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Existence and Uniquenes of Solution for bi-Laplacian Kirchhoff type Equation with Singular Term

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Abstract A Kirchhoff's equations with singular term are determined by bi-Laplacian operator. Using variational methods and critical points theory, we show the existence of a unique weak solution of the following singular bi-harmonic problems of Kirchhoff type involving critical Sobolev exponent:

$$\begin{cases} \left(a\int_\Omega |\Delta u|^2 dx + b\right)^{\theta-1} \Delta^2 u = f(x)|u|^{-\gamma} - \lambda |u|^{p-2}u & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$ $(n \geq 5)$, with Δ^2 is the bi-Laplacian operator. Here $\gamma \in (0,1)$ and $\theta \in (1,2)$ are a constants, $\lambda > 0$, $0 < p \leq 2^\sharp$, $a, b \geq 0$, $a + b > 0$ are parameters, and f belongs to a given Lebesgue space.

1 Introduction and Motivation:

In this paper, we are devoted to investigate the existence and the uniqueness of positive solution with negative energy for the following biharmonic-Kirchhoff type problem with singular term:

$$\begin{cases} \left(a\int_\Omega (\Delta u)^2 dx + b\right)^{\theta-1} \Delta^2 u = f(x)|u|^{-\gamma} - \lambda |u|^{p-2}u & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω a bounded domain of $\mathbb{R}^n (n \geq 5)$ with Δ^2 is the bi-Laplacian operator and

$$\Delta u := -div(\nabla u)$$

is the Laplacian, and $\lambda > 0$ is a real parameter.
Here $0 < \gamma < 1$, is constant, $0 < p \leq 2^\# - 1, a, b \geq 0, a + b > 0$ are real parameters. The weight function $f : \Omega \to \mathbb{R}$ is in $L^q(\Omega)$ such that $q := \frac{2^\#}{2^\# + \gamma - 1}$ with $f(x) > 0$ for almost every $x \in \Omega$ and $2^\# := \frac{2n}{n-4}$ denote the critical Sobolev exponent in the embedding $H^2(\Omega) \hookrightarrow L^{2^\#}(\Omega)$.

The Kirchhoff equation [10], goes back in 1883. It was proposed as an extention of the classical d'Allembert wave equation for the vibration of elastic strings. The problem (1.1) is related to the stationary analogue of the evolution equation of Kirchhoff type:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \left(a\int_\Omega (\Delta u)^2 dx + b\right) \Delta^2 u = f(x,u), & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } \partial\Omega \times (0,T), \\ u(x,0) = v(x), \frac{\partial u(x,0)}{\partial t} = w(x), & \text{on } \partial\Omega, \end{cases}$$

where the parameters in above equation have physical significant meanings as follows: T is a positive constant, v and w are two given functions. The equation is one dimensional, time dependent, and it was written as

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \left(a\int_\Omega (\Delta u)^2 dx + b\right)^{\theta-1} \Delta^2 u = f(x,u), & \text{in } \Omega\times(0,T), \\ u = 0 & \text{in } \partial\Omega\times(0,T), \\ u(x,0) = v(x), \frac{\partial u(x,0)}{\partial t} = w(x), & \text{on } \partial\Omega \end{cases}$$

where $\theta \in (1,2)$, which is used to describe some phenomenon appeared in physics and engineering. Due to this, it is regarded as a good approximation for describing vibrations of beams or plates, see [7] and [8].

In fact, the related problems have been studied extensively, especially on the existence of the positive solutions, multiple solutions, ground state solutions, and least energy sign-changing solutions, we cite here, in particular [17], which treats the following the Kirchhoff equations type:

$$\begin{cases} -\left(a\int_\Omega |\nabla u|^2 dx + b\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the minimax methods, they obtained the multiplicity result of solutions with sign-changing solution.

In [23], the authors have considered the existence and multiplicity of solutions to a class of bi-laplacian Kirchhoff type equation:

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u) & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{cases}$$

Using Pass-Montain theorem, they obtained the existence result of positive non trivial solution of the problem $(\mathcal{P}_4)$.

In [21], K. Tahri and F. Yazid have considered in a bounded domain Ω of $\mathbb{R}^n (n \geq 5)$ a fourth order Kirchhoff type equation:

$$\begin{cases} \Delta^2 u - \left(a\int_\Omega |\nabla u|^2 dx + b\right)\Delta u + cu = f(x)|u|^{-\gamma} - \lambda|u|^{p-2}u & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{cases}$$

Under some assumptions and by variational methods, they obtained the uniqueness of the solution. For more details, we refer the reader to [1, 2, 3, 4, 5, 6, 9, 11, 12, 13, 14, 15, 16, 18, 19, 22].

