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INTERVAL-VALUED NEUTROSOPHIC N-STRUCTURES IN D-ALGEBRA

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Abstract In this article we introduce the notion of novel set interval-valued neutrosophic N-structure and define its fundamental operations union, intersection, containment, and complement along with examples. Furthermore, we apply interval-valued neutrosophic N-structure concept to algebraic structure d-algebra and introduce interval-valued neutrosophic N-d-subalgebra. We provide some characteristics of interval-valued neutrosophic N-d-subalgebra.

1 Introduction

In the branch of abstract algebra, K. Iseki and Y. Imai (see [5], [4]) established two exceptional structures known as BCK-algebras and BCI-algebras. Particularly, BCK-algebras are a special type of BCI-algebra. Advancing from this groundwork, J. Negger and H. S. Kim (see [12]) suggested d-algebras, an expanded approach that generalizes the scope of BCK-algebras. Y. B. Jun, J. Negger, and H. S. Kim (see [13]). examined the theory of ideals in d-algebras. Zadeh, L. A. (see [21]), initially introduced the concept of fuzzy sets. Later, Xi, O. G. (see [20]) utilized these fuzzy sets to study BCK algebras. Recently, M. M. Jansisani and S. Sujatha (see [11]) introduced Łukasiewicz fuzzy ideals in BCK/BCI-algebras, while M. Balamurugan et al. (see [3]) introduced anti-fuzzy B-ideals of BCI-algebras.

Smarandache's neutrosophic set theory (see [14], [15], [16]) offers a dominant structure in the field of set theory. This structure extends classical sets, fuzzy sets, and different generalizations, alike IvFSs, IFSs, and IvIFSs. The application of NSs expands to various fields like control theory, topology, and algebra. They are also helpful in medicine, decision-making, and real-life practical challenges. Developing further on this idea, Wang et al. (see [19]) gave the idea of interval-valued neutrosophic sets, which give higher adjustability and precision compared to single-valued neutrosophic sets.

The extension of crisp sets to fuzzy sets focused on transforming the precise value 1 into the interval [0, 1], representing positive data. Since this transformation didn't cover negative information, there is a development need to solve it. To address this, Jun et al. (see [6]) introduced a novel concept called negative-valued function and created N-structures. Negative-valued functions are crucial mathematical tools for representing loss, decrease, or opposition in various real-life situations. The mapping to the range [-1, 0] plays an essential role in characterizing complex systems where quantities decrease. This type of function may be utilized in various fields like modeling energy loss in systems, decrease in pollutant levels, data reduction, decline in infected individuals, and negative returns on stocks. These N-structures were further extended by Khan et al. (see [10]),who introduced the idea of neutrosophic N-structure and applied it to a semigroup. Jun et al. (see [7]) applied the idea of neutrosophic N-structure to BCK/BCI-algebras. In 2020, M. Al-Tahan et al. introduced the concept of Neutrosophic N–Ideals (N-Sub algebras) of Subtraction Algebra (see [1]). Smarandache and Al-Tahan (see [17], [18]) introduced NeutroAlgebraic Structures and NeutroGeometry, which provides a new approach for studying neutrality in algebraic structure.

The progress of neutrosophic set theory has seen powerful improvements in recent years. In this study, we introduce the concept of interval-valued neutrosophic N-structure by integrating interval-valued neutrosophic logic and neutrosophic N-structure. Further, we apply interval-valued neutrosophic N-structure to d-algebras and introduce interval-valued neutrosophic N-d-subalgebra with an example. We provided some characteristics of interval-valued neutrosophic N-d-subalgebra.

2 Preliminaries

In this section, we present fundamental definitions that are essential for further discussion.

Definition 2.1. [12] Let $\mathcal{D}$ $(\neq \varnothing)$ be a set with a constant '0' and a binary operation '$*$'. Then $\mathcal{D}$ is called a d-algebra if it satisfies the following conditions for all $ȡ_1, ȡ_2 \in \mathcal{D}$
(d-A 1) $ȡ_1 * ȡ_1 = 0$
(d-A 2) $0 * ȡ_1 = 0$
(d-A 3) $ȡ_1 * ȡ_2 = 0$ and $ȡ_2 * ȡ_1 = 0$ implies $ȡ_1 = ȡ_2$.

We will refer to $ȡ_1 \leq ȡ_2$ if and only if $ȡ_1 * ȡ_2 = 0$.

Definition 2.2. [13] Let $\mathcal{D}$ be a d-algebra with binary operation '$*$' and $\mathcal{P} \subseteq \mathcal{D}$. Then, $\mathcal{P}$ is said to be a d-subalgebra of $\mathcal{D}$, if $ȡ_1, ȡ_2 \in \mathcal{P}$ implies $ȡ_1 * ȡ_2 \in \mathcal{P}$.

Definition 2.3. [21] A FS α in a set $\mathcal{D}$ $(\neq \varnothing)$ is a function from $\mathcal{D}$ into a $[0, 1]$.

Definition 2.4. [2] A FS α in a d-algebra $\mathcal{D}$ is called a fuzzy d-subalgebra of $\mathcal{D}$ if it satisfies $\alpha(ȡ_1 * ȡ_2) \geq min\{\alpha(ȡ_1), \alpha(ȡ_2)\}$, for all $ȡ_1, ȡ_2 \in \mathcal{D}$.

Definition 2.5. A mapping $f : \mathcal{D} \to \mathcal{E}$ of d-algebras is called a homomorphism if $f(ȡ_1 * ȡ_2) = f(ȡ_1) * f(ȡ_2)$, for all $ȡ_1, ȡ_2 \in \mathcal{D}$.

Note that if $f : \mathcal{D} \to \mathcal{E}$ is a homomorphism of d-algebras, then $f(0) = 0$.

Definition 2.6. [14] A neutrosophic set over a universal set $\mathcal{D}$ is defined as follows

$$\mathcal{N} = \{\langle ȡ_1; \alpha(ȡ_1), \beta(ȡ_1), \gamma(ȡ_1)\rangle \mid ȡ_1 \in \mathcal{D}\} \tag{2.1}$$

where $\alpha(ȡ_1) : \mathcal{D} \to]^-0, 1^+[$, $\beta(ȡ_1) : \mathcal{D} \to]^-0, 1^+[$, and $\gamma(ȡ_1) : \mathcal{D} \to]^-0, 1^+[$ are the truth, indeterminacy and false degree value of $\mathcal{D}$ and $^-0 \leq \alpha(ȡ_1) + \beta(ȡ_1) + \gamma(ȡ_1) \leq 1^+$.

Consider $\mathcal{F}(\mathcal{D}, [-1, 0])$ to be the set of all functions mapping elements from a set $\mathcal{D}$ to the interval $[-1, 0]$. We define an element of $\mathcal{F}(\mathcal{D}, [-1, 0]\ )$ as a negative-valued function from $\mathcal{D}$ to $[-1, 0]$, and is abbreviated as an N-function (see [8]).

Definition 2.7. [9] A neutrosophic N-structure over $\mathcal{D}$ is defined to be the structure

$$\mathcal{N} = \{\langle ȡ_1; \alpha(ȡ_1), \beta(ȡ_1), \gamma(ȡ_1)\rangle \mid ȡ_1 \in \mathcal{D}\} \tag{2.2}$$

where $\alpha(ȡ_1) : \mathcal{D} \to [-1, 0]$, $\beta(ȡ_1) : \mathcal{D} \to [-1, 0]$, and $\gamma(ȡ_1) : \mathcal{D} \to [-1, 0]$ are N-functions on $\mathcal{D}$ which are called the negative truth membership function, the negative indeterminacy membership function, and the negative falsity membership function, respectively, on $\mathcal{D}$ and $-3 \leq \alpha(ȡ_1) + \beta(ȡ_1) + \gamma(ȡ_1) \leq 0$.

