}] \to k[\mathrm{G}]$ defined by

$$[x, y] = xy - yx$$

endows $k[\mathrm{G}]$ with the structure of a Lie algebra. For each $g \in \mathrm{G}$, define the element $\widehat{g} := g - g^{-1} \in k[\mathrm{G}]$. Following [3], the *Plesken Lie algebra* $\mathcal{L}[\mathrm{G}]$ is the Lie subalgebra of $k[\mathrm{G}]$ linearly spanned over k by the $\widehat{g}$.

This raises a very natural question: *what is the isomorphism type of* $\mathcal{L}[\mathrm{G}]$*?* In particular, we can ask whether there is a natural relation between the Lie algebra structure of $\mathcal{L}[\mathrm{G}]$ and the group structure of G. Since $\mathcal{L}[\mathrm{G}]$ might not be a simple Lie algebra, we state our main question more formally as follows.

Question 1. What are the isomorphism types (as Lie algebras over k) of the simple composition factors of $\mathcal{L}[\mathrm{G}]$?

A natural starting point is when $k = \mathbf{C}$, since there is a well-understood classical theory of complex simple Lie algebras. Indeed, in [3] the authors show that when $k = \mathbf{C}$, the Lie algebra $\mathcal{L}[\mathrm{G}]$ is semisimple (owing to the fact that $\mathbf{C}[\mathrm{G}]$ is a direct sum of matrix algebras) and admits a direct sum decomposition into Lie subalgebras according to the degrees and Frobenius-Schur indicators of the complex irreducible representations of G. We review their main theorem and give an illustrative example in Section 3 below.

Since the structure of $\mathcal{L}[\mathrm{G}]$ is known when $k = \mathbf{C}$, we switch perspective and ask what happens for other fields k, in particular when the characteristic of k divides the order of G. Here, there appear to be interesting connections with the modular representation theory of G which have yet to be studied. After reviewing the current literature below, we propose new questions and conjectures for further research based on extensive calculation.

2 History of the Problem

Before working with $\mathcal{L}[\mathrm{G}]$ directly, let us first consider $k[\mathrm{G}]$ as a Lie algebra. If $k = \mathbf{C}$, then it is well known that $\mathbf{C}[\mathrm{G}]$ decomposes as a direct sum of matrix algebras, which coincides with its decomposition into Lie algebras. Suppose now that the characteristic of k divides the order of G. Then Passi, Passman, and Sehgal determined in [8] the exact conditions under which $k[G]$ is a nilpotent Lie algebra, and under which it is a solvable Lie algebra; they showed that the nilpotency/solvability can be determined purely from the group-theoretic properties of G, foreshadowing the potential for deep connections between the group theory of G and the Lie algebra theory of $\mathcal{L}[\mathrm{G}]$.

Roughly a decade later, Smirnov and Zalesskii in [10] systematically studied the Lie-algebraic properties of $\mathcal{L}[\mathrm{G}]$; to the best of our knowledge this is the first work that concerns $\mathcal{L}[\mathrm{G}]$ explicitly. In that paper the authors show (among other things) that

(i) if k has characteristic 0 and $\mathcal{L}[\mathrm{G}]$ is solvable, then G is solvable, and

(ii) if k has characteristic different than 2 and $\mathcal{L}[\mathrm{G}]$ is nilpotent, then G^2 (the subgroup of G generated by squares) is nilpotent.

We remark that item (1) does not generalize to positive characteristic (as the authors point out in [10, §5.1]). Additionally, item (2) appears to be the first structure theorem relating the Lie-algebraic properties of $\mathcal{L}[\mathrm{G}]$ to the group theory of G in the modular setting (excluding characteristic 2).

Returning to the case $k = \mathbf{C}$, in 2007 Cohen and Taylor determined in [3] the direct-sum decomposition of $\mathcal{L}[\mathrm{G}]$ into complex simple Lie algebras of classical type (we review the exact decomposition in Section 3). Subsequently, in [7], Marin considered a more general setup, which is also briefly addressed in [10], as follows. Let $\alpha : \mathrm{G} \to \mathbf{C}^\times$ be a multiplicative character, and let $\mathcal{L}_\alpha[\mathrm{G}]$ denote the span of the elements $g - \alpha(g)g^{-1}$. Then $\mathcal{L}_\alpha[\mathrm{G}]$ is also a Lie subalgebra of $k[\mathrm{G}]$ when endowed with the bracket operation. When $\alpha = \mathbf{1}$, we recover $\mathcal{L}_{\mathbf{1}}[\mathrm{G}] = \mathcal{L}[\mathrm{G}]$. The main result of [7] is the determination of $\mathcal{L}_\alpha[\mathrm{G}]$ as a direct sum of complex, simple Lie algebras of classical type and contextualizes the study of the Plesken algebra within the field of harmonic analysis. More recently, Chaudhuri determined in 2020 in [2] the direct-sum decomposition of $\mathcal{L}[\mathrm{G}]$ when k is an algebraic extension of $\mathbf{Q}$. And in 2023 Arjun and Romeo studied in [1] the Lie algebra representation theory of $\mathcal{L}[\mathrm{G}]$ over $\mathbf{C}$.