Recently, $K.Tahri$, S. Benmansour and Kh. Tahri in [20], showed the existence, nonexistence and multiplicity results for the following p -Laplacian Kirchhoff equation:

$$\begin{cases} -\left(a\int_\Omega |\nabla u|^p dx + b\right)\Delta_p u = \mu|u|^{p^*-2}u + \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n (n > 3)$, $\lambda, \mu > 0$ are real parameters and $a, b \geq 0 : a+b > 0$ are positive constants, $\Delta_p u$ is the p-Laplacian operator, that is,

$$\Delta_p u := -div\left(|\nabla u|^{p-2}\nabla u\right) = \sum_{1\leq i\leq n} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right),$$

for $1 < p < n$, and $p^* := \frac{np}{n-p}$ is the Sobolev critical exponent of the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$. They established the following results:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n (n > 3)$. Assume $a, b \geq 0 : a + b > 0$ and $p^* = 4$. Then the following assertions are true:*
(i). Assume that $a > 0, b > 0, 0 < \mu < aK(n,p)^2$ and $0 < \lambda < b\lambda_1$. Then the equation (1.2) has no positive

nontrivial solution.
(ii). Assume that $a \geq 0, b > 0, 0 < \mu < aK(n,p)^2$ *and* $\lambda > b\lambda_1$. *Then the equation (1.2) has a positive nontrivial solution.*
(iii). Assume that $a \geq 0, b > 0, 0 < \mu < aK(n,p)^2$. *Then for any* $k \in \mathbb{Z}^+$, *there exists* $\Lambda_k > 0$ *such that the equation (1.2) has at least k pairs of nontrivial solutions for* $\lambda > \Lambda_k > 0$.

Motivated by the above works, the aim of the present paper is going to discuss the biharmonic-Kirchhoff type problem with singular term for $\theta \in (1,2)$. Our main result is the following:

Theorem 1.2. *Let* $\Omega \subset \mathbb{R}^n (n \geq 5)$ *be a smooth bounded domain and assuming that* $(A_i)_{1 \leq i \leq 3}$ *are hold. Then the problem (1.1) possesses a positive solution. Moreover, this solution is a global minimizer solution.*

The paper is organized as follows: in Sec. 2, some notations and preliminaries are given, including lemmas that are required to obtain our main Theorem, and finally in Sec. 3 we provide the proof of Theorem.

2 Variational Setting and Assumptions:

In this paper, we make use the following notation:
$L^p(\Omega)$ for $1 \leq p < \infty$, denote the Lebesgue spaces, the norm in L^p is given by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$

$H_0^1(\Omega)$ denotes the completion of the space $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2 dx,$$

and $H^2(\Omega)$ denotes the completion of the space $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{H^2(\Omega)} = \int_\Omega |\Delta u|^2 dx.$$

S is the best Sobolev constant to the Sobolev embedding $H^2(\Omega) \hookrightarrow L^{2^\#}(\Omega)$, that is

$$S := \inf \frac{\int_\Omega |\Delta u|^2 dx.}{\int_\Omega |u|^{2^\#} dx} := \inf_{\left\{u \in H^2(\Omega):\ \int_\Omega |u|^{2^\#} dx = 1\right\}} \int_\Omega |\Delta u|^2 dx.$$

It is well known that for $\Omega = \mathbb{R}^n$ the best constant S is attained by the radial functions

$$u_\epsilon(x) := \alpha_n \left(\frac{\epsilon}{|x - x_0|^2 + \epsilon^2} \right)^{\frac{n-4}{2}},$$

where

$$\alpha_n := \left(n(n-4) \left(n^2 - 4 \right) \right)^{\frac{n-4}{8}}.$$

Consider the eigenvalue problem

$$\begin{cases} \Delta^2 \phi = \lambda \phi & \text{in } \Omega, \\ \Delta \phi = \phi = 0 & \text{on } \Omega. \end{cases}$$

According to the work developped by K.Tahri and F. Yazid in [21], has shown that the first eigenvalue λ_1 is given by

$$\lambda_1 := \inf_{\phi \in W} \frac{\int_\Omega (\Delta \phi)^2 dx}{\int_\Omega \phi^2 dx}.$$

