Palestine Journal of Mathematics

Vol 14(3)(2025) , 1339–1352

Approximation by Kantorovich type q-Mittag Leffler Operators

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Communicated by S. A. Mohiuddine

MSC 2010 Classification: 41A10, 41A25, 41A36, 40A30

Keywords: q-Mittag Leffler operators; Kantorovich operators; $\alpha\beta$-statistical convergence, Rate of convergence, Lipschitz class.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract The purpose of this article is to introduce the Kantorovich variant of q-analogue of Mittag-Leffler operators and study their approximating properties. We obtain basic estimates for these operators and present convergence theorems. The weighted $\alpha\beta-$statistical convergence is discussed for these operators. The rate of convergence in Lipschitz type space is analyzed and relavent theorems are proved.

1 Introduction and Preliminaries

We begin by listing some q-calculus notions. We assume q to be a real number such that $0 < q \leq 1$. For $\mathfrak{s} \in \mathbb{N}$,

$$[\mathfrak{s}]_q = [\mathfrak{s}] := \left\{ \begin{array}{cc} (1-q^{\mathfrak{s}})/(1-q), & q \neq 1 \\ \mathfrak{s} & , q = 1 \end{array} \right\},$$

$$[\mathfrak{s}]_q! = [\mathfrak{s}!] := \left\{ \begin{array}{cc} [\mathfrak{s}][\mathfrak{s}-1].....[1], & s \geq 1 \\ 1 & , s = 0 \end{array} \right\},$$

and

$$(1+y)_q^{\mathfrak{s}} := \left\{ \begin{array}{cc} \prod_{j=0}^{\mathfrak{s}-1}(1+q^j y), & s = 1, 2, ... \\ 1 & , s = 0 \end{array} \right\}.$$

The q-analogue for binomial cofficient is given as

$$\left[\begin{array}{c} \mathfrak{s} \\ \mathfrak{l} \end{array} \right] = \frac{[\mathfrak{s}]!}{[\mathfrak{l}]![\mathfrak{s}-\mathfrak{l}]!}.$$

Introcuced by Jackson [11], the integral q-analogue is defined as

$$\int_0^a g(y) d_q y := a(1-q) \sum_{\mathfrak{s}=0}^{\infty} g\left(aq^{\mathfrak{s}}\right) q^{\mathfrak{s}}, 0 < q < 1 \text{ and } a > 0.$$

The q- analogue of improper integral is defined as

$$\int_0^{\infty/B} g(y) d_q y := (1-q) \sum_{\mathfrak{s}=0}^{\infty} g\left(\frac{q^{\mathfrak{s}}}{B}\right) \frac{q^{\mathfrak{s}}}{B}, B > 0,$$

given the fact that the sum converges absolutely.
The q-Gamma functions are given by

$$\Gamma_q(y) = \int_0^{1/1-q} t^{y-1} E_q(-qt) d_q t \text{ and } \gamma_q^A(y) = \int_0^{\infty/A(1-q)} t^{y-1} e_q(-t) d_q t. \tag{1.1}$$

In [12], q-analogues of the exponential function are given in two ways

$$e_q(y) = \sum_{j=0}^{\infty} \frac{y^j}{[j]!} = \frac{1}{(1-(1-q)y)_q^{\infty}}, |y| < \frac{1}{1-q}, |q| < 1,$$

and

$$E_q(y) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{y^k}{[k]!} = (1+(1-q)y)_q^{\infty}, |q| < 1.$$

In [11], Jackson established the q-analogue of Beta function as

$$B_q(t,r) = \frac{\Gamma_q(r)\Gamma_q(t)}{\Gamma_q(r+t)}.$$

Also, the q- analogue of the integral representation of q-Beta function, which is a q-analogue of Euler's formula, is

$$B_q(t,r) = \int_0^1 y^{t-1}(1-qy)_q^{r-1} d_q y, t, r > 0. \tag{1.2}$$

The Mittag-Leffler function was introduced in 1903 by G.M. Mittag-Leffer by

$$E_\gamma(\mathfrak{z}) = \sum_{j=0}^{\infty} \frac{\mathfrak{z}^j}{\Gamma(\gamma j+1)}; (\mathfrak{z} \in \mathbb{C}, Re(\gamma) > 0),$$

where $\mathbb{C}$ denotes the set of complex numbers. In 1905, A. Wiman [27] gave the two-index Mittag-Leffler function definition by

$$E_{\gamma,\eta}(\mathfrak{z}) = \sum_{j=0}^{\infty} \frac{\mathfrak{z}^j}{\Gamma(\gamma j+\eta)}; (\mathfrak{z}, \eta \in \mathbb{C}, Re(\gamma) > 0).$$

If we put $\eta = 1$ then one obtains $E_{\gamma,1}(\mathfrak{z}) = E_\gamma(\mathfrak{z})$.
Also

$$E_{1,1}(\mathfrak{z}) = e^{\mathfrak{z}}, \quad E_{1,2}(\mathfrak{z}) = \frac{e^{\mathfrak{z}}-1}{\mathfrak{z}}, \quad E_{1,m+1}(\mathfrak{z}) = \frac{e^{\mathfrak{z}} - \sum_{j=0}^{m-1} \frac{\mathfrak{z}^j}{j!}}{\mathfrak{z}^m}.$$

The relation between the two-index Mittag-Leffler functions and the generalised Bernoulli numbers $\mathfrak{B}_l$ is as

$$\frac{1}{E_{1,2}(\mathfrak{z})} = \sum_{l=0}^{\infty} \mathfrak{B}_l \frac{\mathfrak{z}^l}{l!}.$$

Moreover, we have

$$\frac{1}{E_{1,m+1}(\mathfrak{z})} = \sum_{l=0}^{\infty} \mathfrak{B}_l^{(m)} \frac{\mathfrak{z}^l}{l!}.$$

The aforementioned Mittag-Leffler operators were studied by M.A. Özarslan in [23]

$$\mathcal{L}_m^{(\eta)}(h;u) = \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m}\right)} \sum_{j=0}^{\infty} h\left(\frac{j\mathfrak{b}_m}{m}\right) \frac{(mu)^j}{\mathfrak{b}_m^j \Gamma(j+\eta)}. \tag{1.3}$$