It is evident that there is an established and significant body of work surrounding the Plesken Lie algebra in characteristic 0, relating the ordinary representation theory and group structure of G to the classical Lie algebra structure of $\mathcal{L}[\mathrm{G}]$. Before addressing the modular theory in Section 4, we review the main theorem of [3] along with some new generalizations.

2.1 Notation

For the remainder of the paper we use Atlas [4] notation for finite simple groups. We recall the simple Lie algebras of classical type in Table 1 and their corresponding dimensions. Even though we will not use it, we record the $ABCD$ labeling for the reader's convenience.

$ABCD$ Label	Isomorphism Type	Dimension	n
A_n	$\mathfrak{sl}_{n+1}$	$(n+1)^2 - 1$	$n \geq 1$
B_n	$\mathfrak{so}_{2n+1}$	$2n^2 + n$	$n \geq 2$
C_n	$\mathfrak{sp}_{2n}$	$2n^2 + n$	$n \geq 3$
D_n	$\mathfrak{so}_{2n}$	$2n^2 - n$	$n \geq 4$

Table 1. Simple Lie Algebras of Classical Type

For small values of n, we have the following coincidences: $\mathfrak{sl}_2 = \mathfrak{so}_2 = \mathfrak{so}_3 = \mathfrak{sp}_2$, $\mathfrak{so}_4 = \mathfrak{sl}_2 \times \mathfrak{sl}_2$, and $\mathfrak{so}_6 = \mathfrak{sl}_4$. If p is a prime number dividing $n+1$, then type $\mathfrak{sl}_{n+1}$ is not simple in characteristic p, since the scalar matrices have trace 0. We write $\mathfrak{psl}_{n+1}$ for the codimension-1 simple factor of $\mathfrak{sl}_{n+1}$ in this case. Finally, we write $\mathsf{Ab}(n)$ for an abelian Lie algebra of dimension n.

3 The Ordinary Plesken Lie Algebra

In [3] the authors show that if t is the number of involutions of G, then

$$\dim \mathcal{L}[\mathrm{G}] = (|\mathrm{G}| - t - 1)/2. \tag{3.1}$$

The irreducible characters of a finite group G are partitioned according to their Frobenius-Schur indicator $+$, $-$, or 0. We denote these sets by $\mathcal{X}_+$, $\mathcal{X}_-$, and $\mathcal{X}_0$, respectively, and recall the direct-sum decomposition proved in [3, Thm. 5.1]:

$$\mathcal{L}[\mathrm{G}] = \bigoplus_{\chi\in\mathcal{X}_+} \mathfrak{so}_{\chi(1)} \bigoplus_{\chi\in\mathcal{X}_-} \mathfrak{sp}_{\chi(1)} \bigoplus_{\chi\in\mathcal{X}_0} {}'\mathfrak{gl}_{\chi(1)}, \tag{3.2}$$

where the prime signifies that there is just one summand $\mathfrak{gl}_{\chi(1)}$ for each pair $\{\chi, \overline{\chi}\}$ from $\mathcal{X}_0$. The summands of $\mathfrak{gl}_n$ type are not simple, but contain $\mathfrak{sl}_n$ as a codimension-1 summand. Let us now consider an illustrative example that shows how one can generalize (3.2) to finite fields in ordinary characteristic.