An interval number is defined as a closed subinterval $\breve{\propto} = [\propto^{\mathcal{L}}, \propto^{\mathcal{U}}]$ within the interval $[-1, 0]$, where $-1 \leq \propto^{\mathcal{L}} \leq \propto^{\mathcal{U}} \leq 0$. Let I represent the set of all such interval numbers. We define the refined minimum (denoted by rmin) and refined maximum (denoted by rmax) for any two members in I. Further, we define the symbols "$\preccurlyeq$", "$\succcurlyeq$", and "=" for two members in I. For two interval numbers $\breve{\propto}_1 = [{\propto_1}^{\mathcal{L}}, {\propto_1}^{\mathcal{U}}]$ and $\breve{\propto}_2 = [{\propto_2}^{\mathcal{L}}, {\propto_2}^{\mathcal{U}}]$:
$rmin\{\breve{\propto}_1, \breve{\propto}_2\} = [min\{{\propto_1}^{\mathcal{L}}, {\propto_2}^{\mathcal{L}}\}, min\{{\propto_1}^{\mathcal{U}}, {\propto_2}^{\mathcal{U}}\}]$
$rmax\{\breve{\propto}_1, \breve{\propto}_2\} = [max\{{\propto_1}^{\mathcal{L}}, {\propto_2}^{\mathcal{L}}\}, max\{{\propto_1}^{\mathcal{U}}, {\propto_2}^{\mathcal{U}}\}]$
$\breve{\propto}_1 \preccurlyeq \breve{\propto}_2 \Leftrightarrow {\propto_1}^{\mathcal{L}} \leq {\propto_2}^{\mathcal{L}},\ {\propto_1}^{\mathcal{U}} \leq {\propto_2}^{\mathcal{U}}$, and likewise, $\breve{\propto}_1 \succcurlyeq \breve{\propto}_2$ and $\breve{\propto}_1 = \breve{\propto}_2$.

3 Interval-valued Neutrosophic N-Structures

In this section, we introduce the interval-valued neutrosophic N-structure and its basic operations, such as union, intersection, and complement, with examples.

Definition 3.1. Consider $\mathcal{D}$ is a set of objects (or points), where each object in $\mathcal{D}$ is represented by $\mathfrak{d}$. An IvNSN-S over $\mathcal{D}$ is characterized as the set

$$\mathcal{N} = \left\{ \frac{(\check{\alpha} = [\alpha^{\mathcal{L}}(\mathfrak{d}_1), \alpha^{\mathcal{U}}(\mathfrak{d}_1)], \check{\beta} = [\beta^{\mathcal{L}}(\mathfrak{d}_1), \beta^{\mathcal{U}}(\mathfrak{d}_1)], \check{\gamma} = [\gamma^{\mathcal{L}}(\mathfrak{d}_1), \gamma^{\mathcal{U}}(\mathfrak{d}_1)])}{\mathfrak{d}_1} \mid \mathfrak{d}_1 \in \mathcal{D} \right\} \tag{3.1}$$

Where

$$\check{\alpha}(\mathfrak{d}_1) : \mathcal{D} \to I[-1,0], \ \check{\beta}(\mathfrak{d}_1) : \mathcal{D} \to I[-1,0], \text{ and } \check{\gamma}(\mathfrak{d}_1) : \mathcal{D} \to I[-1,0]$$

are functions on $\mathcal{D}$ which are called the negative interval-valued degree of membership, the negative interval-valued degree of indeterminacy, and the negative interval-valued degree of non-membership, respectively, on $\mathcal{D}$.

For simplicity we denote IvNSN-S as $\mathcal{N} = (\check{\alpha}, \check{\beta}, \check{\gamma})$

Remark 3.2. Every IvNSN-S $\mathcal{N}$ over $\mathcal{D}$ satisfies the condition for all $\mathfrak{d}_1 \in \mathcal{D}$, $-6 \leq \alpha^{\mathcal{L}}(\mathfrak{d}_1) + \beta^{\mathcal{L}}(\mathfrak{d}_1) + \gamma^{\mathcal{L}}(\mathfrak{d}_1) + \alpha^{\mathcal{U}}(\mathfrak{d}_1) + \beta^{\mathcal{U}}(\mathfrak{d}_1) + \gamma^{\mathcal{U}}(\mathfrak{d}_1) \leq 0$.

Definition 3.3 (Containment). An IvNSN-S $\mathcal{N}_1 = (\check{\alpha}_1, \check{\beta}_1, \check{\gamma}_1)$ is contained in the other IvNSN-S $\mathcal{N}_2 = (\check{\alpha}_2, \check{\beta}_2, \check{\gamma}_2)$, denoted by $\mathcal{N}_1 \subseteq \mathcal{N}_2$, if and only if, $\check{\alpha}_1(\mathfrak{d}_1) \succeq \check{\alpha}_2(\mathfrak{d}_1)$, $\check{\beta}_1(\mathfrak{d}_1) \preceq \check{\beta}_2(\mathfrak{d}_1)$, and $\check{\gamma}_1(\mathfrak{d}_1) \succeq \check{\gamma}_2(\mathfrak{d}_1)$ for all $\mathfrak{d}_1 \in \mathcal{D}$.

Definition 3.4 (Equality). Two IvNSN-Ss $\mathcal{N}_1$ and $\mathcal{N}_2$ are said to be equal, denoted by $\mathcal{N}_1 = \mathcal{N}_2$, if and only if $\mathcal{N}_1 \subseteq \mathcal{N}_2$ and $\mathcal{N}_2 \subseteq \mathcal{N}_1$.

Definition 3.5 (Complement). The complement of an IvNSN-S $\mathcal{N}$ is denoted by $\mathcal{N}^C = (\check{\alpha}^c, \check{\beta}^c, \check{\gamma}^c)$ and is defined by $\check{\alpha}^c(\mathfrak{d}_1) = -\check{1} - \check{\alpha}(\mathfrak{d}_1)$, $\check{\beta}^c(\mathfrak{d}_1) = -\check{1} - \check{\beta}(\mathfrak{d}_1)$, and $\check{\gamma}^c(\mathfrak{d}_1) = -\check{1} - \check{\gamma}(\mathfrak{d}_1)$ for all $\mathfrak{d}_1 \in \mathcal{D}$.

Definition 3.6 (Union). The union of two IvNSN-Ss $\mathcal{N}_1$ and $\mathcal{N}_2$ is an IvNSN-S $\mathcal{N}$, denoted by $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$, and is defined as

$$\mathcal{N}_1 \cup \mathcal{N}_2 = \left\{ (\mathfrak{d}_1; rmin(\check{\alpha}_1(\mathfrak{d}_1), \check{\alpha}_2(\mathfrak{d}_1)), rmax(\check{\beta}_1(\mathfrak{d}_1), \check{\beta}_2(\mathfrak{d}_1)), rmin(\check{\gamma}_1(\mathfrak{d}_1), \check{\gamma}_2(\mathfrak{d}_1))) \right\}.$$

Definition 3.7 (Intersection). The intersection of two IvNSN-Ss $\mathcal{N}_1$ and $\mathcal{N}_2$ is an IvNSN-S $\mathcal{N}$, denoted by $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$, and is defined as

$$\mathcal{N}_1 \cap \mathcal{N}_2 = \left\{ (\mathfrak{d}_1; rmax(\check{\alpha}_1(\mathfrak{d}_1), \check{\alpha}_2(\mathfrak{d}_1)), rmin(\check{\beta}_1(\mathfrak{d}_1), \check{\beta}_2(\mathfrak{d}_1)), rmax(\check{\gamma}_1(\mathfrak{d}_1), \check{\gamma}_2(\mathfrak{d}_1))) \right\}$$