Let W be the space $H^2(\Omega) \cap H_0^1(\Omega)$ endowed with the norm:

$$\|u\|^2_W := \int_\Omega |\Delta u|^2 dx.$$

A function $u \in W$ is said to be a weak solution of the problem (1.1) if $u > 0$ in Ω:

$$\left(a\int_\Omega |\Delta u|^2 dx + b\right)^{\theta-1} \int_\Omega \Delta u \cdot \Delta\varphi\, dx - \int_\Omega f(x)|u|^{-\gamma}\cdot\varphi\, dx + \int_\Omega \lambda|u|^{p-2}u\cdot\varphi\, dx = 0,$$

for all $\varphi \in W$. We shall look for (weak) solutions of (1.1) by finding critical points of the energy functional $I_\lambda : W \to \mathbb{R}$ given by

$$I_\lambda(u) = \frac{1}{2a\theta}\left(a\int_\Omega |\Delta u|^2 dx + b\right)^\theta + \frac{\lambda}{p}\int_\Omega |u|^p dx + \frac{1}{\gamma-1}\int f(x)|u|^{1-\gamma}dx.$$

Throughout this paper, we make the following assumptions

$(\mathbf{A_1})$ $0 < \gamma < 1,\ 0 < p \le 2^\# := \frac{2n}{n-4}$ and $1 < \theta < 2$.

$(\mathbf{A_2})$ $f \in L^q(\Omega)$ with $q = \frac{2^\sharp}{2^\sharp+\gamma-1}$ satisfying $f(x) > 0$ for almost every $x \in W$.

$(\mathbf{A_3})$ For all $x \in W$:

$$\|u\|_W^2 := \int_\Omega |\Delta u|^2 dx.$$

3 Technical Lemmas and Main Result:

In this section, we introduce some technical lemmas which will be used for the proof of the main result.

Lemma 3.1. *The energy functional I_λ is coercive and bounded from below on W.*

Proof. Since $0 < \gamma < 1$, $\lambda > 0$, by the Hölder inequality, we have

$$\begin{aligned}
&\left|\int_\Omega f(x)|u|^{1-\gamma}dx\right| \\
&\le \left(\int_\Omega |f(x)|^{\frac{2^\#}{2^\#+\gamma-1}}\right)^{\frac{2^\#+\gamma-1}{2^\#}} \times \left(\int_\Omega |u|^{(1-\gamma)\frac{2^\#}{1-\gamma}}\right)^{\frac{1-\gamma}{2^\#}} \\
&= \|f\|_{L^q(\Omega)} \times \|u\|^{1-\gamma}_{L^{2^\#}(\Omega)}, \quad with \quad q := \frac{2^\#}{2^\#+\gamma-1}.
\end{aligned}$$

Furthermore, by the Sobolev embedding theorem, we obtain that there exists a constant $\beta > 0$ such that

$$\|u\|^{1-\gamma}_{L^{2^\#}(\Omega)} \leqslant (1-\gamma)\beta\|u\|_W^{1-\gamma}.$$

Hence,

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2a\theta}(a\|u\|_W^2 + b)^\theta + \frac{\lambda}{p}\int_\Omega |u|^p dx + \frac{1}{\gamma-1}\int f(x)|u|^{1-\gamma}dx, \\
&\ge \frac{1}{2a\theta}(a^\theta\|u\|_W^{2\theta}) - \beta\|u\|_W^{1-\gamma}.
\end{aligned} \tag{3.1}$$

This implies that I_λ is coercive.

To prove that I_λ is bounded from below on W, we consider two cases:

Case I: if $\|u\|_W \geqslant 1$ and $0 < \gamma < 1$,$1 < \theta < 2$, then

$$I_\lambda(u) \ge \left(\frac{1}{2a\theta} - \beta\right)\|u\|_W^{1-\gamma}.$$

Case II: if $0 < \zeta \le \|u\|_W \le 1$ and $0 < \gamma < 1$,$1 < \theta < 2$, then

$$I_\lambda(u) \ge \left(\frac{a^\theta}{2a\theta} - \beta\right)\zeta^{2\theta}.$$

Thus, I_λ is bounded from below on W. □

Lemma 3.2. *The energy functional I_λ has a minimum c in W with $c < 0$.*

Proof. Since I_λ is coercive and bounded from below on W in Lemma 3.1

$$c = \inf_{u \in W} I_\lambda(u),$$

is well defined.
Moreover, since $0 < \gamma < 1$, $f(x) > 0$ for almost every $x \in \Omega$, we have $I_\lambda(t\delta) < 0$ for all $\delta \neq 0$ and small $t > 0$.
Thus, we obtain

$$c = \inf_{u \in W} I_\lambda(u) < 0.$$

The proof is completed. □

The validity of the next lemma will be crucial in the sequel.