Two q-analogues for the well-known Mittag Leffler operators have been defined by M.H. Annaby and Z.S. Mansour in [3] as

$$\mathfrak{e}_{\gamma,\eta}(\mathfrak{z};q) = \sum_{j=0}^{\infty} \frac{\mathfrak{z}^j}{\Gamma_q(\gamma j+\eta)}, |\mathfrak{z}(1-q)^\gamma| < 1,$$

and

$$\mathfrak{E}_{\gamma,\eta}(\mathfrak{z};q) = \sum_{j=0}^{\infty} \frac{q^{\gamma\frac{j(j-1)}{2}} \mathfrak{z}^j}{\Gamma_q(\gamma j+\eta)}, \mathfrak{z} \in \mathbb{C},$$

where $\eta \in \mathbb{C}$ and $Re(\gamma) > 0$. The q-analogue of the Mittag-Leffler function is defined for $\gamma = 1$ as

$$\mathfrak{E}_{1,\eta}\left(\frac{m\mathfrak{z}}{\mathfrak{b}_m}; q\right) = \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{m\mathfrak{z}}{\mathfrak{b}_m}\right)^j.$$

In [10], G. İcöz and B. cekim, established the following sequence of operators

$$\mathcal{L}_{m,q}^{(\eta)}(h; u) = \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m}; q\right)} \sum_{j=0}^{\infty} h\left(\frac{[j]\mathfrak{b}_m}{m}\right) \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j, \tag{1.4}$$

where $\eta > 1$ is fixed, m is a natural number and $< \mathfrak{b}_m >$ is a real positive sequence and

$$h \in \mathcal{G} := \left\{ h \in \mathfrak{C}[0, \infty) : \lim_{u \to \infty} \frac{h(u)}{1+u^2} \text{ is finite} \right\},$$

where $\mathfrak{C}[0, \infty)$ being the space containing all continuous functions on $[0, \infty)$ and the norm is defined as

$$\|h\|_* := \sup_{u \in [0,\infty)} \frac{|h(u)|}{1+u^2}.$$

The operators defined in (1.4) can approximate continous functions on the interval $[0, \infty)$. These operators have constraints while approximating the functions with singularities or discontinuities. To deal with the Lebesgue integrable functions which include functions with singularities, we introduce the following operators. For the positive real sequence $< \mathfrak{b}_m >$,

$$\mathcal{K}_{m,q}^{*(\eta)}(h; u) = \frac{m}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m}; q\right)} \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j \int_{\frac{[j]\mathfrak{b}_m}{m}}^{\frac{([j]+1)\mathfrak{b}_m}{m}} h(s)ds. \tag{1.5}$$

For more on Kantorovich type modified operators, one can refer to ([5], [2], [16]). It is evident that the sequence of operators $\mathcal{K}_{m,q}^{*(\eta)}(h; u)$ defined in (1.5) are positive and linear. Many researchers have carried out similar studies on important positive linear operators ([14], [24]).

The following lemma was proved by G. İcöz and B. cekim in [10] for the operators $\mathcal{L}_{m,q}^{(\eta)}(h; u)$ defined in (1.4)

Lemma 1.1. *For each* $0 < q < 1, u \geq 0$ *and* $m \in \mathbb{N}$, *we have*

$$\mathcal{L}_{m,q}^{(\eta)}(1; u) = 1,$$

$$\left|\mathcal{L}_{m,q}^{(\eta)}(t; u) - u\right| \leq \frac{[\eta-1]\mathfrak{b}_m}{m},$$

$$\left|\mathcal{L}_{m,q}^{(\eta)}\left(t^2; u\right) - u^2\right| \leq \frac{\mathfrak{b}_m}{m}\left(u + \frac{[\eta-1]\mathfrak{b}_m}{m}\right) + 2u\frac{\mathfrak{b}_m[\eta-1]}{m},$$

and

$$\mathcal{L}_{m,q}^{(\eta)}\left((t-u)^2; u\right) \leq \frac{\mathfrak{b}_m}{m}(1 + 4[\eta-1])u + \left(\frac{\mathfrak{b}_u}{u}\right)^2 [\eta-1].$$

To prove our main result, we require the following lemma.

Lemma 1.2. *For each* $0 < q < 1, u \geq 0$ *and* $m \in \mathbb{N}$, *we have*

$$(a)\ \mathcal{K}_{m,q}^{*(\eta)}(1; u) = 1,$$

$$(b)\ \left|\mathcal{K}_{m,q}^{*(\eta)}(t; u) - u\right| \leq \frac{[\eta-1]\mathfrak{b}_m}{m} + \frac{\mathfrak{b}_m}{2m},$$

$$(c)\ \left|\mathcal{K}_{m,q}^{*(\eta)}\left(t^2; u\right) - u^2\right| \leq u\left(\frac{2[\eta-1]+2}{m}\right)\mathfrak{b}_m + \frac{(2[\eta-1]+\frac{1}{3})\mathfrak{b}_m^2}{m^2}$$

and

$$(d)\ \mathcal{K}_{m,q}^{*(\eta)}\left((t-u)^2; u\right) \leq \frac{\mathfrak{b}_m}{m}(1 + 4[\eta-1])u + \left(\frac{\mathfrak{b}_u}{u}\right)^2 [\eta-1].$$

Proof. It is trivial that $\mathcal{K}^{*(\eta)}_{m,q}(1;u) = \mathcal{L}^{(\eta)}_{m,q}(1;u) = 1$.

Using $[k] = [k+\eta-1] - q^k[\eta-1]$ from (1.4), we have

$$\begin{aligned}
\mathcal{K}^{*(\eta)}_{m,q}(t;u) =& \frac{m}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j \int_{\frac{[j]\mathfrak{b}_m}{m}}^{\frac{([j]+1)\mathfrak{b}_m}{m}} sds \\
=& \frac{m}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{[j]\mathfrak{b}_m^2}{m^2} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j \\
+& \frac{m}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{[j]\mathfrak{b}_m^2}{2m^2} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j \\
=& \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{[j]\mathfrak{b}_m}{m} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j \\
+& \frac{\mathfrak{b}_m}{2m} \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j \\
=& \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=1}^{\infty} \frac{\mathfrak{b}_m}{m} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta-1)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j \\
-& \frac{[\eta-1]}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{\mathfrak{b}_m}{m} \frac{q^{\frac{j(j-1)}{2}} q^j}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j + \frac{\mathfrak{b}_m}{2m} \\
\leq& \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{\mathfrak{b}_m}{m} \frac{q^{\frac{j(j+1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^{j+1} \\
+& \frac{[\eta-1]}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{\mathfrak{b}_m}{m} \frac{q^{\frac{j(j-1)}{2}} q^j}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j + \frac{\mathfrak{b}_m}{2m}.
\end{aligned}$$