Example 3.8. Consider a universe of discourse $\mathcal{D} = \{\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3\}$. Then an IvNSN-S is represented as

$$\mathcal{N} = \left\{ \begin{array}{c} \frac{([\alpha^{\mathcal{L}}(\mathfrak{d}_1), \alpha^{\mathcal{U}}(\mathfrak{d}_1)],\ [\beta^{\mathcal{L}}(\mathfrak{d}_1), \beta^{\mathcal{U}}(\mathfrak{d}_1)],\ [\gamma^{\mathcal{L}}(\mathfrak{d}_1), \gamma^{\mathcal{U}}(\mathfrak{d}_1)])}{\mathfrak{d}_1}, \\ \frac{([\alpha^{\mathcal{L}}(\mathfrak{d}_2), \alpha^{\mathcal{U}}(\mathfrak{d}_2)],\ [\beta^{\mathcal{L}}(\mathfrak{d}_2), \beta^{\mathcal{U}}(\mathfrak{d}_2)],\ [\gamma^{\mathcal{L}}(\mathfrak{d}_2), \gamma^{\mathcal{U}}(\mathfrak{d}_2)])}{\mathfrak{d}_2}, \\ \frac{([\alpha^{\mathcal{L}}(\mathfrak{d}_3), \alpha^{\mathcal{U}}(\mathfrak{d}_3)],\ [\beta^{\mathcal{L}}(\mathfrak{d}_3), \beta^{\mathcal{U}}(\mathfrak{d}_3)],\ [\gamma^{\mathcal{L}}(\mathfrak{d}_3), \gamma^{\mathcal{U}}(\mathfrak{d}_3)])}{\mathfrak{d}_3} \end{array} \right\}$$

For example,

$$\mathcal{N}_1 = \left\{ \begin{array}{c} \frac{([-0.6,-0.3],\ [-0.8,-0.5],\ [-0.5,-0.2])}{\mathfrak{d}_1}, \\ \frac{([-0.4,-0.2],\ [-1.0,-0.8],\ [-0.8,-0.5])}{\mathfrak{d}_2}, \\ \frac{([-0.5,-0.1],\ [-0.9,-0.6],\ [-0.9,-0.5])}{\mathfrak{d}_3} \end{array} \right\}$$

$$\mathcal{N}_2 = \left\{ \begin{array}{c} \frac{([-0.9,-0.4],\ [-0.5,-0.2],\ [-0.5,-0.1])}{\text{ȡ}_1}, \\ \frac{([-0.5,-0.2],\ [-0.7,-0.3],\ [-0.9,-0.6])}{\text{ȡ}_2}, \\ \frac{([-0.8,-0.1],\ [-0.7,-0.2],\ [-0.3,-0.1])}{\text{ȡ}_3} \end{array} \right\}$$

Then,

$$\mathcal{N}_1 \cup \mathcal{N}_2 = \left\{ \begin{array}{c} \frac{([-0.9,-0.4],\ [-0.5,-0.2],\ [-0.5,-0.2])}{\text{ȡ}_1}, \\ \frac{([-0.5,-0.2],\ [-0.7,-0.3],\ [-0.9,-0.6])}{\text{ȡ}_2}, \\ \frac{([-0.8,-0.1],\ [-0.7,-0.2],\ [-0.9,-0.5])}{\text{ȡ}_3} \end{array} \right\}$$

$$\mathcal{N}_1 \cap \mathcal{N}_2 = \left\{ \begin{array}{c} \frac{([-0.6,-0.3],\ [-0.8,-0.5],\ [-0.5,-0.1])}{\text{ȡ}_1}, \\ \frac{([-0.4,-0.2],\ [-1.0,-0.8],\ [-0.8,-0.5])}{\text{ȡ}_2}, \\ \frac{([-0.5,-0.1],\ [-0.9,-0.6],\ [-0.3,-0.1])}{\text{ȡ}_3} \end{array} \right\}$$

$$\mathcal{N}_1^c = \left\{ \begin{array}{c} \frac{([-0.7,-0.4],\ [-0.5,-0.2],\ [-0.8,-0.5])}{\text{ȡ}_1}, \\ \frac{([-0.8,-0.6],\ [-0.2,0.0],\ [-0.5,-0.2])}{\text{ȡ}_2}, \\ \frac{([-0.9,-0.5],\ [-0.4,-0.1],\ [-0.5,-0.1])}{\text{ȡ}_3} \end{array} \right\}$$

4 Interval-valued Neutrosophic N-d-subalgebra

In this section, we present interval-valued neutrosophic N-d-subalgebra and their related properties.

Definition 4.1. Let $\mathcal{D}$ be a d-algebra. An IvNSN-S $\mathcal{N} = (\check{\alpha}, \check{\beta}, \check{\gamma})$ over $\mathcal{D}$ is called an Interval-valued neutrosophic N-d-subalgebra if it satisfies

(IvNSN-d-SA 1) $\check{\alpha}(\text{ȡ}_1 * \text{ȡ}_2) \preccurlyeq rmax\{\check{\alpha}(\text{ȡ}_1), \check{\alpha}(\text{ȡ}_2)\}$

(IvNSN-d-SA 2) $\check{\beta}(\text{ȡ}_1 * \text{ȡ}_2) \succcurlyeq rmin\left\{\check{\beta}(\text{ȡ}_1), \check{\beta}(\text{ȡ}_2)\right\}$

(IvNSN-d-SA 3) $\check{\gamma}(\text{ȡ}_1 * \text{ȡ}_2) \preccurlyeq rmax\{\check{\gamma}(\text{ȡ}_1), \check{\gamma}(\text{ȡ}_2)\}$ for all $\text{ȡ}_1, \text{ȡ}_2 \in \mathcal{D}$.

Example 4.2. Consider a set $\mathcal{D} = \{0, 1, 2, 3\}$ in which the binary operation "$*$" is defined as shown in the following Cayley table 1.

Table 1. d-algebra

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	2
2	2	2	0	0
3	3	3	3	0

Then $(\mathcal{D}, *, 0)$ is a d-algebra. Let $\mathcal{N} = (\check{\alpha}, \check{\beta}, \check{\gamma})$ be an IvNSN-S in $\mathcal{D}$ as defined in the following table 2.

Then $\mathcal{N} = (\check{\alpha}, \check{\beta}, \check{\gamma})$ is an IvNSN-d-SA.

Example 4.3. Consider a set $\mathcal{D} = \{0, 1, 2, 3\}$ in which the binary operation "$*$" is defined as shown in the following Cayley table 3.

Then $(\mathcal{D}, *, 0)$ is a d-algebra. Let $\mathcal{N} = (\check{\alpha}, \check{\beta}, \check{\gamma})$ be an IvNSN-S in $\mathcal{D}$ as defined in the following table 4.

where $\check{a} \preccurlyeq \check{b}$, $\check{c} \succcurlyeq \check{d}$, $\check{e} \preccurlyeq \check{f}$ and $a_i, b_i, c_i, d_i, e_i, f_i \in [-1, 0]$ for $i = 1, 2$. Then $\mathcal{N} = (\check{\alpha}, \check{\beta}, \check{\gamma})$ is an IvNSN-d-SA.