Example 1. Let $\mathrm{G} = L_2(8)$, the simple group of order $504 = 2^3 \cdot 3^2 \cdot 7$. Using Atlas notation [4, p. 6], the character table of G is given in Table 2.

	ind	1A	2A	3A	7A	B*2	C*4	9A	B*2	C*4
χ_1	+	1	1	1	1	1	1	1	1	1
χ_2	+	7	−1	−2	0	0	0	1	1	1
χ_3	+	7	−1	1	0	0	0	−y9	*2	*4
χ_4	+	7	−1	1	0	0	0	*4	−y9	*2
χ_5	+	7	−1	1	0	0	0	*2	*4	−y9
χ_6	+	8	0	−1	1	1	1	−1	−1	−1
χ_7	+	9	1	0	y7	*2	*4	0	0	0
χ_8	+	9	1	0	*4	y7	*2	0	0	0
χ_9	+	9	1	0	*2	*4	y7	0	0	0

Table 2. Character Table of $L_2(8)$

Here, y9 is a root of the polynomial $f_9(x) = x^3 - 3x - 1$ and y7 a root of $f_7(x) = x^3 + x^2 - 2x - 1$. Both y9 and y7 are linear combinations of certain roots of unity:

$$\mathtt{y9} = -\zeta_9^5 - \zeta_9^4$$
$$\mathtt{y7} = -\zeta_7^4 + \zeta_7^3,$$

and '$*m$' denotes replacing a root of unity ζ by ζ^m. By Equation (3.1), $\dim \mathcal{L}[\mathrm{G}] = 220$, and by Equation (3.2), we see that that $\mathcal{L}[\mathrm{G}]$ admits the direct-sum decomposition

$$\mathcal{L}[\mathrm{G}] = \mathfrak{so}_7^4 \oplus \mathfrak{so}_8 \oplus \mathfrak{so}_9^3.$$

Let us now consider the Plesken Lie algebra from another perspective. Let p be a prime number and define $\mathcal{L}_p[\mathrm{G}]$ to be the Lie subalgebra of $\mathbf{F}_p[\mathrm{G}]$ spanned by the $\widehat{g}$.

Lemma 1. For all prime numbers p, we have $\dim \mathcal{L}_p[\mathrm{G}] = (|G| - t - 1)/2$, where t is the number of involutions of G.

Proof. The proof is identical to the case of characteristic 0. If $g \neq h$, then $\widehat{g}$ is independent from $\widehat{h}$, so it remains to determine when $\widehat{g} = 0$. In odd characteristic, this is equivalent to $g = g^{-1}$ (so g is an involution) and in characteristic 2 it is the same as well (g is an involution if and only if $g = g^{-1}$ if and only if $\widehat{g} = g - g^{-1} = g + g = 2g = 0$.) □

In spite of the dimension of $\mathcal{L}_p[\mathrm{G}]$ being independent of p, we will restrict to p odd for the remainder of the paper, due to the fact that in characteristic 2, symmetric and alternating bilinear forms coincide, complicating a uniform decomposition theorem that is based on distinguishing between these forms.

For the remainder of the section we let p be an odd prime number coprime to $|\mathrm{G}|$, so that $\mathbf{F}_p[\mathrm{G}]$ is semisimple (we call p an *ordinary* prime). Since p is odd, the Frobenius-Schur indicators $+$ and $-$ do not coincide. Granting this, the proof of [3, Thm. 5.1] applies identically to ordinary positive characteristic, provided $\mathbf{F}_p$ contains enough roots of unity so that all irreducible representations of G over $\mathbf{F}_p$ are absolutely irreducible (we say that $\mathbf{F}_p$ is a *splitting field* of G). If $\mathbf{F}_p$ is not a splitting field for G, then the absolutely irreducible Galois-conjugate representations that are not realizable over $\mathbf{F}_p$ coalesce to an irreducible representation over $\mathbf{F}_p$. Given this caveat, the proof of [3, Thm. 5.1] carries through as well, though the dimensions of the simple factors are potentially different than over a splitting field. We illustrate this with several examples.

(i) Let $p = 71$. Then p is ordinary for G, both $f_9(x)$ and $f_7(x)$ split modulo p, and we have

$$\mathcal{L}_p[\mathrm{G}] = \mathfrak{so}_7^4 \oplus \mathfrak{so}_8 \oplus \mathfrak{so}_9^3,$$

where all Lie algebras in the sum are defined over the field $\mathbf{F}_{71}$.