For $q^j \leq 1$, we get

$$\begin{aligned}
\mathcal{K}^{*(\eta)}_{m,q}(t;u) \leq& \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{\mathfrak{b}_m}{m} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^{j+1} \\
&+ \frac{[\eta-1]}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{\mathfrak{b}_m}{m} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j + \frac{\mathfrak{b}_m}{2m} \\
&\leq u + \frac{[\eta-1]\mathfrak{b}_m}{m} + \frac{\mathfrak{b}_m}{2m}.
\end{aligned}$$

So, we can write

$$\left|\mathcal{K}^{*(\eta)}_{m,q}(t;u) - u\right| \leq \frac{[\eta-1]\mathfrak{b}_m}{m} + \frac{\mathfrak{b}_m}{2m}. \tag{1.6}$$

Using definition of the operators $\mathcal{K}^{*(\eta)}_{m,q}$, we have

$$\mathcal{K}^{*(\eta)}_{m,q}(t^2;u) = \frac{m}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j \int_{\frac{[j]\mathfrak{b}_m}{m}}^{\frac{([j]+1)\mathfrak{b}_m}{m}} s^2 ds$$

$$= \frac{m}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{[j]^2\mathfrak{b}_m^3}{m^3} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j$$

$$+ \frac{m}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{[j]\mathfrak{b}_m^3}{m^3} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j$$

$$+ \frac{m}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{\mathfrak{b}_m^3}{3m^3} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j$$

$$= \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \left(\frac{[j]\mathfrak{b}_m}{m}\right)^2 \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j$$

$$+ \frac{\mathfrak{b}_m}{m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \left(\frac{[j]\mathfrak{b}_m}{m}\right) \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j$$

$$+ \frac{\mathfrak{b}_m^2}{3m^2 E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j.$$

Using the fact $[k] = q[k-1]+1$ and $[k] = [k+\eta-1] - q^k[\eta-1]$, we obtain

$$\mathcal{K}_{m,q}^{*(\eta)}(t^2;u) = \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \left(\frac{\mathfrak{b}_m}{m}\right)^2 (q[j-1][j]+[j]) \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j$$

$$+ \frac{\mathfrak{b}_m}{m}\left(\mathcal{K}_{m,q}^{*(\eta)}(t;u) - \frac{\mathfrak{b}_m}{2m}\right) + \frac{\mathfrak{b}_m^2}{3m^2}$$

$$= \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=2}^{\infty} \left(\frac{\mathfrak{b}_m}{m}\right)^2 [j-1][j] \frac{q^{\frac{j(j-1)}{2}+1}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j$$

$$+ \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \left(\frac{\mathfrak{b}_m}{m}\right)^2 [j] \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j + \frac{\mathfrak{b}_m}{m}\left(\mathcal{K}_{m,q}^{*(\eta)}(t;u) - \frac{\mathfrak{b}_m}{2m}\right) + \frac{\mathfrak{b}_m^2}{3m^2}$$

$$= \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=2}^{\infty} \left(\frac{\mathfrak{b}_m}{m}\right)^2 [j-1]\left([j+\eta-1] - q^j[\eta-1]\right) \frac{q^{\frac{j(j-1)}{2}+1}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j$$

$$+ \frac{\mathfrak{b}_m}{m}\left(\mathcal{K}_{m,q}^{*(\eta)}(t;u) - \frac{\mathfrak{b}_m}{2m}\right) + \frac{\mathfrak{b}_m^2}{3m^2} + \frac{\mathfrak{b}_m}{m}\mathcal{K}_{m,q}^{*(\eta)}(t;u)$$

$$= \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=2}^{\infty} \left(\frac{\mathfrak{b}_m}{m}\right)^2 [j-1] \frac{q^{\frac{j(j-1)}{2}+1}}{\Gamma_q(j+\eta-1)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j + \frac{\mathfrak{b}_m}{m}\left(2\mathcal{K}_{m,q}^{*(\eta)}(t;u) - \frac{\mathfrak{b}_m}{2m}\right)$$

$$+ \frac{\mathfrak{b}_m^2}{3m^2} - \frac{[\eta-1]}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=1}^{\infty} \left(\frac{\mathfrak{b}_m}{m}\right)^2 [j-1] \frac{q^{\frac{j(j-1)}{2}+1+j}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j$$

$$= \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=2}^{\infty} \left(\frac{\mathfrak{b}_m}{m}\right)^2 \left([j+\eta-2] - q^{j-1}[\eta-1]\right) \frac{q^{\frac{j(j-1)}{2}+1}}{\Gamma_q(j+\eta-1)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j$$

$$- \frac{[\eta-1]}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=2}^{\infty} \left(\frac{\mathfrak{b}_m}{m}\right)^2 [j-1] \frac{q^{\frac{j(j-1)}{2}+1+j}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j + \frac{\mathfrak{b}_m}{m}\left(2\mathcal{K}_{m,q}^{*(\eta)}(t;u) - \frac{\mathfrak{b}_m}{2m}\right)$$

$$+ \frac{\mathfrak{b}_m^2}{3m^2}$$

$$= \frac{\mathfrak{b}_m}{m}\left(2\mathcal{K}^{*(\eta)}_{m,q}(t;u) - \frac{\mathfrak{b}_m}{2m}\right) + \frac{\mathfrak{b}_m^2}{3m^2} + \frac{1}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)}\sum_{j=2}^{\infty}\left(\frac{\mathfrak{b}_m}{m}\right)^2 \frac{q^{\frac{j(j-1)}{2}+1}}{\Gamma_q(j+\eta-2)}\left(\frac{mu}{\mathfrak{b}_m}\right)^j$$
$$- \frac{[\eta-1]}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)}\sum_{j=2}^{\infty}\left(\frac{\mathfrak{b}_m}{m}\right)^2 \frac{q^{\frac{j(j-1)}{2}+j}}{\Gamma_q(j+\eta-1)}\left(\frac{mu}{\mathfrak{b}_m}\right)^j$$
$$- \frac{[\eta-1]}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)}\sum_{j=1}^{\infty}\left(\frac{\mathfrak{b}_m}{m}\right)^2 [j-1]\frac{q^{\frac{j(j-1)}{2}+1+j}}{\Gamma_q(j+\eta)}\left(\frac{mu}{\mathfrak{b}_m}\right)^j$$
$$\leq \left(u + \frac{[\eta-1]\mathfrak{b}_m}{m}\right)\frac{\mathfrak{b}_m}{m} + \frac{\mathfrak{b}_m^2}{3m^2} + \frac{u^2}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)}\sum_{j=0}^{\infty}\frac{q^{\frac{j(j-1)}{2}}q^{3j+3}}{\Gamma_q(j+\eta)}\left(\frac{mu}{\mathfrak{b}_m}\right)^j$$
$$+ \frac{[\eta-1]}{E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)}\sum_{j=2}^{\infty}\left(\frac{\mathfrak{b}_m}{m}\right)^2 \frac{q^{\frac{j(j-1)}{2}+j}}{\Gamma_q(j+\eta-1)}\left(1 + \frac{q[j-1]}{[j+\eta-1]}\right)\left(\frac{mu}{\mathfrak{b}_m}\right)^j.$$