Table 2. Interval-valued neutrosophic N-d-subalgebra

$\mathcal{D}$	$\check{\alpha}(d_1)$	$\check{\beta}(d_1)$	$\check{\gamma}(d_1)$
0	$[-0.91, -0.53]$	$[-0.57, -0.34]$	$[-0.85, -0.62]$
1	$[-0.91, -0.53]$	$[-0.57, -0.34]$	$[-0.85, -0.62]$
2	$[-0.75, -0.21]$	$[-0.95, -0.47]$	$[-0.57, -0.11]$
3	$[-0.75, -0.21]$	$[-0.95, -0.47]$	$[-0.57, -0.11]$

Table 3. d-algebra

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	1	0

Proposition 4.4. *If an IvNSN-S in $\mathcal{D}$ is an IvNSN-d-SA of $\mathcal{D}$, then $\check{\alpha}(0) \preccurlyeq \check{\alpha}(d_1)$, $\check{\beta}(0) \succcurlyeq \check{\beta}(d_1)$, and $\check{\gamma}(0) \preccurlyeq \check{\gamma}(d_1)$ for all $d_1 \in \mathcal{D}$.*

Proof. Let $d_1 \in \mathcal{D}$. Then by utilizing Definition 4.1 and $d_1 * d_1 = 0$, we obtained

$\check{\alpha}(0) = \check{\alpha}(d_1 * d_1) \preccurlyeq rmax\,\{\check{\alpha}(d_1), \check{\alpha}(d_1)\} = \check{\alpha}(d_1)$

$\check{\beta}(0) = \check{\beta}(d_1 * d_1) \succcurlyeq rmin\left\{\check{\beta}(d_1), \check{\beta}(d_1)\right\} = \check{\beta}(d_1)$

$\check{\gamma}(0) = \check{\gamma}(d_1 * d_1) \preccurlyeq rmax\,\{\check{\gamma}(d_1), \check{\gamma}(d_1)\} = \check{\gamma}(d_1)$, for all $d_1 \in \mathcal{D}$. □

Theorem 4.5. *If $\{\mathcal{N}_i : i \in \Omega\}$ is an arbitrary family of IvNSN-d-SAs of $\mathcal{D}$, then $\cap\mathcal{N}_i$ is an IvNSN-d-SA of $\mathcal{D}$, where*

$$\cap\mathcal{N}_i = \left\{\left\langle d_1, rmax(\check{\alpha}_i(d_1)), rmin(\check{\beta}_i(d_1)), rmax(\check{\gamma}_i(d_1))\right\rangle |d_1 \in \mathcal{D}\right\}.$$

Proof. Let $d_1, d_2 \in \mathcal{D}$. Then,

$$rmax\check{\alpha}_i(d_1 * d_2) \preccurlyeq rmax\,\{rmax\,\{\check{\alpha}_i(d_1), \check{\alpha}_i(d_2)\}\} = rmax\,\{rmax(\check{\alpha}_i(d_1)), rmax(\check{\alpha}_i(d_1))\}$$

$$rmin\check{\beta}_i(d_1 * d_2) \succcurlyeq rmin\left\{rmin\left\{\check{\beta}_i(d_1), \check{\beta}_i(d_2)\right\}\right\} = rmin\left\{rmin(\check{\beta}_i(d_1)), rmin(\check{\beta}_i(d_1))\right\}$$

$$rmax\check{\gamma}_i(d_1 * d_2) \preccurlyeq rmax\,\{rmax\,\{\check{\gamma}_i(d_1), \check{\gamma}_i(d_2)\}\} = rmax\,\{rmax(\check{\gamma}_i(d_1)), rmax(\check{\gamma}_i(d_1))\}$$

Hence $\cap\mathcal{N}_i = \left\{\left\langle d_1, rmax(\check{\alpha}_i(d_1)), rmin(\check{\beta}_i(d_1)), rmax(\check{\gamma}_i(d_1))\right\rangle |d_1 \in \mathcal{D}\right\}$ is an IvNSN-d-SA of $\mathcal{D}$. □

Theorem 4.6. *If an IvNSN-S in $\mathcal{D}$ is an IvNSN-d-SA of $\mathcal{D}$, then the sets*

$\mathcal{N}_{\check{\alpha}} = \{d_1 \in \mathcal{D} \mid \check{\alpha}(d_1) = \check{\alpha}(0)\}$, $\mathcal{N}_{\check{\beta}} = \left\{d_1 \in \mathcal{D} \mid \check{\beta}(d_1) = \check{\beta}(0)\right\}$, *and*

$\mathcal{N}_{\check{\gamma}} = \{d_1 \in \mathcal{D} \mid \check{\gamma}(d_1) = \check{\gamma}(0)\}$ *are d-subalgebras of $\mathcal{D}$.*

Proof. Assume that $\mathcal{N} = (\check{\alpha}, \check{\beta}, \check{\gamma})$ be an IvNSN-d-SA of $\mathcal{D}$. Let $d_1, d_2 \in \mathcal{N}_{\check{\alpha}}$. Therefore $\check{\alpha}(d_1) = \check{\alpha}(0)$, $\check{\alpha}(d_2) = \check{\alpha}(0)$. Now by definition 4.1 $\check{\alpha}(d_1 * d_2) \preccurlyeq rmax\,\{\check{\alpha}(d_1), \check{\alpha}(d_2)\} = rmax\,\{\check{\alpha}(0), \check{\alpha}(0)\} = \check{\alpha}(0) \Rightarrow \check{\alpha}(d_1 * d_2) \preccurlyeq \check{\alpha}(0)$. From proposition 4.4, we have $\check{\alpha}(0) \preccurlyeq \check{\alpha}(d_1 * d_2)$. Therefore $\check{\alpha}(d_1 * d_2) = \check{\alpha}(0) \Rightarrow d_1 * d_2 \in \mathcal{N}_{\check{\alpha}}$. Therefore, the set $\mathcal{N}_{\check{\alpha}} = \{d_1 \in \mathcal{D} \mid \check{\alpha}(d_1) = \check{\alpha}(0)\}$ is a d-subalgebra of $\mathcal{D}$. Similarly, we can show that $\mathcal{N}_{\check{\beta}}$ and $\mathcal{N}_{\check{\gamma}}$ are d-subalgebras of $\mathcal{D}$. □

Definition 4.7. Let $\mathcal{N} = (\check{\alpha}, \check{\beta}, \check{\gamma})$ be an IvNSN-S in $\mathcal{D}$ and let $\check{r} = [r^{\mathcal{L}}, r^{\mathcal{U}}]$, $\check{s} = [s^{\mathcal{L}}, s^{\mathcal{U}}]$, $\check{t} = [t^{\mathcal{L}}, t^{\mathcal{U}}] \in I[-1, 0]$. Then, we define the following level sets for all $d_1 \in \mathcal{D}$

$$L_1(\check{\alpha}, \check{r}) = \left\{d_1 \in \mathcal{D} : \check{\alpha}(d_1) \preccurlyeq [r^{\mathcal{L}}, r^{\mathcal{U}}]\right\}$$

$$U(\check{\beta}, \check{s}) = \left\{d_1 \in \mathcal{D} : \check{\beta}(d_1) \succcurlyeq [s^{\mathcal{L}}, s^{\mathcal{U}}]\right\}$$

$$L_2(\check{\gamma}, \check{t}) = \left\{d_1 \in \mathcal{D} : \check{\gamma}(d_1) \preccurlyeq [t^{\mathcal{L}}, t^{\mathcal{U}}]\right\}$$

Table 4. Interval-valued neutrosophic N-d-subalgebra

$\mathcal{D}$	$\breve{\alpha}(\mathfrak{d}_1)$	$\breve{\beta}(\mathfrak{d}_1)$	$\breve{\gamma}(\mathfrak{d}_1)$
0	$\breve{a}=[a_1,a_2]$	$\breve{c}=[c_1,c_2]$	$\breve{e}=[e_1,e_2]$
1	$\breve{a}=[a_1,a_2]$	$\breve{c}=[c_1,c_2]$	$\breve{e}=[e_1,e_2]$
2	$\breve{b}=[b_1,b_2]$	$\breve{d}=[d_1,d_2]$	$\breve{f}=[f_1,f_2]$
3	$\breve{a}=[a_1,a_2]$	$\breve{c}=[c_1,c_2]$	$\breve{e}=[e_1,e_2]$

Theorem 4.8. *If an IvNSN-S in $\mathcal{D}$ is an IvNSN-d-SA of $\mathcal{D}$, then the level sets $L_1(\breve{\alpha},\breve{r})$, $U(\breve{\beta},\breve{s})$, and $L_2(\breve{\gamma},\breve{t})$ of $\mathcal{N}$ are d-subalgebras of $\mathcal{D}$.*

