

# Generalized Central Sets Theorem for partial semigroups and VIP systems

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**Abstract.** The Central Sets Theorem, a fundamental result in Ramsey theory, was originally introduced by H. Furstenberg using methods from topological dynamics. Later, N. Hindman and V. Bergelson extended the theorem using the algebraic structure of the Stone-Čech compactification  $\beta S$  of a semigroup  $S$ . An algebraic characterization of central sets in semigroups was subsequently developed. H. Shi and H. Yang established a topological dynamical characterization of central sets in an arbitrary semigroup  $(S, +)$  and showed it to be equivalent to the usual algebraic characterization. D. De, N. Hindman, and D. Strauss later proved a stronger version of the Central Sets Theorem for semigroups. D. Phulara further generalized the result for commutative semigroups. In this work, we extend the theorem to arbitrary adequate partial semigroups and VIP systems.

## 1 Introduction

Central sets are important Ramsey-theoretic objects, as whenever  $\mathbb{N}$  is partitioned into finitely many cells, one of the cells will be central. The Central Sets Theorem shows its combinatorial importance. Following the foundational results of van der Waerden's and Hindman's theorems, a natural question arises: can one find a joint extension of both of these theorems? In [3], using methods of topological dynamics, Furstenberg introduced the notion of central sets and proved that they contain important combinatorial structure.

**Definition 1.1.** The following are definitions and notations used throughout the paper:

- (i) Given a nonempty set  $A$ ,  $\mathcal{P}_f(A)$  denote the collection of nonempty finite subsets of  $A$  i.e.

$$\mathcal{P}_f(A) = \{F : \emptyset \neq F \subseteq A \text{ and } F \text{ is finite}\}$$

- (ii)  $\mathcal{J}_m = \{t \in \mathbb{N}^m : t(1) < t(2) < \dots < t(m)\}$

- (iii) For  $d \in \mathbb{N}$ , let  $\mathcal{F}_d$  denote the family of nonempty subsets of  $\mathbb{N}$  having cardinality at most  $d$  i.e.

$$\mathcal{F}_d = \{A \subset \mathbb{N} : |A| \leq d\}$$

- (iv) Let  $\langle H_n \rangle_{n=1}^\infty$  be a sequence of finite sets, then

$$FU(\langle H_n \rangle_{n=1}^\infty) = \{\cup_{n \in F} H_n : F \in \mathcal{P}_f(\mathbb{N})\}$$

- (v) Let  $\langle H_n \rangle_{n=1}^\infty, H_n \in \mathcal{P}_f(\mathbb{N})$  by  $H_n < H_{n+1}$  we mean  $\max H_n < \min H_{n+1}$
- (vi) An IP ring  $\mathcal{F}^{(1)}$  is a set of the form  $\mathcal{F}^{(1)} = FU(\langle \alpha_n \rangle_{n=1}^\infty)$ , where  $\langle \alpha_n \rangle_{n=1}^\infty$  is a sequence of members of  $\mathcal{P}_f(\mathbb{N})$  such that  $\max \alpha_n < \min \alpha_{n+1}$  for each  $n$ .
- (vii) By  ${}^{\mathbb{N}}S$ , we mean collection of all sequences with elements from  $S$ .

**Theorem 1.2.** *Let  $l \in \mathbb{N}$  and for each  $i \in \{1, 2, \dots, l\}$ , let  $\langle y_{i,n} \rangle_{n=1}^\infty$  be a sequence in  $\mathbb{Z}$ . Let  $C$  be a central subset of  $\mathbb{N}$ . Then there exist sequences  $\langle a_n \rangle_{n=1}^\infty$  in  $\mathbb{N}$  and  $\langle H_n \rangle_{n=1}^\infty$  in  $\mathcal{P}_f(\mathbb{N})$  such that*

- (i) for all  $n, \max H_n < \min H_{n+1}$  and
- (ii) for all  $F \in \mathcal{P}_f(\mathbb{N})$  and all  $i \in \{1, 2, \dots, l\}, \sum_{n \in F} (a_n + \sum_{t \in H_n} y_{i,t}) \in C$ .