For $q^j \leq 1$ and $\frac{[j-1]}{[j+\eta-1]} \leq 1$, we have

$$\mathcal{K}^{*(\eta)}_{m,q}(t^2;u) \leq u^2 + \frac{\mathfrak{b}_m}{m}\left(u + \frac{[\eta-1]\mathfrak{b}_m}{m}\right) + 2u\frac{\mathfrak{b}_m[\eta-1]}{m} + \left(u + \frac{[\eta-1]\mathfrak{b}_m}{m}\right)\frac{\mathfrak{b}_m}{m} + \frac{\mathfrak{b}_m^2}{3m^2}.$$

So, we can write

$$\left|\mathcal{K}^{*(\eta)}_{m,q}(t^2;u) - u^2\right| \leq \frac{(2[\eta-1]+2)\mathfrak{b}_m}{m}u + \frac{(2[\eta-1]+\frac{1}{3})}{m^2}\mathfrak{b}_m^2. \tag{1.7}$$

Finally, using (1.6) and (1.7), we obtain

$$\begin{aligned}\mathcal{K}^{*(\eta)}_{m,q}((t-u)^2;u) &= \mathcal{K}^{*(\eta)}_{m,q}(t^2;u) - 2u\mathcal{K}^{*(\eta)}_{m,q}(t;u) + u^2\mathcal{K}^{*(\eta)}_{m,q}(1;u) \\ &\leq \left|\mathcal{K}^{*(\eta)}_{m,q}(t^2;u) - u^2\right| + 2u\left|\mathcal{K}^{*(\eta)}_{m,q}(t;u) - u\right| \\ &\leq \frac{(2[\eta-1]+2)\mathfrak{b}_m}{m}u + \frac{(2[\eta-1]+\frac{1}{3})}{m^2}\mathfrak{b}_m^2 - 2u\left(\frac{[\eta-1]\mathfrak{b}_m}{m} + \frac{\mathfrak{b}_m}{2m}\right) \\ &= \frac{\mathfrak{b}_m}{m}u + (6[\eta-1]+1)\frac{\mathfrak{b}_m^2}{3m^2}.\end{aligned}$$

This proves the lemma. □

2 Statistical Convergence

If the set of natural numbers is denoted by S and $\eta(S)$ denotes the density of S, then

$$\eta(S) = \lim_{m\to\infty}\frac{|S_m|}{m}, \tag{2.1}$$

where $S_m := \{l \leq m : l \in S\}$ and $|S_m|$ denotes the cardinality of S_m. H. Fast [7] presented the idea of statistical convergence in the following manner using the density function provided in (2.1). A sequence $< z_m >$ is referred to as statistically convergent to s and represented by $st - \lim_{m\to\infty} z_m = s$, if for every $\epsilon > 0$, $\eta(S_\epsilon) = 0$, where $\eta(S_\epsilon) := \{l \leq m : |z_m - s| \geq \epsilon\}$ (see [8], [9], [20]).

Kolke then established that the definition given in (2.1) may be applied to any non-negative regular summability matrix $A = (\mathfrak{a})_{jk}$ by taking into account the connection between statistical convergence and the Cesaro matrix $\mathfrak{C}_1$ ([4],[15], [19]) of order one.

In this case a sequence $< z_m >$ is said to be A-statistically convergent to s and denoted by $st_A - \lim_{m\to\infty} z_m = s$, if for every $\epsilon > 0$, $\eta_A(S_\epsilon) = 0$ where

$$\eta_A(S) = \lim_j \frac{|S_m|}{m}.$$

Several researchers have used statistical convergence to approximate the characteristics of positive linear operators during the last two decades at a growing rate (see [21], [22], [26], [13], [17], [18]).
The $\alpha\beta-$convergence was presented by Aktuglu in [1]. The concept of $\alpha\beta-$statistical convergence can be seen as a generalization of the classical statistical convergence. Assume that $\alpha(m)$ and $\beta(m)$ are two sequences of positive numbers of real numbers satisfying the following conditions:
(A_1) $\alpha(m)$ and $\beta(m)$ are both non-decreasing,
(A_2) $\beta(m) \geq \alpha(m)$ for all natural numbers m,
(A_3) $\beta(m) - \alpha(m) \to \infty$ as $m \to \infty$.
Let us represent the set of pairs (α, β) that satisfies (A_1), (A_2) and (A_3) by Δ. For every pair $(\alpha, \beta) \in \Delta$, and $B \subset \mathbb{N}$, we define $\eta^{\alpha,\beta}(B)$ in the following manner:

$$\eta^{\alpha,\beta}(B) = \lim_{m\to\infty} \frac{\left|B \cap J_m^{\alpha,\beta}\right|}{(\beta(m) - \alpha(m) + 1)}, \tag{2.2}$$

where $J_m^{\alpha,\beta} = [\alpha(m), \beta(m)]$. As a result of (2.1), one can obtain the next lemma stated in [1].

Lemma 2.1. *Let B and L be two subsets of $\mathbb{N}$ and $(\alpha, \beta) \in \Delta$. Then the following properties hold true.*
i) $\eta^{\alpha,\beta}(\psi) = 0$,
ii) $\eta^{\alpha,\beta}(\mathbb{N}) = 1$,
iii) For a finite set B, $\eta^{\alpha,\beta}(B) = 0$,
iv) if $B \subset L$, then $\eta^{\alpha,\beta}(B) \leq \eta^{\alpha,\beta}(L)$,
v) $\eta^{\alpha,\beta}(B \cup L) \leq \eta^{\alpha,\beta}(B) + \eta^{\alpha,\beta}(L)$.