Proof. Assume that $\mathcal{N}=(\breve{\alpha},\breve{\beta},\breve{\gamma})$ be an IvNSN-d-SA of $\mathcal{D}$. Let for any $\mathfrak{d}_1,\mathfrak{d}_2\in L_1(\breve{\alpha},\breve{r})\cap U(\breve{\beta},\breve{s})\cap L_2(\breve{\gamma},\breve{t})$, we have

$$\begin{pmatrix} \breve{\alpha}(\mathfrak{d}_1)\preccurlyeq[r^{\mathcal{L}},r^{\mathcal{U}}] \text{ and } \breve{\alpha}(\mathfrak{d}_2)\preccurlyeq[r^{\mathcal{L}},r^{\mathcal{U}}] \\ \breve{\beta}(\mathfrak{d}_1)\succcurlyeq[s^{\mathcal{L}},s^{\mathcal{U}}] \text{ and } \breve{\beta}(\mathfrak{d}_2)\succcurlyeq[s^{\mathcal{L}},s^{\mathcal{U}}] \\ \breve{\gamma}(\mathfrak{d}_1)\preccurlyeq[t^{\mathcal{L}},t^{\mathcal{U}}] \text{ and } \breve{\gamma}(\mathfrak{d}_2)\preccurlyeq[t^{\mathcal{L}},t^{\mathcal{U}}] \end{pmatrix}$$

Now utilizing (IvNSN-d-SA 1), (IvNSN-d-SA 2), and (IvNSN-d-SA 3), we obtain

$$\begin{pmatrix} \breve{\alpha}(\mathfrak{d}_1*\mathfrak{d}_2)\preccurlyeq rmax\{\breve{\alpha}(\mathfrak{d}_1),\breve{\alpha}(\mathfrak{d}_2)\}\preccurlyeq rmax\{[r^{\mathcal{L}},r^{\mathcal{U}}],[r^{\mathcal{L}},r^{\mathcal{U}}]\}=[r^{\mathcal{L}},r^{\mathcal{U}}] \\ \breve{\beta}(\mathfrak{d}_1*\mathfrak{d}_2)\succcurlyeq rmin\left\{\breve{\beta}(\mathfrak{d}_1),\breve{\beta}(\mathfrak{d}_2)\right\}\succcurlyeq rmin\{[s^{\mathcal{L}},s^{\mathcal{U}}],[s^{\mathcal{L}},s^{\mathcal{U}}]\}=[s^{\mathcal{L}},s^{\mathcal{U}}] \\ \breve{\gamma}(\mathfrak{d}_1*\mathfrak{d}_2)\preccurlyeq rmax\{\breve{\gamma}(\mathfrak{d}_1),\breve{\gamma}(\mathfrak{d}_2)\}\preccurlyeq rmax\{[t^{\mathcal{L}},t^{\mathcal{U}}],[t^{\mathcal{L}},t^{\mathcal{U}}]\}=[t^{\mathcal{L}},t^{\mathcal{U}}] \end{pmatrix}$$

$\Rightarrow \mathfrak{d}_1*\mathfrak{d}_2\in L_1(\breve{\alpha},\breve{r})$, $\mathfrak{d}_1*\mathfrak{d}_2\in U(\breve{\beta},\breve{s})$, and $\mathfrak{d}_1*\mathfrak{d}_2\in L_2(\breve{\gamma},\breve{t})$. Therefore, the level sets $L_1(\breve{\alpha},\breve{r})$, $U(\breve{\beta},\breve{s})$, and $L_2(\breve{\gamma},\breve{t})$ of $\mathcal{N}$ are d-subalgebras of $\mathcal{D}$. □

Theorem 4.9. *Let $\mathcal{N}=(\breve{\alpha},\breve{\beta},\breve{\gamma})$ be an IvNSN-S in $\mathcal{D}$ such that the level sets $L_1(\breve{\alpha},\breve{r})$, $U(\breve{\beta},\breve{s})$, and $L_2(\breve{\gamma},\breve{t})$ of $\mathcal{N}$ are d-subalgebras of $\mathcal{D}$. Then $\mathcal{N}=(\breve{\alpha},\breve{\beta},\breve{\gamma})$ is an IvNSN-d-SA of $\mathcal{D}$.*

Proof. Assume that for any $\breve{r},\breve{s},\breve{t}\in I[-1,0]$ the sets $L_1(\breve{\alpha},\breve{r})$, $U(\breve{\beta},\breve{s})$, and $L_2(\breve{\gamma},\breve{t})$ of $\mathcal{N}$ are d-subalgebras of $\mathcal{D}$. Let us take $\zeta_1,\zeta_2\in L_1(\breve{\alpha},\breve{r})$ such that $\breve{\alpha}(\zeta_1*\zeta_2)\succ rmax\{\breve{\alpha}(\zeta_1),\breve{\alpha}(\zeta_2)\}$. Suppose that $\breve{\alpha}(\zeta_1*\zeta_2)=\breve{r}_1=[r_1^{\mathcal{L}},r_1^{\mathcal{U}}]$, $\breve{\alpha}(\zeta_1)=\breve{r}_2=[r_2^{\mathcal{L}},r_2^{\mathcal{U}}]$, and $\breve{\alpha}(\zeta_2)=\breve{r}_3=[r_3^{\mathcal{L}},r_3^{\mathcal{U}}]$. Then,

$[r_1^{\mathcal{L}},r_1^{\mathcal{U}}]\succ rmax\{[r_2^{\mathcal{L}},r_2^{\mathcal{U}}],[r_3^{\mathcal{L}},r_3^{\mathcal{U}}]\}=[max\{r_2^{\mathcal{L}},r_3^{\mathcal{L}}\},max\{r_2^{\mathcal{U}},r_3^{\mathcal{U}}\}]$ and so,
$r_1^{\mathcal{L}}>max\{r_2^{\mathcal{L}},r_3^{\mathcal{L}}\}$ and $r_1^{\mathcal{U}}>max\{r_2^{\mathcal{U}},r_3^{\mathcal{U}}\}$.
Taking $\breve{r}_4=[r_4^{\mathcal{L}},r_4^{\mathcal{U}}]=\frac{1}{2}[\breve{\alpha}(\zeta_1*\zeta_2)+rmax\{\breve{\alpha}(\zeta_1),\breve{\alpha}(\zeta_2)\}]$
$=\frac{1}{2}[[r_1^{\mathcal{L}},r_1^{\mathcal{U}}]+[max\{r_2^{\mathcal{L}},r_3^{\mathcal{L}}\},max\{r_2^{\mathcal{U}},r_3^{\mathcal{U}}\}]]$
$=[\frac{1}{2}(r_1^{\mathcal{L}}+max\{r_2^{\mathcal{L}},r_3^{\mathcal{L}}\}),\frac{1}{2}(r_1^{\mathcal{U}}+max\{r_2^{\mathcal{U}},r_3^{\mathcal{U}}\})]$
It follows that
$r_1^{\mathcal{L}}>r_4^{\mathcal{L}}=\frac{1}{2}(r_1^{\mathcal{L}}+max\{r_2^{\mathcal{L}},r_3^{\mathcal{L}}\})>max\{r_2^{\mathcal{L}},r_3^{\mathcal{L}}\}$,
$r_1^{\mathcal{U}}>r_4^{\mathcal{U}}=\frac{1}{2}(r_1^{\mathcal{U}}+max\{r_2^{\mathcal{U}},r_3^{\mathcal{U}}\})>max\{r_2^{\mathcal{U}},r_3^{\mathcal{U}}\}$.
Hence, $[max\{r_2^{\mathcal{L}},r_3^{\mathcal{L}}\},\ max\{r_2^{\mathcal{U}},r_3^{\mathcal{U}}\}]\prec[r_4^{\mathcal{L}},r_4^{\mathcal{U}}]\prec[r_1^{\mathcal{L}},r_1^{\mathcal{U}}]=\breve{\alpha}(\zeta_1*\zeta_2)$.
Therefore, $\zeta_1*\zeta_2\notin L_1(\breve{\alpha},\ \breve{r}_4)$. On the other hand
$\breve{\alpha}(\zeta_1)=\breve{r}_2=[r_2^{\mathcal{L}},r_2^{\mathcal{U}}]\preccurlyeq[max\{r_2^{\mathcal{L}},r_3^{\mathcal{L}}\},max\{r_2^{\mathcal{U}},r_3^{\mathcal{U}}\}]\prec[r_4^{\mathcal{L}},r_4^{\mathcal{U}}]=\breve{r}_4$,
$\breve{\alpha}(\zeta_2)=\breve{r}_3=[r_3^{\mathcal{L}},r_3^{\mathcal{U}}]\preccurlyeq[max\{r_2^{\mathcal{L}},r_3^{\mathcal{L}}\},\ max\{r_2^{\mathcal{U}},r_3^{\mathcal{U}}\}]\prec[r_4^{\mathcal{L}},r_4^{\mathcal{U}}]=\breve{r}_4$.
i.e., $\zeta_1,\zeta_2\in L_1(\breve{\alpha},\ \breve{r}_4)$.
This a contradiction and therefore $\breve{\alpha}(\zeta_1*\zeta_2)\preccurlyeq rmax\{\breve{\alpha}(\zeta_1),\breve{\alpha}(\zeta_2)\}$ for all $\zeta_1,\zeta_2\in\mathcal{D}$.