Lemma 3.3. *Assume that the conditions* (A_1) *and* (A_2) *hold. Then* I_λ *attains the global minimizer in* W, *that is, there exists* $u^* \in W$ *such that*

$$I_\lambda(u^*) = c < 0.$$

Proof. From Lemma 3.1 there exists a minimizing sequence $\{u_n\} \subset W$ such that

$$\lim_{n \to \infty} I_\lambda(u_n) = c < 0.$$

By (3.1), the sequence $\{u_n\}$ is bounded in W. Since W is reflexive, and the compact embedding theorem, we may extract a subsequence that for simplicity we call again $\{u_n\}$, for which there exists $u_* \geq 0$ such that

$$\begin{cases} u_n \to u^*, & \text{weakly in } W, \\ u_n \to u^*, & \text{strongly in } L^p(\Omega), \quad 1 \leq p < 2^\#, \\ u_n(x) \to u^*(x), & \text{a.e. in } \Omega, \end{cases} \tag{3.2}$$

as $n \to \infty$ As usual, letting $w_n = u_n - u_*$, we need to prove that $\|w_n\| \to 0$ as $n \to \infty$. By Vitali's theorem [17], we find

$$\lim_{n \longrightarrow +\infty} \int_\Omega f(x)|u_n|^{1-\gamma} dx = \int_\Omega f(x)|u^*|^{1-\gamma} dx. \tag{3.3}$$

Moreover, by the weak convergence of $\{u_n\}$ in W and Brézis-Lieb's Lemma (see [6]), one obtains

$$||u_n||_W^2 - ||w_n||_W^2 = ||u^*||_W^2 + o(1), \tag{3.4}$$

and

$$\int_\Omega h(x)|u_n|^{2^\#} dx = \int_\Omega h(x)|w_n|^{2^\#} dx + \int_\Omega h(x)|u^*|^{2^\#} dx + o(1), \tag{3.5}$$

where $o(1)$ is an infinitesimal as $n \to \infty$. Hence, in the case when $1 < p < 2^\#$ from (3.2)-(3.4), we deduce that

$$\begin{aligned}
c &= \lim_{n \to +\infty} J_\lambda(u_n) \\
&= \lim_{n \to +\infty} \left[\frac{1}{2a\theta}(a\|u_n\|_W^2 + b)^\theta + \frac{\lambda}{p} \int_\Omega |u_n|^p dx + \frac{1}{\gamma - 1} \int f(x)|u_n|^{1-\gamma} dx \right] \\
&= \lim_{n \to +\infty} \left[\frac{1}{2a\theta}(a\|w_n\|_W^2 + \|u^*\|_W^2 + b)^\theta + \frac{\lambda}{p} \int_\Omega |u^*|^p dx + \frac{1}{\gamma - 1} \int f(x)|u^*|^{1-\gamma} dx \right] \\
&= \lim_{n \to +\infty} \left[\frac{1}{2a\theta}(a\|w_n\|_W^2 + \|u^*\|_W^2 + b)^\theta + \frac{\lambda}{p} \int_\Omega |u^*|^p dx \right. \\
&\left. + \frac{1}{\gamma - 1} \int f(x)|u^*|^{1-\gamma} dx + \frac{1}{2a\theta}(a\|u^*\|_W^2 + b)^\theta - \frac{1}{2a\theta}(a\|u^*\|_W^2 + b)^\theta \right] \\
&= I_\lambda(u^*) + \lim_{n \to +\infty} \left[\frac{1}{2a\theta}(a\|w_n\|_W^2 + \|u^*\|_W^2 + b)^\theta - \frac{1}{2a\theta}(a\|u^*\|_W^2 + b)^\theta \right] \\
&\geq I_\lambda(u^*) \\
&\geq \inf_{u \in X} I_\lambda(u_n) = c.
\end{aligned}$$