Definition 2.2. A sequence $<z_m>$ is $\alpha\beta$-statistically convergent to s, written as $st_{\alpha\beta} - \lim_{m\to\infty} z_m = s$, if for every $\varepsilon > 0$,

$$\eta^{\alpha,\beta}\left(\left\{l \in J_m^{\alpha,\beta} : |z_l - s| \geq \varepsilon\right\}\right) = \lim_{m\to\infty} \frac{\left|\left\{l \in J_m^{\alpha,\beta} : |z_l - s| \geq \varepsilon\right\}\right|}{(\beta(m) - \alpha(m) + 1)} = 0.$$

Remark 2.3. Take $\alpha(m) = 1$ and $\beta(m) = m$. Then $J_m^{\alpha,\beta} = [1, m]$ and

$$\eta^{\alpha,\beta}\left(\{l \in [1, m] : |z_m - s| \geq \varepsilon\}\right) = \lim_{m\to\infty} \frac{|\{l \leq m : |z_l - s| \geq \varepsilon\}|}{m}.$$

Thus for $\alpha(m) = 1$ and $\beta(m) = m, \alpha\beta$-statistical convergence implies statistical convergence. Depending on the choice of α and $\beta, \alpha\beta$-statistical convergence is a non-trivial extension of both ordinary and statistical convergence. Here are two examples to show this claim.

Example 1 Consider the sequence $< g_m >$, where

$$g_m := \begin{cases} m^2, & \text{if } m = p^2 \text{ for some } p \in \mathbb{N} \\ \frac{1}{m}, & \text{otherwise .} \end{cases} \tag{2.3}$$

So, $\alpha\beta$-statistical convergence can be viewed as a special case of statistical convergence, where $\alpha(m) = 1$ and $\beta(m) = m$. Therefore $st_{\alpha\beta} - \lim_{m\to\infty} g_m = 0$.

Example 2 Take $\alpha(m) = 2^{2m-1}, \beta(m) = 2^{2m}$ and consider the sequence $< h_m >$, where

$$h_m := \begin{cases} 0, & \text{if } m \in \left[2^{2p-1}, 2^{2p} - 1\right] \text{ for } p = 1, 2, \ldots \\ 1, & \text{otherwise.} \end{cases} \tag{2.4}$$

Then, the sequence $< h_m >$ does not converge in statistical sense. Since

$$\frac{\left|\left\{l \in J_m^{\alpha,\beta} : |h_m| \geq \varepsilon\right\}\right|}{(\beta(m) - \alpha(m) + 1)} \leq \frac{1}{(\beta(m) - \alpha(m) + 1)},$$

we have $st_{\alpha\beta}- \lim_{m\to\infty} h_m = 0$.
The following lemma is required for rest of the paper.

Lemma 2.4. *Let* $(\alpha,\beta)\in\Delta$ *and let* (b_m) *be a sequence with st* $_{\alpha\beta}-\lim_{m\to\infty} b_m=0$*; then,*
i) $\mathrm{st}_{\alpha\beta}-\lim_{n\to\infty} lb_m=0$*, where* l *is a fixed number,*
ii) $\mathrm{st}_{\alpha\beta}-\lim_{m\to\infty}(b_m)^2=0$*.*

Theorem 2.5. *Let* $(\alpha,\beta)\in\Delta$ *and* $\nu>0$ *and* $D>0$ *be fixed. If*

$$\mathrm{st}_{\alpha\beta}-\lim_m \frac{b_m}{m}=0$$

then

$$\mathrm{st}_{\alpha\beta}-\lim_m \left\|\mathcal{K}^{*(\eta)}_{m,q}(h;u)-h(u)\right\|_{C[0,D]}=0$$

holds for every $h\in\mathcal{G}$*.*

Proof. Using Lemma 2.4 (a), $\left\|\mathcal{K}^{*(\eta)}_{m,q}(1;\cdot)-1\right\|_{C[0,D]}=0.$
From Lemma 2.4, we have

$$\mathrm{st}_{\alpha\beta}-\lim_m \left\|\mathcal{K}^{*(\eta)}_{m,q}(\mathrm{t};-)-u\right\|_{C[0,D]}\le \mathrm{st}_{\alpha\beta}-\lim_m\left([\eta-1]+\frac{1}{2}\right)\frac{b_m}{m}=0,$$

and

$$\left\|\mathcal{K}^{*(\eta)}_{m,q}(\mathrm{t}^2;)-u^2\right\|_{C[0,D]}\le\frac{(2[\eta-1]+2)Db_m}{m}+\frac{\left(2[\eta-1]+\frac{1}{3}\right)b_m^2}{m^2}.$$

Let $\eta>0$ be a fixed number. For a given $\mathfrak{s}>0$, choose $\varepsilon>0$ such that $\varepsilon<\mathfrak{s}$ and define the following sets:

$$W:=\left\{l\in J_m^{\alpha,\beta}:\left\|\mathcal{K}^{*(\eta)}_{l,q}(\mathrm{t}^2;\cdot)-u^2\right\|_{C[0,D]}\ge\mathfrak{s}\right\},$$
$$W_1:=\left\{l\in J_m^{\alpha,\beta}:\frac{b_l}{l}\ge\frac{\mathfrak{s}}{2(2[\eta-1]+2)D}\right\},$$
$$W_2:=\left\{l\in J_m^{\alpha,\beta}:\left(\frac{b_l}{l}\right)^2\ge\frac{\mathfrak{s}}{2\left(2[\eta-1]+\frac{1}{3}\right)}\right\}.$$

It is obvious that $V\subseteq W_1\cup W_2$. Therefore Lemma 2.1 implies that

$$\zeta^{\alpha,\beta}(W)\le\zeta^{\alpha,\beta}(W_1)+\zeta^{\alpha,\beta}(W_2).$$

Using $\mathrm{st}_{\alpha\beta}-\lim_m\frac{b_m}{m}=0$, and Lemma 2.4 , we obtain

$$\zeta^{\alpha,\beta}(W)=0.$$

Hence, using [1: Theorem 2], the proof is completed. □

Consider the following function spaces:

$$\mathfrak{D}_\sigma:=\{h:\mathbb{R}\to D\subset\mathbb{R}:|h(u)|\le N_h\sigma(u)\},$$
$$\mathfrak{C}_\sigma:=\{h\in\mathfrak{D}_\sigma:h\text{ is continuous over }\mathbb{R}\},$$

where $\sigma(u)$ denotes the weight function such that $\sigma(u)$ is a continuous function on $\mathbb{R}$ with $\lim_{|u|\to\infty}\sigma(u)=\infty$ and $\sigma(u)\ge1$ for all $u\in\mathbb{R}$ and N_h is a positive constant that depends on h. These are Banach spaces under the norm

$$\|h\|_\sigma=\sup_{u\in\mathbb{R}}\frac{|h(u)|}{\sigma(u)}.$$

Now, we will prove the following theorem by using the concept of O. Duman and C. Orhan [6].