Suppose that $\breve{\beta}(\zeta_1*\zeta_2)\prec rmin\left\{\breve{\beta}(\zeta_1),\breve{\beta}(\zeta_2)\right\}$ for some $\zeta_1,\zeta_2\in U(\breve{\beta},\ \breve{s})$. Let us take $\breve{\beta}(\zeta_1*\zeta_2)=\breve{s}_1=[s_1^{\mathcal{L}},s_1^{\mathcal{U}}]$, $\breve{\beta}(\zeta_1)=\breve{s}_2=[s_2^{\mathcal{L}},s_2^{\mathcal{U}}]$, and $\breve{\beta}(\zeta_2)=\breve{s}_3=[s_3^{\mathcal{L}},s_3^{\mathcal{U}}]$. Then,
$[s_1^{\mathcal{L}},s_1^{\mathcal{U}}]\prec rmin\{[s_2^{\mathcal{L}},s_2^{\mathcal{U}}],[s_3^{\mathcal{L}},s_3^{\mathcal{U}}]\}=[min\{s_2^{\mathcal{L}},s_3^{\mathcal{L}}\},min\{s_2^{\mathcal{U}},s_3^{\mathcal{U}}\}]$ and so,
$s_1^{\mathcal{L}}<min\{s_2^{\mathcal{L}},s_3^{\mathcal{L}}\}$ and $s_1^{\mathcal{U}}<min\{s_2^{\mathcal{U}},s_3^{\mathcal{U}}\}$.
Taking $\breve{s}_4=[s_4^{\mathcal{L}},s_4^{\mathcal{U}}]=\frac{1}{2}\left[\breve{\beta}(\zeta_1*\zeta_2)+rmin\left\{\breve{\beta}(\zeta_1),\breve{\beta}(\zeta_2)\right\}\right]$
$=\frac{1}{2}[[s_1^{\mathcal{L}},s_1^{\mathcal{U}}]+[min\{s_2^{\mathcal{L}},s_3^{\mathcal{L}}\},min\{s_2^{\mathcal{U}},s_3^{\mathcal{U}}\}]]$

$= \left[\frac{1}{2}(s_1^{\mathcal{L}} + min\left\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\right\}), \frac{1}{2}(s_1^{\mathcal{U}} + min\left\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\right\})\right]$

It follows that

$s_1^{\mathcal{L}} < s_4^{\mathcal{L}} = \frac{1}{2}(s_1^{\mathcal{L}} + min\left\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\right\}) < min\left\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\right\}$,

$s_1^{\mathcal{U}} < s_4^{\mathcal{U}} = \frac{1}{2}(s_1^{\mathcal{U}} + min\left\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\right\}) < min\left\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\right\}$.

Hence, $\left[min\left\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\right\},\ min\left\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\right\}\right] \succ \left[s_4^{\mathcal{L}}, s_4^{\mathcal{U}}\right] \succ \left[s_1^{\mathcal{L}}, s_1^{\mathcal{U}}\right] = \check{\beta}(\zeta_1 * \zeta_2)$.

Therefore, $\zeta_1 * \zeta_2 \notin U(\check{\beta},\ \check{s_4})$. On the other hand

$\check{\beta}(\zeta_1) = \check{s_2} = \left[s_2^{\mathcal{L}}, s_2^{\mathcal{U}}\right] \succcurlyeq \left[min\left\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\right\}, min\left\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\right\}\right] \succ \left[s_4^{\mathcal{L}}, s_4^{\mathcal{U}}\right] = \check{s_4}$,

$\check{\beta}(\zeta_2) = \check{s_3} = \left[s_3^{\mathcal{L}}, s_3^{\mathcal{U}}\right] \succcurlyeq \left[min\left\{s_2^{\mathcal{L}}, s_3^{\mathcal{L}}\right\},\ min\left\{s_2^{\mathcal{U}}, s_3^{\mathcal{U}}\right\}\right] \succ \left[s_4^{\mathcal{L}}, s_4^{\mathcal{U}}\right] = \check{s_4}$.

i.e., $\zeta_1, \zeta_2 \in U(\check{\beta},\ \check{s_4})$. This a contradiction and therefore

$\check{\beta}(\zeta_1 * \zeta_2) \succcurlyeq rmin\left\{\check{\beta}(\zeta_1), \check{\beta}(\zeta_2)\right\}$ for all $\zeta_1, \zeta_2 \in \mathcal{D}$.

Let us take $\zeta_1, \zeta_2 \in L_2(\check{\gamma},\ \check{t})$ such that $\check{\gamma}(\zeta_1 * \zeta_2) \succ rmax\left\{\check{\gamma}(\zeta_1), \check{\gamma}(\zeta_2)\right\}$. Suppose that $\check{\gamma}(\zeta_1 * \zeta_2) = \check{t_1} = \left[t_1^{\mathcal{L}}, t_1^{\mathcal{U}}\right]$, $\check{\gamma}(\zeta_1) = \check{t_2} = \left[t_2^{\mathcal{L}}, t_2^{\mathcal{U}}\right]$, and $\check{\gamma}(\zeta_2) = \check{t_3} = \left[t_3^{\mathcal{L}}, t_3^{\mathcal{U}}\right]$. Then,

$\left[t_1^{\mathcal{L}}, t_1^{\mathcal{U}}\right] \succ rmax\left\{\left[t_2^{\mathcal{L}}, t_2^{\mathcal{U}}\right], \left[t_3^{\mathcal{L}}, t_3^{\mathcal{U}}\right]\right\} = \left[max\left\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\right\}, max\left\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\right\}\right]$ and so,

$t_1^{\mathcal{L}} > max\left\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\right\}$ and $t_1^{\mathcal{U}} > max\left\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\right\}$.

Taking $\check{t_4} = \left[t_4^{\mathcal{L}}, t_4^{\mathcal{U}}\right] = \frac{1}{2}\left[\check{\gamma}(\zeta_1 * \zeta_2) + rmax\left\{\check{\gamma}(\zeta_1), \check{\gamma}(\zeta_2)\right\}\right]$

$= \frac{1}{2}\left[\left[t_1^{\mathcal{L}}, t_1^{\mathcal{U}}\right] + \left[max\left\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\right\}, max\left\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\right\}\right]\right]$

$= \left[\frac{1}{2}(t_1^{\mathcal{L}} + max\left\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\right\}), \frac{1}{2}(t_1^{\mathcal{U}} + max\left\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\right\})\right]$

It follows that

$t_1^{\mathcal{L}} > t_4^{\mathcal{L}} = \frac{1}{2}(t_1^{\mathcal{L}} + max\left\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\right\}) > max\left\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\right\}$,

$t_1^{\mathcal{U}} > t_4^{\mathcal{U}} = \frac{1}{2}(t_1^{\mathcal{U}} + max\left\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\right\}) > max\left\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\right\}$.