This implies

$$c = I_\lambda(u^*).$$

In the case when $p = 2^\#$, it follows from (3.3)-(3.5) that

$$c = J_\lambda(u^*) + \lim_{n\to+\infty}\left[\frac{1}{2a\theta}(a\|w_n\|_W^2 + \|u^*\|_W^2 + b)^\theta\right.$$
$$\left. -\frac{1}{2a\theta}(a\|u^*\|_W^2 + b)^\theta + \frac{\lambda}{p}\int_\Omega h(x)|w_n|^p dx\right]$$
$$\geq I_\lambda(u_n)$$
$$\geq c.$$

This yields

$$I_\lambda(u_n) = c.$$

Thus,

$$\inf_{u_n\in W} I_\lambda(u_n) = I_\lambda(u^*)$$

and this completes the proof of Lemma 3.2 . □

Now, we are ready to state the main result of this paper.

Theorem 3.4. *Assume that conditions ($\mathbf{A_1}$), ($\mathbf{A_2}$) and ($\mathbf{A_3}$) hold. Then problem (1.1) possesses a positive and a unique weak solution. Moreover, this solution is a global minimizer solution.*

We are now in a position to prove **Theorem 3.4**

4 Proof of the Main Result:

The main aim of this section is to prove that u^* is a weak solution of (1.1) and $u^* > 0$ in Ω. Firstly, we show that u^* is a weak solution of (1.1). From Lemma 3.2, we see that

$$\min I_\lambda(u^* + t\varphi) = I_\lambda(u^* + t\varphi)|_{t=0} = I_\lambda(u^*), \forall\varphi \in W.$$

This implies that

$$\left(a\int_\Omega |\Delta u|^2 dx + b\right)^{\theta-1}\int_\Omega \Delta u\cdot\Delta\varphi\, dx - \int_\Omega f(x)|u|^{-\gamma}\cdot\varphi\, dx + \int_\Omega \lambda|u|^{p-2}u\cdot\varphi\, dx = 0, \tag{4.1}$$

for all $\varphi \in W$. Thus, there is a weak solution of (1.1).

Secondly, we prove that $u^* > 0$ for almost every $x \in \Omega$. Since $I_\lambda(u^*) = c < 0$, we obtain $u^* \geq 0$ and $u^* \not\equiv 0$.

Then, $\forall\phi \in W$, $\phi \geq 0$ and $t > 0$, we have

$$\begin{aligned} 0 &\leq \frac{I_\lambda(u^* + t\phi) - I_\lambda(u^*)}{t} \\ &= \frac{1}{2a\theta}\left[\frac{\left(a\int_\Omega |\Delta u + tv|^2 dx + b\right)^\theta - \left(a\int_\Omega |\Delta u|^2 dx + b\right)^\theta}{t}\right] \\ &+ \frac{\lambda}{p}\int_\Omega h(x)\left(\frac{|u^* + t\phi|^p - |u^*|^p}{t}\right)dx \\ &- \frac{1}{1-\gamma}\int_\Omega f(x)\left(\frac{|u^* + t\phi|^{1-\gamma} - |u^*|^{1-\gamma}}{t}\right)dx. \end{aligned} \tag{4.2}$$

Using the Lebesgue dominated convergence theorem, we have

$$\frac{1}{p}\lim_{t\longrightarrow 0^+}\int_\Omega h(x)\left(\frac{|u^* + t\phi|^p - |u^*|^p}{t}\right)dx = \int_\Omega h(x)|u^*|^{p-2}u^*\phi dx. \tag{4.3}$$

For any $x \in \Omega$, we denote

$$g(t) := f(x)\left(\frac{|u^*(x) + t\phi(x)|^{1-\gamma} - |u^*(x)|^{1-\gamma}}{(\gamma-1)t}\right).$$

Then

$$g'(t) := f(x)\left(\frac{(u^*(x))^{1-\gamma} - (\gamma t\phi(x) + u^*(x))|u^*(x) + t\phi(x)|^{-\gamma}}{(1-\gamma)t^2}\right) \leq 0,$$

which implies that $g(t)$ is non increasing for $t > 0$.