Theorem 2.6. *Suppose $(\alpha,\beta)\in\Delta$ and let σ_1 and σ_2 are the weight functions which satisfy $\lim_{|u|\to\infty}\frac{\sigma_1(u)}{\sigma_2(u)}=0$. Assume that Y_m is a sequence of linear positive operators acting from C_{σ_1} to B_{σ_2}. Then for all $f\in C_{\sigma_1}$*

$$\mathrm{st}_{\alpha\beta}-\lim_{m}\|Y_m(h;\cdot)-h\|_{\sigma_2}=0$$

if and only if

$$\mathrm{st}_{\alpha\beta}-\lim_{m}\|Y_m(H_\nu;\cdot)-H_\nu\|_{\sigma_1}=0,$$

where $H_\nu=\frac{x^\nu\mathfrak{p}_1(u)}{1+u^2}(\nu=0,1,2)$.

Now, consider the weight functions $\sigma_1(u)=1+u^2,\sigma_2(u)=1+u^{2+\gamma}$ for $\gamma>0$ and $u\in[0,\infty)$.

Lemma 2.7. *Let $\left(\frac{b_m}{m}\right)$ be a bounded sequence of positive numbers and $\eta>0$ be fixed. Then there exists a constant $M(\eta)$ such that*

$$\frac{1}{\sigma_2(u)}\mathcal{K}^{*(\eta)}_{m,q}(\sigma_1(u);u)\le M(\eta)$$

holds for all $u\in[0,\infty)$ and $m\in\mathbb{N}$. Furthermore, for all $h\in C_{\sigma_1}$, we have

$$\left\|\mathcal{K}^{*(\eta)}_{m,q}(h)\right\|_{\sigma_2}\le M(\eta)\|h\|_{\sigma_1}.$$

Proof. By Lemma 1.2, we have

$$\begin{aligned}&\frac{1}{\sigma_2(u)}\mathcal{K}^{*(\eta)}_{m,q}(\sigma_1(u);u)\\ =&\frac{1}{1+u^{2+\gamma}}\left[\mathcal{K}^{*(\eta)}_{m,q}(1;u)+\mathcal{K}^{*(\eta)}_{m,q}(t^2;u)\right]\\ \le&\frac{1}{1+u^{2+\gamma}}\left[1+u^2+\frac{(2[\eta-1]+2)b_m}{m}u+\frac{\left(2[\eta-1]+\frac{1}{3}\right)b_m^2}{m^2}\right]\le M(\eta).\end{aligned}$$

Now proving the second inequality, we write

$$\frac{1}{\sigma_2(u)}\left|\mathcal{K}^{*(\eta)}_{m,q}(h;u)\right|=\frac{1}{\sigma_2(u)}\left|\mathcal{K}^{*(\eta)}_{m,q}\left(\sigma_1\frac{h}{\sigma_1};u\right)\right|\le\frac{\|h\|_{\sigma_1}}{\sigma_2(u)}\mathcal{K}^{*(\eta)}_{m,q}(\sigma_1(u);u)\le M(\eta)\|h\|_{\sigma_1}.$$

Taking supremum over $u\in[0,\infty)$, we prove the result. □

The last inequality in Lemma 2.7 shows that $\mathcal{K}^{*(\eta)}_{m,q}:\mathfrak{C}_{\sigma_1}\to\mathfrak{D}_{\sigma_2}$.
For our operators (1.5), we have the following result based on weighted $\alpha\beta$-statistical approximation.

Theorem 2.8. *Let $(\alpha,\beta)\in\Delta$ and $\eta>0$ be fixed. If*

$$\mathrm{st}_{\alpha\beta}-\lim_{m}\frac{b_m}{m}=0$$

then for each $h\in C_{1+u^2}[0,\infty)$ and $\gamma>0$, we have

$$\mathrm{st}_{\alpha\beta}-\lim_{m}\left\|\mathcal{K}^{*(\eta)}_{m,q}(h;\cdot)-h\right\|_{1+u^{2+\gamma}}=0.$$

Proof. From Lemma 1.2 (a), we have $\mathrm{st}_{\alpha\beta}-\lim_m\left\|\mathcal{K}^{*(\eta)}_{m,q}(1;\cdot)-1\right\|_{1+u^2}=0$.
By Lemma 1.2 (b), we get

$$\begin{aligned}\mathrm{st}_{\alpha\beta}-\lim_{m}\left\|\mathcal{K}^{*(\eta)}_{m,q}(t;\cdot)-u\right\|_{1+u^2}&\le\mathrm{st}_{\alpha\beta}-\lim_{m}\sup_{u\in[0,\infty)}\frac{1}{1+u^2}\left([\eta-1]+\frac{1}{2}\right)\frac{b_m}{m}\\&=\mathrm{st}_{\alpha\beta}-\lim_{m}\left([\eta-1]+\frac{1}{2}\right)\frac{b_m}{m}=0.\end{aligned}$$

Finally, by Lemma 1.2 (c), we have

$$\begin{aligned}
&\left\|\mathcal{K}_{m,q}^{*(\eta)}(t^2;\cdot)-u^2\right\|_{1+u^2}\\
&=\sup_{u\in[0,\infty)}\frac{1}{1+u^2}\left(u\left(\frac{2[\eta-1]+2}{m}\right)\mathfrak{b}_m+\frac{(2[\eta-1]+\frac{1}{3})\mathfrak{b}_m^2}{m^2}\right)\\
&\leq\left(\frac{2[\eta-1]+2}{m}\right)\mathfrak{b}_m+\frac{(2[\eta-1]+\frac{1}{3})\mathfrak{b}_m^2}{m^2}.
\end{aligned}$$