Hence, $\left[max\left\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\right\},\ max\left\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\right\}\right] \prec \left[t_4^{\mathcal{L}}, t_4^{\mathcal{U}}\right] \prec \left[t_1^{\mathcal{L}}, t_1^{\mathcal{U}}\right] = \check{\gamma}(\zeta_1 * \zeta_2)$.

Therefore, $\zeta_1 * \zeta_2 \notin L_1(\check{\gamma},\ \check{t_4})$. On the other hand

$\check{\gamma}(\zeta_1) = \check{t_2} = \left[t_2^{\mathcal{L}}, t_2^{\mathcal{U}}\right] \preccurlyeq \left[max\left\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\right\}, max\left\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\right\}\right] \prec \left[t_4^{\mathcal{L}}, t_4^{\mathcal{U}}\right] = \check{t_4}$,

$\check{\gamma}(\zeta_2) = \check{t_3} = \left[t_3^{\mathcal{L}}, t_3^{\mathcal{U}}\right] \preccurlyeq \left[max\left\{t_2^{\mathcal{L}}, t_3^{\mathcal{L}}\right\},\ max\left\{t_2^{\mathcal{U}}, t_3^{\mathcal{U}}\right\}\right] \prec \left[t_4^{\mathcal{L}}, t_4^{\mathcal{U}}\right] = \check{t_4}$.

i.e., $\zeta_1, \zeta_2 \in L_1(\check{\gamma},\ \check{t_4})$. This a contradiction and therefore

$\check{\gamma}(\zeta_1 * \zeta_2) \preccurlyeq rmax\left\{\check{\gamma}(\zeta_1), \check{\gamma}(\zeta_2)\right\}$ for all $\zeta_1, \zeta_2 \in \mathcal{D}$. This completes the proof. □

Theorem 4.10. *Any d-subalgebra of $\mathcal{D}$ can be realized as a level d-subalgebra of some IvNSN-d-SA of $\mathcal{D}$.*

Proof. Let $\mathcal{J}$ be a d-subalgebra of $\mathcal{D}$ and let $\mathcal{N} = (\check{\alpha}, \check{\beta}, \check{\gamma})$ be an IvNSN-S in $\mathcal{D}$ defined by $\check{\alpha}(\text{đ}_1) = \begin{cases} \check{r} = \left[r^{\mathcal{L}}, r^{\mathcal{U}}\right], & if\ \text{đ}_1 \in \mathcal{J}, \\ \check{0} = [0,0], & otherwise, \end{cases}$, $\check{\beta}(\text{đ}_1) = \begin{cases} \check{s} = \left[s^{\mathcal{L}}, s^{\mathcal{U}}\right], & if\ \text{đ}_1 \in \mathcal{J}, \\ \check{-1} = [-1,-1], & otherwise, \end{cases}$, and $\check{\gamma}(\text{đ}_1) = \begin{cases} \check{t} = \left[t^{\mathcal{L}}, t^{\mathcal{U}}\right], & if\ \text{đ}_1 \in \mathcal{J}, \\ \check{0} = [0,0], & otherwise, \end{cases}$ for all $\text{đ}_1 \in \mathcal{J}$, where $r^{\mathcal{L}}$, $r^{\mathcal{U}}$, $s^{\mathcal{L}}$, $s^{\mathcal{U}}$, $t^{\mathcal{L}}$, $t^{\mathcal{U}} \in (-1,0)$.

Let $\text{đ}_1, \text{đ}_2 \in \mathcal{J}$, then $\text{đ}_1 * \text{đ}_2 \in \mathcal{J}$. Hence

$$\begin{pmatrix} \check{\alpha}(\text{đ}_1 * \text{đ}_2) = \left[r^{\mathcal{L}}, r^{\mathcal{U}}\right] = rmax\left\{\left[r^{\mathcal{L}}, r^{\mathcal{U}}\right], \left[r^{\mathcal{L}}, r^{\mathcal{U}}\right]\right\} = rmax\left\{\check{\alpha}(\text{đ}_1), \check{\alpha}(\text{đ}_2)\right\} \\ \check{\beta}(\text{đ}_1 * \text{đ}_2) = \left[s^{\mathcal{L}}, s^{\mathcal{U}}\right] = rmin\left\{\left[s^{\mathcal{L}}, s^{\mathcal{U}}\right], \left[s^{\mathcal{L}}, s^{\mathcal{U}}\right]\right\} = rmin\left\{\check{\beta}(\text{đ}_1), \check{\beta}(\text{đ}_2)\right\} \\ \check{\gamma}(\text{đ}_1 * \text{đ}_2) = \left[t^{\mathcal{L}}, t^{\mathcal{U}}\right] = rmax\left\{\left[t^{\mathcal{L}}, t^{\mathcal{U}}\right], \left[t^{\mathcal{L}}, t^{\mathcal{U}}\right]\right\} = rmax\left\{\check{\gamma}(\text{đ}_1), \check{\gamma}(\text{đ}_2)\right\} \end{pmatrix}$$

If $\text{đ}_1 \in \mathcal{J}$ and $\text{đ}_2 \notin \mathcal{J}$, then $\check{\alpha}(\text{đ}_1) = \left[r^{\mathcal{L}}, r^{\mathcal{U}}\right]$, $\check{\beta}(\text{đ}_1) = \left[s^{\mathcal{L}}, s^{\mathcal{U}}\right]$, $\check{\gamma}(\text{đ}_1) = \left[t^{\mathcal{L}}, t^{\mathcal{U}}\right]$, $\check{\alpha}(\text{đ}_2) = [0,0]$, $\check{\beta}(\text{đ}_2) = [-1,-1]$, and $\check{\gamma}(\text{đ}_2) = [0,0]$. Therefore,

$$\begin{pmatrix} \check{\alpha}(\text{đ}_1 * \text{đ}_2) \preccurlyeq [0,0] = rmax\left\{\left[r^{\mathcal{L}}, r^{\mathcal{U}}\right], [0,0]\right\} = rmax\left\{\check{\alpha}(\text{đ}_1), \check{\alpha}(\text{đ}_2)\right\} \\ \check{\beta}(\text{đ}_1 * \text{đ}_2) \succcurlyeq [-1,-1] = rmin\left\{\left[s^{\mathcal{L}}, s^{\mathcal{U}}\right], [-1,-1]\right\} = rmin\left\{\check{\beta}(\text{đ}_1), \check{\beta}(\text{đ}_2)\right\} \\ \check{\gamma}(\text{đ}_1 * \text{đ}_2) \preccurlyeq [0,0] = rmax\left\{\left[t^{\mathcal{L}}, t^{\mathcal{U}}\right], [0,0]\right\} = rmax\left\{\check{\gamma}(\text{đ}_1), \check{\gamma}(\text{đ}_2)\right\} \end{pmatrix}$$