Moreover, we have

$$\lim_{t\longrightarrow 0^+} g(t) = \left(\left[u^*(x)+t\phi(x)\right]^{1-\gamma}\right)'\Big|_{t=0} = f(x)(u^*(x))^{-\gamma}\phi(x),$$

for every $x \in \Omega$, which may be $+\infty$ when $u^*(x) = 0$ and $\phi(x) > 0$.
Consequently, by the monotone convergence theorem, we obtain

$$\lim_{t\longrightarrow 0^+}\left[\frac{1}{\gamma-1}\int_\Omega f(x)\left(\frac{|u^*+t\phi|^{1-\gamma}-|u^*|^{1-\gamma}}{t}\right)dx\right]$$
$$=\int_\Omega f(x)(u^*(x))^{-\gamma}\phi(x)dx,$$

which may equal to $+\infty$.
Combining this with (4.3), let $t \longrightarrow 0^+$. Then, it follows from (4.2) that

$$0 \le \left(a\int_\Omega |\Delta u^*|^2dx+b\right)^{\theta-1}\int_\Omega \Delta u^*\cdot\Delta\phi(x)\,dx+\lambda\int_\Omega h(x)|u^*|^{p-2}u^*\phi(x)dx$$
$$-\int_\Omega f(x)(u^*(x))^{-\gamma}\phi(x)dx.$$

Then, we have

$$\int_\Omega f(x)(u^*(x))^{-\gamma}\phi(x)dx$$
$$\le \left(a\int_\Omega |\Delta u^*|^2dx+b\right)^{\theta-1}\int_\Omega \Delta u^*\cdot\Delta\phi(x)\,dx+\lambda\int_\Omega h(x)|u^*|^{p-2}u^*\phi(x)dx, \tag{4.4}$$

for all $\phi \in W$ with $\phi > 0$.
Let $e_1 \in W$ be the first eigenfunction of the operator Δ^2 with $e_1 > 0$ and $||e_1||_W = 1$.
Particularly, taking $\phi = e_1$ in (4.1), one gets

$$\int_\Omega f(x)(u^*(x))^{-\gamma}e_1dx$$
$$\le \left(a\int_\Omega |\Delta u^*|^2dx+b\right)^{\theta-1}\int_\Omega \Delta u^*\cdot\Delta e_1\,dx+\lambda\int_\Omega h(x)|u^*|^{p-2}u^*e_1dx$$
$$<\infty,$$

which implies that $u^* > 0$ for almost every $x \in \Omega$.
Moreover, according to Lemma 3.2 we have

$$I_\lambda(u^*) = \inf_{u_n\in W} I_\lambda(u).$$

Thus, u^* is a global minimizer solution. Finally, we prove the uniqueness of solutions of the problem (**??**). Assume that v^* is another solution of the problem (**??**). Then it follows from (4.4) that

$$\left(a\|u^*\|_W^2+b\right)^{\theta-1}\int_\Omega \Delta u^*\Delta(u^*-v^*)\,dx-\int_\Omega f(x)|u^*|^{-\gamma}(u^*-v^*)\,dx$$
$$+\lambda\int_\Omega |u^*|^{p-2}u^*(u^*-v^*)\,dx=0 \tag{4.5}$$

and

$$\left(a\|v^*\|_W^2+b\right)^{\theta-1}\int_\Omega \Delta v^*\Delta(u^*-v^*)\,dx-\int_\Omega f(x)|v^*|^{-\gamma}(u^*-v^*)\,dx$$
$$+\lambda\int_\Omega |v^*|^{p-2}v^*(u^*-v^*)\,dx=0. \tag{4.6}$$

From (4.5) and (4.6), one obtains

$$\left(a\left\|u^*\right\|_W^2+b\right)^{\theta-1}\left[\left\|u^*\right\|_W^2-\int_\Omega \Delta u^*\Delta v^*dx\right]$$
$$+\left(a\left\|v^*\right\|_W^2+b\right)^{\theta-1}\left[\left\|v^*\right\|_W^2-\int_\Omega \Delta v^*\Delta u^*dx\right]$$
$$-\int_\Omega f(x)\left(\left|u^*\right|^{-\gamma}-\left|v^*\right|^{-\gamma}\right)\left(u^*-v^*\right)dx+\lambda\int_\Omega\left(\left|u^*\right|^{p-2}u^*-\left|v^*\right|^{p-2}v^*\right)\left(u^*-v^*\right)dx=0.$$