(ii) Let $p = 17$. Then p is ordinary for G, but only $f_9(x)$ splits modulo p, while $f_7(x)$ is irreducible. The three irreducible representations of degree 9 are not realizable over $\mathbf{F}_p$, but coalesce to an irreducible degree 27 representation. Explicitly, this representation is given by the reduction modulo 17 of the degree 27 representation of $L_2(8)$ over $\mathbf{Z}$ in the online Atlas:

https://brauer.maths.qmul.ac.uk /Atlas/v3/lin/L28/

We then have

$$\mathcal{L}_p[\mathrm{G}] = \mathfrak{so}_7^4 \oplus \mathfrak{so}_8 \oplus \mathfrak{m},$$

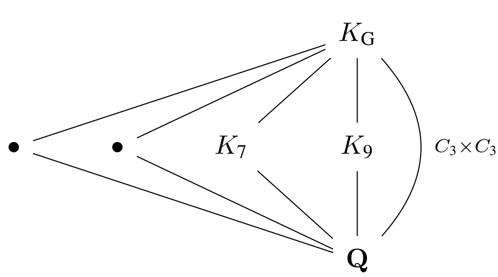
where the Lie algebras corresponding to χ_7, χ_8, and χ_9 coalesce to form the 108-dimensional irreducible Lie algebra $\mathfrak{m}$ over $\mathbf{F}_{17}$.

(iii) Let $p = 5$. Then p is ordinary for G, neither $f_9(x)$ nor $f_7(x)$ split modulo p, and we have

$$\mathcal{L}_p[\mathrm{G}] = \mathfrak{so}_7 \oplus \mathfrak{so}_8 \oplus \mathfrak{l} \oplus \mathfrak{m}.$$

The Lie algebras corresponding to χ_3, χ_4, and χ_5 coalesce to form the 63-dimensional Lie algebra $\mathfrak{l}$ and, similar to the case where $p = 17$, the Lie algebras associated to χ_7, χ_8, and χ_9 coalesce to form $\mathfrak{m}$.

We can generalize Items (i), (ii), and (iii) to number fields as follows. Denote by K_7 and K_9 the splitting fields over $\mathbf{Q}$ of the polynomials $f_7(x)$ and $f_9(x)$, respectively. Both K_7 and K_9 are cyclic Galois extensions over $\mathbf{Q}$. Let K_{G} be the minimal splitting field over $\mathbf{Q}$ of G. Then K_{G} is isomorphic to the compositum K_7K_9 with $\mathrm{Gal}(K_{\mathrm{G}}/\mathbf{Q}) \simeq C_3 \times C_3$. Basic field theory equips us with the diagram



where '•' stands for a cubic extension of $\mathbf{Q}$ that does not concern this exposition. The distribution of ordinary primes by the factorization pattern of the defining polynomials f_7 and f_9 determines the classical Lie direct-sum type of $\mathcal{L}_p[\mathrm{G}]$, with natural densities given by the Artin symbol clases in $\mathrm{Gal}(K_{\mathrm{G}}/\mathbf{Q}) \simeq C_3 \times C_3$, as shown in Table 3. This example generalizes in a natural way to any finite group G and is also a natural generalization of the results of [2] and [3] to characteristic p fields when p is ordinary.

Proportion of Primes	Decomposition of $\mathcal{L}_p[G]$	$f_7(x)$	$f_9(x)$
1/9	$\mathfrak{so}_7 \oplus \mathfrak{so}_8 \oplus \mathfrak{so}_7^3 \oplus \mathfrak{so}_9^3$	split	split
4/9	$\mathfrak{so}_7 \oplus \mathfrak{so}_8 \oplus \mathfrak{l}_1 \oplus \mathfrak{l}_2$	irred.	irred.
2/9	$\mathfrak{so}_7 \oplus \mathfrak{so}_8 \oplus \mathfrak{so}_7^3 \oplus \mathfrak{l}_2$	split	irred.
2/9	$\mathfrak{so}_7 \oplus \mathfrak{so}_8 \oplus \mathfrak{l}_1 \oplus \mathfrak{so}_9^3$	irred	split

Table 3. Splitting of $\mathcal{L}_p[G]$ for ordinary primes

4 The Modular Plesken Lie Algebra

In this section we fix an odd prime p dividing $|G|$ and let k be an algebraically closed field of characteristic p. Then $k[G]$ is no longer semisimple, but decomposes as a direct sum of blocks:

$$k[G] = b_1 \oplus \cdots \oplus b_r.$$

Then each b_j, being a two-sided ideal of $k[G]$, inherits a Lie algebra structure over k from $k[G]$. The goal of this section is to prove a structure theorem on the maximal semisimple quotient of $\mathcal{L}_p[G]$ and to introduce a conjecture on the structure of the nilpotent radical. Taken together, these provide a description of $\mathcal{L}_p[G]$ in modular characteristic. We begin by recalling some facts from the modular representation theory of finite groups.