Choose $\epsilon>0$, for a given $\mathfrak{r}>0$ such that $\epsilon<\mathfrak{r}$. For a fixed $\eta>0$, we define the sets:

$$\begin{aligned}
V&:=\left\{l\in J_m^{\alpha,\beta}:\left\|\mathcal{K}_{l,q}^{*(\eta)}(t^2;\cdot)-u^2\right\|_{1+u^2}\geq\mathfrak{r}\right\},\\
V_1&:=\left\{l\in J_m^{\alpha,\beta}:\frac{b_l}{l}\geq\frac{\mathfrak{r}}{2(2[\eta-1]+2)}\right\},\\
V_2&:=\left\{l\in J_m^{\alpha,\beta}:\left(\frac{b_l}{l}\right)^2\geq\frac{\mathfrak{r}}{2\left(2[\eta-1]+\frac{1}{3}\right)}\right\}.
\end{aligned}$$

Since $V\subseteq V_1\cup V_2$, so

$$\zeta^{\alpha,\beta}(V)\leq\zeta^{\alpha,\beta}(V_1)+\zeta^{\alpha,\beta}(V_2).$$

Using limit $\mathrm{st}_{\alpha\beta}-\lim_m\frac{b_m}{m}=0$ and Lemma 2.4 , we have

$$\zeta^{\alpha,\beta}(V)=0.$$

Therefore, by using Theorem 2.6, the proof is completed. □

Remark 2.9. The $\alpha\beta$-statistical convergence is an extension of statistical convergence for $\alpha(m)=1$ and $\beta(m)=m$, as mentioned in Remark 2.3. Hence, the results presented here are valid in statistical sense also.

Remark 2.10. Consider a non-decreasing sequence of positive numbers ρ_m tending to ∞ such that

$$\rho_{m+1}\leq\rho_m+1,\ \text{and}\ \rho_1=1.$$

Then, by choosing $\alpha(m)=m-\rho_m+1$ and $\beta(m)=m$, it is clear that $J_m^{\alpha,\beta}=[m-\rho_m+1,m]$. Moreover,

$$\zeta^{\alpha,\beta}\left(\{l:|u_l-L|\geq\epsilon\}\right)=\lim_{m\to\infty}\frac{|\{l\in[m-\rho_m+1,m]:|u_l-L|\geq\epsilon\}|}{\rho_m}.$$

That proves that if we take $\alpha(m)=m-\rho_m+1$ and $\beta(n)-n$, $\alpha\beta$-statistical convergence brings down to ρ-statistical convergence. Therefore, by chosing $\alpha(m)=m-\rho_m+1$ and $\beta(m)=m$ all the results obtained here are also valid in ρ-statistical sense.

Remark 2.11. Let $\xi=\{x_s\}$ be a lacunary sequence then for $\alpha(s)=x_{s-1}+1$ and $\beta(s)=x_s$, then $J_s^{\alpha,\beta}=[x_{s-1}+1,x_s]$. But since $(x_{s-1},x_s]\cap\mathbb{N}=[x_{s-1}+1,x_s]\cap\mathbb{N}$, we have

$$\begin{aligned}
\zeta^{\alpha,\beta}\left(\{x:|u_x-L|\geq\epsilon\}\right)&=\lim_{m\to\infty}\frac{|\{x\in[x_{s-1}+1,x_s|:|u_x-L|\geq\epsilon]|}{h_r}\\
&=\lim_{m\to\infty}\frac{|\{x\in(x_{s-1},x_s|:|u_x-L|\geq\epsilon]|}{h_r},
\end{aligned}$$

which shows that in case $\alpha(s)=x_{s-1}+1$ and $\beta(s)=x_s$, $\alpha\beta$-statistical convergence and lacunary statistical convergence behave the same. Therefore, on taking $\alpha(s)=x_{s-1}+1$ and $\beta(s)=x_s$, we find that all the results obtained here are true in lacunary statistical sanse.

3 Rate of convergence in Lipschitz type spaces

In this section, we determine the rate of convergence of the sequence of operators $\mathcal{K}_{m,q}^{*(\eta)}$ defined for locally Lipschitz functions. Notice that the space of locally Lipschitz functions is a subspace of $C_H[0,\infty)$, which represents the space of bounded continuous functions on $[0,\infty)$. The order of approximation for modified Lipschitz class functions introduced by O. Szasz [25], is also given.

Theorem 3.1. *Let $0 < \delta \leq 1$ and K be any bounded subset of the interval $[0,\infty)$. Then, if $h \in C_B[0,\infty)$ which is locally $Lip(\delta)$, i.e., if the condition*

$$|h(v) - h(u)| \leq N|v-u|^{\delta}, \quad y \in K \text{ and } u \in [0,\infty), \tag{3.1}$$

holds, then for each $u \in [0,\infty)$ and fixed $\eta > 0$, we have

$$\begin{aligned} &\left|\mathcal{K}_{m,q}^{*(\eta)}(h;u) - h(u)\right| \\ &\leq N\left(\frac{(1+4[\eta-1])\mathfrak{b}_m}{m}u + \frac{[\eta-1]\mathfrak{b}_m{}^2}{m^2}\right)^{\alpha/2} + 2(d(z,S))^{\alpha}, \end{aligned}$$

where N is a constant depending on δ and h; and $d(u,S)$ is the dtsiance berween x and S defined as $d(u,K) = \inf\{|v-u| : v \in K\}$.

Proof. Let $\bar{K}$ represents the closure of the set K in $[0,\infty)$. Then, there exists a point $u_0 \in \bar{K}$ such that $|u-u_0| = d(u,K)$. By applying the triangle inequality, we get

$$|h(v) - h(u)| \leq |h(v) - h(u_0)| + |h(u) - h(u_0)|.$$

Hence, by using (3.1), we get

$$\begin{aligned} &\left|\mathcal{K}_{m,q}^{*(\eta)}(h;u) - h(u)\right| \leq \mathcal{K}_{m,q}^{*(\eta)}(|h(v) - h(u)| : u) \\ &\leq \frac{m}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j \int_{\frac{[j]\mathfrak{b}_m}{m}}^{\frac{([j]+1)\mathfrak{b}_m}{m}} |h(t) - h(u_0)|\,dt + |h(u) - h(u_0)|^{\delta} \\ &\leq \frac{mN}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j \int_{\frac{[j]\mathfrak{b}_m}{m}}^{\frac{([j]+1)\mathfrak{b}_m}{m}} |t-u_0|\,dt + N|u-u_0|^{\delta} \\ &\leq \frac{mN}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)} \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)} \left(\frac{mu}{\mathfrak{b}_m}\right)^j \int_{\frac{[j]\mathfrak{b}_m}{m}}^{\frac{([j]+1)\mathfrak{b}_m}{m}} |t-u_0|\,dt + 2N|u-u_0|^{\delta}. \end{aligned}$$