Then

$$\begin{aligned}&\left(a\left\|u^*\right\|_W^2+b\right)^{\theta-1}\left\|u^*\right\|_W^2+\left(a\left\|v^*\right\|_W^2+b\right)^{\theta-1}\left\|v^*\right\|_W^2\\&-\left[\left(a\left\|u^*\right\|_W^2+b\right)^{\theta-1}+\left(a\left\|v^*\right\|_W^2+b\right)^{\theta-1}\right]\int_\Omega \Delta u^*\Delta v^*dx\\&-\int_\Omega f(x)\left(\left|u^*\right|^{-\gamma}-\left|v^*\right|^{-\gamma}\right)\left(u^*-v^*\right)dx+\lambda\int_\Omega\left(\left|u^*\right|^{p-2}u^*-\left|v^*\right|^{p-2}v^*\right)\left(u^*-v^*\right)dx=0.\end{aligned}\tag{4.7}$$

Denote

$$\begin{aligned}g\left(u^*,v^*\right):=&\left(a\left\|u^*\right\|_W^2+b\right)^{\theta-1}\left\|u^*\right\|_W^2+\left(a\left\|v^*\right\|_W^2+b\right)^{\theta-1}\left\|v^*\right\|_W^2\\&-\left[\left(a\left\|u^*\right\|_W^2+b\right)^{\theta-1}+\left(a\left\|v^*\right\|_W^2+b\right)^{\theta-1}\right]\int_\Omega \Delta u^*\Delta v^*dx.\end{aligned}$$

Using the Hölder inequality, one has

$$\begin{aligned}&\left[\left(a\left\|u^*\right\|_W^2+b\right)^{\theta-1}+\left(a\left\|v^*\right\|_W^2+b\right)^{\theta-1}\right]\int_\Omega \Delta u^*\Delta v^*dx\le\\&\left[\left(a\left\|u^*\right\|_W^2+b\right)^{\theta-1}+\left(a\left\|v^*\right\|_W^2+b\right)^{\theta-1}\right]\left\|u^*\right\|_W\times\left\|v^*\right\|_W.\end{aligned}$$

Then

$$\begin{aligned}g\left(u^*,v^*\right)\ge&\left(a\left\|u^*\right\|_W^2+b\right)^{\theta-1}\left\|u^*\right\|_W^2+\left(a\left\|v^*\right\|_W^2+b\right)^{\theta-1}\left\|v^*\right\|_W^2\\&-\left[\left(a\left\|u^*\right\|_W^2+b\right)^{\theta-1}+\left(a\left\|v^*\right\|_W^2+b\right)^{\theta-1}\right]\left\|u^*\right\|_W\times\left\|v^*\right\|_W\\=&\left[\left\|u^*\right\|_W\left(a\left\|u^*\right\|_W^2+b\right)^{\theta-1}-\left\|v^*\right\|_W\left(a\left\|v^*\right\|_W^2+b\right)^{\theta-1}\right]\left(\left\|u^*\right\|_W-\left\|v^*\right\|_W\right).\end{aligned}$$

We divided in three cases:

1. Case: if $\left\|u^*\right\|_W=\left\|v^*\right\|_W$, then
$$g\left(u^*,v^*\right)\ge 0.$$
2. Case: if $\left\|u^*\right\|_W>\left\|v^*\right\|_W$, then
$$g\left(u^*,v^*\right)>0.$$
3. Case: if $\left\|u^*\right\|_W<\left\|v^*\right\|_W$, then
$$g\left(u^*,v^*\right)>0.$$

Since $0<\gamma<1$ and $p>0$, it is well clear the following elementary inequalities

$$(m^{-\gamma}-n^{-\gamma})(m-n)\le 0,\qquad (m^p-n^p)(m-n)\ge 0,\qquad \forall m,n>0.$$

Thus $$\int_\Omega f(x)\left(\left|u^*\right|^{-\gamma}-\left|v^*\right|^{-\gamma}\right)\left(u^*-v^*\right)dx\le 0,\quad \int_\Omega\left(\left|u^*\right|^{p-2}u^*-\left|v^*\right|^{p-2}v^*\right)\left(u^*-v^*\right)dx\ge 0.$$

Consequently we obtain contradiction with equality 4.7.
Therefore u^* is the unique positive solution of problem (**??**). This completes the proof of Theorem 1.2.

5 Conclusion Remarks

This paper aims to obtain a unique positive weak solution for a Kirchhoff equation of fourth order by using so suitable technique based on the behavior of energy functional associated to the initial problem. Therefore, the results of this work are significant and so it is interesting and maybe to generalize its study in the future for evolution equation.

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