Let $\phi_1, \ldots, \phi_s$ denote the Brauer characters of G, where s is the number of p-regular classes of G. As in the ordinary case, the set of Brauer characters is partitioned by Frobenius-Schur indicator into subsets $\mathcal{X}_+$, $\mathcal{X}_-$, and $\mathcal{X}_0$. Indeed, for any self-dual simple $k[G]$-module V, there exists a unique (by Schur's lemma) G-invariant nondegenerate bilinear form β that is either symmetric or alternating (recall our standing assumption that p is odd so that these two cases do not coincide), which defines an indicator "+" or "−", respectively. If V is not self-dual, then $\beta = 0$ and therefore the Lie algebra generated by commutators is the full matrix algebra. We label these and their conjugates with indicator 0. See [5, Introduction] for more background on the indicators of Brauer characters.

Let $A = k[G]$ and $J = \mathrm{Rad}(A)$ its Jacobson radical. Since p is modular, A is not semisimple. However, A/J is the maximal semisimple quotient of A and so, by the Wedderburn-Artin theorem, we have

$$A/J = \bigoplus_{i=1}^{s} M_{n_i}(k),$$

where the sum is over the Brauer simple modules, $M_{n_i}(k)$ is a matrix algebra over k, and n_i is the degree of the associated Brauer character. Let us define $\overline{\mathcal{L}}_p[G] := (\mathcal{L}_p[G] + J)/J$ to be the image of $\mathcal{L}_p[G]$ in A/J and $N := \mathcal{L}_p[G] \cap J$ the nilpotent radical. The main result of this section is the following structure theorem for $\overline{\mathcal{L}}_p[G]$.

Theorem 1. Let G be a finite group and let k be an algebraically closed field of characteristic $p > 2$ dividing $|G|$. Then, with all notation as above, every nonabelian composition factor of $\mathcal{L}_p[G]$ occurs in $\overline{\mathcal{L}}_p[G]$, hence is of classical type and determined by the self-duality and Frobenius-Schur indicator of the simple $k[G]$-modules. Explicitly, we have

$$\overline{\mathcal{L}}_p[G] = \bigoplus_{\phi \in \mathcal{X}_+} \mathfrak{so}_{\phi(1)} \bigoplus_{\phi \in \mathcal{X}_-} \mathfrak{sp}_{\phi(1)} \bigoplus_{\{\phi,\phi^*\} \in \mathcal{X}_0} \mathfrak{sl}_{\phi(1)}, \tag{4.1}$$

where we replace $\mathfrak{sl}_{\phi(1)}$ with $PA_{\phi(1)}$ if $p \mid (\phi(1)+1)$.

Before proving Theorem 1, we make several observations.

(i) Observe that blocks of defect 0 in the decomposition of $k[G]$ contribute simple summands to $\overline{\mathcal{L}}_p[G]$ corresponding to their Brauer constituents.

(ii) Note that Equation (4.1) coincides with Equation (3.2) if p is an ordinary prime since the Jacobson radical is 0.

(iii) In addition to the classical algebras, there exist exceptional simple Lie algebras in positive characteristic. When $p \geq 5$, the finite-dimensional simple Lie algebras over an algebraically closed field of characteristic p have been fully classified. Any such algebra is either of classical or Cartan type (Witt; Special; Hamiltonian; Contact) for $p \geq 7$ or, additionally, of Melikian type if $p = 5$ [9]. By Theorem 1, these do not occur in the composition series of $\mathcal{L}_p[\mathrm{G}]$.

(iv) The degrees of the Brauer characters do not necessarily agree with those of the ordinary characters, which can result in a very different overall Lie algebra structure of the simple summands $\overline{\mathcal{L}}_p[\mathrm{G}]$ versus $\mathcal{L}[\mathrm{G}]$.

Proof of Theorem 1. Observe that the nilpotent radical N of $\mathcal{L}_p[\mathrm{G}]$ is a subalgebra of J and so its composition series is comprised only of solvable algebras. Hence it contains no non-classical simple factors. Thus, any composition factor of $\mathcal{L}_p[\mathrm{G}]$ that is a simple classical factor must survive reduction modulo J.