If $\text{đ}_1 \notin \mathcal{J}$ and $\text{đ}_2 \in \mathcal{J}$, then $\check{\alpha}(\text{đ}_1) = [0,0]$, $\check{\beta}(\text{đ}_1) = [-1,-1]$, and $\check{\gamma}(\text{đ}_1) = [0,0]$ $\check{\alpha}(\text{đ}_2) = \left[r^{\mathcal{L}}, r^{\mathcal{U}}\right]$, $\check{\beta}(\text{đ}_2) = \left[s^{\mathcal{L}}, s^{\mathcal{U}}\right]$, $\check{\gamma}(\text{đ}_2) = \left[t^{\mathcal{L}}, t^{\mathcal{U}}\right]$. Therefore,

$$\begin{pmatrix} \check{\alpha}(\text{đ}_1 * \text{đ}_2) \preccurlyeq [0,0] = rmax\left\{[0,0], \left[r^{\mathcal{L}}, r^{\mathcal{U}}\right]\right\} = rmax\left\{\check{\alpha}(\text{đ}_1), \check{\alpha}(\text{đ}_2)\right\} \\ \check{\beta}(\text{đ}_1 * \text{đ}_2) \succcurlyeq [-1,-1] = rmin\left\{[-1,-1], \left[s^{\mathcal{L}}, s^{\mathcal{U}}\right]\right\} = rmin\left\{\check{\beta}(\text{đ}_1), \check{\beta}(\text{đ}_2)\right\} \\ \check{\gamma}(\text{đ}_1 * \text{đ}_2) \preccurlyeq [0,0] = rmax\left\{[0,0], \left[t^{\mathcal{L}}, t^{\mathcal{U}}\right]\right\} = rmax\left\{\check{\gamma}(\text{đ}_1), \check{\gamma}(\text{đ}_2)\right\} \end{pmatrix}$$

If $\text{đ}_1 \notin \mathcal{J}$ and $\text{đ}_2 \notin \mathcal{J}$, then $\check{\alpha}(\text{đ}_1) = \check{\alpha}(\text{đ}_2) = [0,0]$, $\check{\beta}(\text{đ}_1) = \check{\beta}(\text{đ}_2) = [-1,-1]$, and

$\check{\gamma}(\mathfrak{d}_1) = \check{\gamma}(\mathfrak{d}_2) = [0,0]$. Therefore,

$$\left(\begin{array}{c} \check{\alpha}(\mathfrak{d}_1 * \mathfrak{d}_2) \preccurlyeq [0,0] = rmax\{[0,0],[0,0]\} = rmax\{\check{\alpha}(\mathfrak{d}_1),\check{\alpha}(\mathfrak{d}_2)\} \\ \check{\beta}(\mathfrak{d}_1 * \mathfrak{d}_2) \succcurlyeq [-1,-1] = rmin\{[-1,-1],[-1,-1]\} = rmin\left\{\check{\beta}(\mathfrak{d}_1),\check{\beta}(\mathfrak{d}_2)\right\} \\ \check{\gamma}(\mathfrak{d}_1 * \mathfrak{d}_2) \preccurlyeq [0,0] = rmax\{[0,0],[0,0]\} = rmax\{\check{\gamma}(\mathfrak{d}_1),\check{\gamma}(\mathfrak{d}_2)\} \end{array}\right)$$

Hence $\mathcal{N} = (\check{\alpha}, \check{\beta}, \check{\gamma})$ is an IvNSN-d-SA of $\mathcal{D}$. Obviously, $L_1(\check{\alpha}, \check{r}) = U(\check{\beta}, \check{s}) = L_2(\check{\gamma}, \check{t}) = \mathcal{J}$. This completes the proof. □

Definition 4.11. Let f be mapping from a d-algebra $\mathcal{D}_1$ to a d-algebra $\mathcal{D}_2$. If $\mathcal{N}$ is an IvNSN-S of $\mathcal{D}_2$, then the preimage of $\mathcal{N}$ under f, denoted by

$$f^{-1}(\mathcal{N}) = (f^{-1}(\check{\alpha}), f^{-1}(\check{\beta}), f^{-1}(\check{\gamma}))$$

is an IvNSN-S of $\mathcal{D}_1$, defined by

$$f^{-1}(\check{\alpha})(\mathfrak{d}_1) = \check{\alpha}(f(\mathfrak{d}_1)), \quad f^{-1}(\check{\beta})(\mathfrak{d}_1) = \check{\beta}(f(\mathfrak{d}_1)), \text{ and } \quad f^{-1}(\check{\gamma})(\mathfrak{d}_1) = \check{\gamma}(f(\mathfrak{d}_1))$$

for all $\mathfrak{d}_1 \in \mathcal{D}_1$.

Theorem 4.12. *Let f be a homomorphism of a d-algebra $\mathcal{D}_1$ into a d-algebra $\mathcal{D}_2$, and $\mathcal{N}$ be an IvNSN-d-SA of $\mathcal{D}_2$. Then $f^{-1}(\mathcal{N}) = (f^{-1}(\check{\alpha}), f^{-1}(\check{\beta}), f^{-1}(\check{\gamma}))$ is an IvNSN-d-SA of $\mathcal{D}_1$, where $f^{-1}(\check{\alpha})(\mathfrak{d}_1) = \check{\alpha}(f(\mathfrak{d}_1))$, $f^{-1}(\check{\beta})(\mathfrak{d}_1) = \check{\beta}(f(\mathfrak{d}_1))$, and $f^{-1}(\check{\gamma})(\mathfrak{d}_1) = \check{\gamma}(f(\mathfrak{d}_1))$ for all $\mathfrak{d}_1 \in \mathcal{D}_1$.*

Proof. For any $\mathfrak{d}_1, \mathfrak{d}_2 \in \mathcal{D}_1$, we have

$$\begin{aligned} f^{-1}(\check{\alpha}(\mathfrak{d}_1 * \mathfrak{d}_2)) &= \check{\alpha}(f(\mathfrak{d}_1 * \mathfrak{d}_2)) = \check{\alpha}(f(\mathfrak{d}_1) * f(\mathfrak{d}_2)) \\ &\preccurlyeq rmax\{\check{\alpha}(f(\mathfrak{d}_1)), \check{\alpha}(f(\mathfrak{d}_2))\} \\ &= rmax\left\{f^{-1}(\check{\alpha}(\mathfrak{d}_1)), f^{-1}(\check{\alpha}(\mathfrak{d}_2))\right\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\check{\beta}(\mathfrak{d}_1 * \mathfrak{d}_2)) &= \check{\beta}(f(\mathfrak{d}_1 * \mathfrak{d}_2)) = \check{\beta}(f(\mathfrak{d}_1) * f(\mathfrak{d}_2)) \\ &\succcurlyeq rmin\left\{\check{\beta}(f(\mathfrak{d}_1)), \check{\beta}(f(\mathfrak{d}_2))\right\} \\ &= rmin\left\{f^{-1}(\check{\beta}(\mathfrak{d}_1)), f^{-1}(\check{\beta}(\mathfrak{d}_2))\right\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\check{\gamma}(\mathfrak{d}_1 * \mathfrak{d}_2)) &= \check{\gamma}(f(\mathfrak{d}_1 * \mathfrak{d}_2)) = \check{\gamma}(f(\mathfrak{d}_1) * f(\mathfrak{d}_2)) \\ &\preccurlyeq rmax\{\check{\gamma}(f(\mathfrak{d}_1)), \check{\gamma}(f(\mathfrak{d}_2))\} \\ &= rmax\left\{f^{-1}(\check{\gamma}(\mathfrak{d}_1)), f^{-1}(\check{\gamma}(\mathfrak{d}_2))\right\}. \end{aligned}$$

Hence, $f^{-1}(\mathcal{N})$ is an interval-valued neutrosophic N-subalgebra of $\mathcal{D}_1$. □

5 Conclusion

In this study, we presented the concept of interval-valued neutrosophic N-structures, a new set-theoretic groundwork that extends neutrosophic N-structures. We defined the basic operations such as containment, complement, union, and intersection for these sets, providing clear examples for better understanding. Additionally, we successfully applied the idea of interval-valued neutrosophic N-structures to d-algebras. This application led to the development of interval-valued neutrosophic N-d-subalgebras. We then examined some important properties of interval-valued neutrosophic N-d-subalgebras, laying the foundation for further studies.

This study, sets the stage for future investigation in applying interval-valued neutrosophic N-structures to different algebraic structures like BCK, BCI, BCH, BG, BH-algebras, etc.

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