Applying the Holder's inequality, with $p = \frac{2}{\delta}, q = \frac{2}{2-\delta}$ we get

$$\left|\mathcal{K}_{m,q}^{*(\eta)}(h;u) - h(u)\right| \leq N\left\{\mathcal{K}_{m,q}^{*(\eta)}((t-u)^2;u)^{\delta} + 2N|u-u_0|^{\delta}\right\}.$$

Finally, from Lemma 1.2, we get the result. □

Now, consider the following Lipschitz-type space

$$\mathrm{Lip}_N^*(\delta) := \left\{h \in \mathcal{C}[0,\infty) : |h(v) - h(u)| \leq N\frac{|v-u|^{\delta}}{(v+u+1)^{\delta/2}}; u,v \in [0,\infty)\right\},$$

where $N > 0$ is a constant and $0 < \delta \leq 1$. Note that the modified version of the space $\mathrm{Lip}_N^*(\delta)$ was considered earlier by O. Szazs [25].
Next, we will have the following theorem.

Theorem 3.2. *For any* $h \in \mathrm{Lip}_N^*(\delta), \delta \in (0,1]$, *and for a fixed* $\eta > 0$ *and for exery* $u \in [0,\infty), m \in \mathbb{N}$, *we have*

$$\begin{aligned}&\left|\mathcal{K}_{m,q}^{*(\eta)}(h;u) - h(u)\right| \\ &\leq N\left\{\left(\frac{(1+4[\eta-1])\mathfrak{b}_m}{m} + \frac{[\eta-1]{\mathfrak{b}_m}^2}{m^2}\right)^{\delta/2}\right\}.\end{aligned}$$

Proof. Let $\delta = 1$. Then, for $h \in \mathrm{Lip}_N^*(\delta)$ and $u \in 0,\infty)$, we have

$$\begin{aligned}&\left|\mathcal{K}_{m,q}^{*(\eta)}(h;u) - h(u)\right| \leq \mathcal{K}_{m,q}^{*(\eta)}|(h(v)-h(u)|;u) \\ &\leq N\frac{m}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)}\sum_{j=0}^{\infty}\frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)}\left(\frac{mu}{\mathfrak{b}_m}\right)^j\int_{\frac{[j]\mathfrak{b}_m}{m}}^{\frac{([j]+1)\mathfrak{b}_m}{m}}\frac{|t-u|}{|t+u+1|^{1/2}}dt \\ &\leq \frac{N}{(u+1)^{1/2}}\frac{m}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)}\sum_{j=0}^{\infty}\frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)}\left(\frac{mu}{\mathfrak{b}_m}\right)^j\int_{\frac{[j]\mathfrak{b}_m}{m}}^{\frac{([j]+1)\mathfrak{b}_m}{m}}|t-u|dt.\end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\mathcal{K}_{m,q}^{*(\eta)}(h;u) - h(u)\right| &\leq \frac{N}{(u+1)^{1/2}}\sqrt{\mathcal{K}_{m,q}^{*(\eta)}\left((t-u)^2;u\right)} \\ &\leq \frac{N}{(u+1)^{1/2}}\left\{\frac{(1+4[\eta-1])\mathfrak{b}_m}{m}u + \frac{[\eta-1]{\mathfrak{b}_m}^2}{m^2}\right\}^{1/2} \\ &\leq N\left\{\frac{(1+4[\eta-1])\mathfrak{b}_m}{m}u + \frac{[\eta-1]{\mathfrak{b}_m}^2}{m^2}\right\}^{1/2}.\end{aligned}$$

Now, let $\delta \in (0,1)$. For $h \in \mathrm{Lip}_N^*(\delta)$ and $u \in [0,\infty)$, we have

$$\begin{aligned}&\left|\mathcal{K}_{m,q}^{*(\eta)}(h;u) - h(u)\right| \leq \mathcal{K}_{m,q}^{*(\eta)}|(h(v)-h(u)|;u) \\ &\leq N\frac{m}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)}\sum_{j=0}^{\infty}\frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)}\left(\frac{mu}{\mathfrak{b}_m}\right)^j\int_{\frac{[j]\mathfrak{b}_m}{m}}^{\frac{([j]+1)\mathfrak{b}_m}{m}}\frac{|t-u|^\delta}{|t+u+1|^{\delta/2}}dt \\ &\leq \frac{N}{(u+1)^{\delta/2}}\frac{m}{\mathfrak{b}_m E_{1,\eta}\left(\frac{mu}{\mathfrak{b}_m};q\right)}\sum_{j=0}^{\infty}\frac{q^{\frac{j(j-1)}{2}}}{\Gamma_q(j+\eta)}\left(\frac{mu}{\mathfrak{b}_m}\right)^j\int_{\frac{[j]\mathfrak{b}_m}{m}}^{\frac{([j]+1)\mathfrak{b}_m}{m}}|t-u|^\delta dt.\end{aligned}$$

By taking the values $p = \frac{2}{\delta}$ and $q = \frac{2}{2-\delta}$ and applying the Holder's inequality, we have

$$\begin{aligned}\left|\mathcal{K}_{m,q}^{*(\eta)}(h;u) - h(u)\right| &\leq \frac{N}{(u+1)^{\delta/2}}\left[\mathcal{K}_{m,q}^{*(\eta)}((t-u)^2;u)\right]^{\delta/2} \\ &\leq \frac{N}{(u+1)^{\delta/2}}\left\{\frac{(1+4[\eta-1])\mathfrak{b}_m}{m}u + \frac{[\eta-1]{\mathfrak{b}_m}^2}{m^2}\right\}^{\delta/2} \\ &\leq N\left\{\frac{(1+4[\eta-1])\mathfrak{b}_m}{m}u + \frac{[\eta-1]{\mathfrak{b}_m}^2}{m^2}\right\}^{\delta/2}\end{aligned}$$

and, hence, the theorem is established. □

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Received: October 18, 2024
Accepted: June 11, 2025