By definition, the algebra $\overline{\mathcal{L}}_p[\mathrm{G}]$ is generated by the images of the $\widehat{g}$ in the matrix algebras $M_{n_i}(k)$. Since A/J is the direct sum of the $M_{n_i}(k)$, the images of the $\widehat{g}$ in each direct summand are independent and hence it suffices to describe them for a fixed Brauer character ϕ_i of degree n_i.

Inside $M_{n_i}(k)$ the $\widehat{g}$ generate a Lie subagebra of matrices $\{x - x^{\top_{\beta_i}}\}$, where ${}^{\top_{\beta_i}}$ denotes the adjoint with respect to the G-invariant form β_i induced from ϕ_i (see the discussion at the beginning of Section 4). Thus, the summand is of $\mathfrak{so}$-type if β_i is symmetric, $\mathfrak{sp}$-type if β_i is alternating, or $\mathfrak{sl}$-type if β_i is not self-dual. In the latter case, either ϕ_i or its conjugate gives rise to the identical $M_{n_i}(k)$, so we count $\mathfrak{sl}$-type Lie algebras by pairs of conjugate characters. This gives us the desired decomposition (4.1). □

We end this section with a conjecture on the structure of the nilpotent radical N. Together with Theorem 1, this would constitute a structure theorem for $\mathcal{L}_p[\mathrm{G}]$ in the modular setting. In the next section we give computational evidence to support our claims.

Conjecture 1. With notation above and p odd, the Lie algebra $N = \mathcal{L}_p[\mathrm{G}] \cap J$ has only abelian composition factors. Equivalently, all non-abelian simple composition factors of $\mathcal{L}_p[\mathrm{G}]$ arise from $\overline{\mathcal{L}}_p[\mathrm{G}] \subseteq A/J$ and are of classical type $\mathfrak{so}, \mathfrak{sp}$, or $\mathfrak{sl}$, determined by the Brauer characters' degrees and Frobenius-Schur indicators.

5 Examples

In this final section of the paper we give illustrative examples that are chosen to highlight different aspects of Conjecture 1. In each case the reader will see that the simple composition factors can be predicted from the first two columns of the Brauer character table in the same manner as the ordinary $\mathcal{L}[\mathrm{G}]$, as claimed. For notational conventions, see Section 2.1.

Example 2. Let $\mathrm{G} = SL_2(5)$ and let $p = 5$. The indicator and degree columns of the Brauer and ordinary character tables are given in Table 4. The group algebra $k[\mathrm{G}]$ decomposes into a direct sum of three blocks

$$k[\mathrm{G}] = b_1 \oplus b_2 \oplus b_3,$$

of defects 1, 1, and 0, respectively. The ordinary characters χ_1, χ_4, χ_5, and χ_7 belong to b_1, while χ_2, χ_3, χ_6, and χ_9 belong to b_2. The Steinberg character χ_8 is the unique character belonging to b_3.

In Magma we compute that $\mathcal{L}_5[\mathrm{G}]$ admits the direct-sum decomposition

$$\mathcal{L}_5[\mathrm{G}] = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{so}_5,$$

where $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are Lie algebras with respective composition factors

$$\mathfrak{a}_1 : \mathsf{Ab}(3), \mathsf{Ab}(3), \mathsf{Ab}(3), \mathfrak{so}_3$$
$$\mathfrak{a}_2 : \mathsf{Ab}(3), \mathsf{Ab}(3), \mathsf{Ab}(8), \mathsf{Ab}(10), \mathfrak{sp}_2, \mathfrak{sp}_4.$$

Brauer (mod 5)	ind	degree	Ordinary	ind	degree
ϕ_1	$+$	1	χ_1	$+$	1
ϕ_2	$-$	2	χ_2	$-$	2
ϕ_3	$+$	3	χ_3	$-$	2
ϕ_4	$-$	4	χ_4	$+$	3
ϕ_5	$+$	5	χ_5	$+$	3
			χ_6	$-$	4
			χ_7	$+$	4
			χ_8	$+$	5
			χ_9	$-$	6

Table 4. Brauer -vs- Ordinary Characters of $SL_2(5)$

Observe that the simple composition factors $\mathfrak{so}_3$, $\mathfrak{sp}_2$, $\mathfrak{so}_5$, $\mathfrak{sp}_4$ that we computed in Magma agree with part (3) of Conjecture 1, where $\mathfrak{so}_3$ corresponds to the Brauer character ϕ_3; $\mathfrak{sp}_2$ to ϕ_2; $\mathfrak{sp}_4$ to ϕ_4; and $\mathfrak{so}_5$ to ϕ_5. Finally, observe that the direct summand $\mathfrak{so}_5$ corresponds to the block b_3 of $k[\mathrm{G}]$ of defect 0.

Example 3. Let $\mathrm{G} = L_2(8)$ as in Example 1. We select modular primes that are different from the defining characteristic (in contrast to Example 2 above).

Brauer (mod 3)	ind	degree	Brauer (mod 7)	ind	degree	Ordinary	ind	degree
ϕ_1	$+$	1	ϕ_1	$+$	1	χ_1	$+$	1
ϕ_2	$+$	7	ϕ_2	$+$	7	χ_2	$+$	7
ϕ_3	$+$	9	ϕ_3	$+$	7	χ_3	$+$	7
ϕ_4	$+$	9	ϕ_4	$+$	7	χ_4	$+$	7
ϕ_5	$+$	9	ϕ_5	$+$	7	χ_5	$+$	7
			ϕ_6	$+$	8	χ_6	$+$	8
						χ_7	$+$	9
						χ_8	$+$	9
						χ_9	$+$	9

Table 5. Brauer -vs- Ordinary Characters of $L_2(8)$

In characteristic 3, we have that $k[\mathrm{G}]$ decomposes into a sum of four blocks

$$k[\mathrm{G}] = b_1 \oplus b_2 \oplus b_3 \oplus b_4,$$

of defects 2, 0, 0, and 0, respectively. The corresponding Lie algebra decomposition, according to Magma, is given by

$$\mathcal{L}_3[\mathrm{G}] = \mathfrak{a}_1 \oplus \mathfrak{so}_9 \oplus \mathfrak{so}_9 \oplus \mathfrak{so}_9,$$

where $\mathfrak{a}_1$ is a Lie algebra with composition factors

$$\mathsf{Ab}(7), \mathsf{Ab}(21), \mathsf{Ab}(21), \mathsf{Ab}(21), \mathsf{Ab}(21), \mathfrak{so}_7.$$

Note that the Lie algebra $\mathfrak{so}_7$ corresponds to the Brauer character ϕ_2, while the three copies of $\mathfrak{so}_9$ correspond to ϕ_3, ϕ_4, and ϕ_5.

In characteristic 7, we have that $k[\mathrm{G}]$ decomposes into a sum of five blocks

$$k[\mathrm{G}] = b_1 \oplus b_2 \oplus b_3 \oplus b_4 \oplus b_5,$$

of defects 1, 0, 0, 0, and 0, respectively. The ordinary characters χ_1, χ_6, χ_7, χ_8, and χ_9 belong to b_1, while b_2, b_3, b_4, b_5 contain only the characters χ_2, χ_3, χ_4, χ_5, respectively, since those characters remain irreducible modulo 7. The corresponding Lie algebra decomposition, according to Magma, is given by

$$\mathcal{L}_7[\mathrm{G}] = \mathfrak{a}_1 \oplus \mathfrak{so}_7 \oplus \mathfrak{so}_7 \oplus \mathfrak{so}_7 \oplus \mathfrak{so}_7,$$

where $\mathfrak{a}_1$ is a Lie algebra with composition factors

$$\mathsf{Ab}(8), \mathsf{Ab}(8), \mathsf{Ab}(8), \mathsf{Ab}(28), \mathsf{Ab}(28), \mathsf{Ab}(28), \mathfrak{so}_8.$$

Here, the $\mathfrak{so}_8$-factor corresponds to the Brauer character ϕ_6, while the $\mathfrak{so}_7$-factors correspond to ϕ_2, ϕ_3, ϕ_4, and ϕ_5. Again, observe that in both characteristic 3 and 7 the direct-sum decomposition and simple composition factors follow the predictions of parts (3) and (4) of Conjecture 1.

Example 4. The purpose of this final example is to show that when there is a large power of p dividing G, the Brauer characters modulo p may be quite different than the ordinary characters over **C**. Let $G = L_2(25)$ and let $p = 5$. Then the dimensions and indicators of the Brauer and ordinary characters are given in Table 6.

Brauer (mod 5)	ind	degree	Ordinary	ind	degree
ϕ_1	+	1	χ_1	+	1
ϕ_2	+	3	χ_2	+	13
ϕ_3	+	3	χ_3	+	13
ϕ_4	+	4	χ_4	+	24
ϕ_5	+	5	χ_5	+	24
ϕ_6	+	5	χ_6	+	24
ϕ_7	+	8	χ_7	+	24
ϕ_8	+	8	χ_8	+	24
ϕ_9	+	9	χ_9	+	24
ϕ_{10}	+	15	χ_{10}	+	25
ϕ_{11}	+	15	χ_{11}	+	26
ϕ_{12}	+	16	χ_{12}	+	26
ϕ_{13}	+	25	χ_{13}	+	26
			χ_{14}	+	26
			χ_{15}	+	26

Table 6. Brauer -vs- Ordinary Characters of $L_2(25)$

The group algebra $k[G]$ decomposes as a direct sum of two blocks $k[G] = b_1 \oplus b_2$ of defect 2 and 0, respectively. The Steinberg character χ_{10} belongs to b_2, while all others belong to b_1. A calculation in Magma shows that the Lie algebra $\mathcal{L}_5[G]$ admits the decomposition

$$\mathcal{L}_5[G] = \mathfrak{a}_1 \oplus \mathfrak{so}_{25},$$

where $\mathfrak{so}_{25}$ corresponds to the Brauer character ϕ_{13}, and $\mathfrak{a}_1$ has composition factors as in Table 7.

Composition Factor	Multiplicity
$\mathsf{Ab}(20), \mathsf{Ab}(25), \mathsf{Ab}(54), \mathsf{Ab}(90), \mathsf{Ab}(210)$ $\mathfrak{so}_9$, $\mathfrak{so}_{16}$	1
$\mathsf{Ab}(16)$, $\mathsf{Ab}(24)$, $\mathsf{Ab}(30)$, $\mathsf{Ab}(48)$, $\mathsf{Ab}(64)$ $\mathsf{Ab}(96)$, $\mathsf{Ab}(144)$, $\mathsf{Ab}(160)$, $\mathsf{Ab}(240)$ $\mathfrak{so}_5$, $\mathfrak{so}_8$, $\mathfrak{so}_{15}$	2
$\mathsf{Ab}(56)$, $\mathsf{Ab}(120)$	3
$\mathfrak{sl}_2$	4
$\mathsf{Ab}(6)$	7
$\mathsf{Ab}(36)$	8

Table 7. Composition Factors of $\mathfrak{a}_1 \subseteq \mathcal{L}_5[G]$

Here, $\mathfrak{so}_9$ corresponds to the Brauer character ϕ_9; $\mathfrak{so}_{16}$ to ϕ_{12}; $\mathfrak{so}_5$ (two copies) to ϕ_5 and ϕ_6; $\mathfrak{so}_8$ to ϕ_7 and ϕ_8; $\mathfrak{so}_{15}$ to ϕ_{10} and ϕ_{11}. Recalling the low-degree isomorphisms from Section 2.1, the four copies of $\mathfrak{sl}_2$ correspond to ϕ_2, ϕ_3 (each corresponding to a $\mathfrak{so}_3$), and ϕ_4 (corresponding to $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2$).

We conclude the paper with an observation and question for future researchers.

(i) Observe that certain ordinary representations remain irreducible upon reduction modulo p. These include the Steinberg representation, as well as any representation whose degree is a multiple of the largest power of p dividing $|G|$. These representations occur in blocks of defect 0. In the case of the Steinberg representation, the Frobenius-Schur indicator is '$+$', hence will give rise to a Lie algebra of type $\mathfrak{so}_{2n+1}$ or $\mathfrak{so}_{2n}$.

(ii) In addition to a proof of Conjecture 1, it would be interesting to determine the precise number and dimensions of the abelian composition factors of $\mathcal{L}_p$[G], as well as investigating any relation between the abelian factors and the defect groups of the blocks.

A Code and Data

All computations in this paper were performed in Magma [6]. For the code below, `G` is a group and `F` is a finite field. This code will build the Plesken Lie Algebra `P` as a subalgebra of the group algebra `FG` and the image `PP` of `P` in the quotient of `FG` by its Jacobson radical `J`.

```
FG := GroupAlgebra(F, G);
A, phi := Algebra(FG);
LA, psi := LieAlgebra(A);
J := JacobsonRadical(LA);

gens := [];
seen := {@ @};
for g in G do
o := Order(g);
if o gt 2 then
h := Inverse(g);
if not (h in seen) then
Include(~seen, g); Include(~seen, h);
Append(~gens, FG!g - FG!h);
end if;
end if;
end for;
img := [ ( (x @ phi) @ psi ) : x in gens ];

P := sub< LA | img >;
PP := quo<P | P meet J>;
```

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