Theorem 1.2 is the Central Sets theorem proved by Furstenberg in 1981 [3]. Later, V. Bergelson and N. Hindman extended this version of the Central Sets Theorem to arbitrary semigroups [5].

**Theorem 1.3.** *Let  $(S, +)$  be a commutative semigroup. Let  $l \in \mathbb{N}$  and for each  $i \in \{1, 2, \dots, l\}$ , let  $\langle y_{i,n} \rangle_{n=1}^\infty$  be a sequence in  $S$ . Let  $C$  be a central subset of  $S$ . Then there exist sequences  $\langle a_n \rangle_{n=1}^\infty$  in  $S$  and  $\langle H_n \rangle_{n=1}^\infty$  in  $\mathcal{P}_f(\mathbb{N})$  such that*

- (i) for all  $n, \max H_n < \min H_{n+1}$  and
- (ii) for all  $F \in \mathcal{P}_f(\mathbb{N})$  and all  $f : F \rightarrow \{1, 2, \dots, l\}$ ,

$$\sum_{n \in F} \left( a_n + \sum_{t \in H_n} y_{f(i),t} \right) \in C.$$

There is another extension of the Central Sets Theorem that takes infinitely many sequences at once, as presented in [5]. In 2008, D. De, N. Hindman and D. Strauss proved a stronger version of the Central Sets Theorem in [2].

**Theorem 1.4.** *Let  $(S, +)$  be a commutative semigroup and let  $C$  be a central subset of  $S$ . Then there exist functions  $\alpha : \mathcal{P}_f({}^{\mathbb{N}}S) \rightarrow S$  and  $H : \mathcal{P}_f({}^{\mathbb{N}}S) \rightarrow \mathcal{P}_f(\mathbb{N})$  such that*

- (i) if  $F, G \in \mathcal{P}_f({}^{\mathbb{N}}S)$  and  $F \subsetneq G$  then  $\max H(F) < \min H(G)$  and
- (ii) if  $m \in \mathbb{N}, G_1, G_2, \dots, G_m \in \mathcal{P}_f({}^{\mathbb{N}}S), G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_m$ , and for each  $i \in \{1, 2, \dots, m\}, \langle y_{i,n} \rangle_{n=1}^\infty \in G_i$ , then

$$\sum_{i=1}^m \left( \alpha(G_i) + \sum_{t \in H(G_i)} y_{i,t} \right) \in C.$$

In 2015, D. Phulara generalized the stronger version of the Central Sets Theorem for commutative semigroups. The theorem is as follows:

**Theorem 1.5.** *Let  $(S, +)$  be a commutative semigroup, let  $r$  be an idempotent in  $J(S)$ , and let  $\langle C_n \rangle_{n=1}^\infty$  be a sequence of members of  $r$ . Then there exist  $\alpha : \mathcal{P}_f({}^{\mathbb{N}}S) \rightarrow S$  and  $H : \mathcal{P}_f({}^{\mathbb{N}}S) \rightarrow \mathcal{P}_f(\mathbb{N})$  such that*

- (i) if  $F, G \in \mathcal{P}_f({}^{\mathbb{N}}S)$  and  $F \subsetneq G$ , then  $\max H(F) < \min H(G)$  and
- (ii) whenever  $t \in \mathbb{N}, G_1, G_2, \dots, G_t \in \mathcal{P}_f({}^{\mathbb{N}}S), G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_t, |G_1| = m$  and for each  $i \in \{1, 2, \dots, t\}, f_i \in G_i$ , then

$$\sum_{i=1}^t \left( \alpha(G_i) + \sum_{s \in H(G_i)} f_i(s) \right) \in C_m.$$

Later, in 2021, N. Hindman and K. Pleasant proved the Central Sets Theorem for adequate partial semigroups in [7]. Here, we generalize the theorem by K. Pleasant and N. Hindman in D. Phulara’s way. In [13], Poddar and Pal did the same thing for near zero. Apart from that, we generalize the Central Sets Theorem for VIP systems in commutative adequate partial semigroups. We now give a concise overview of VIP systems, which are polynomial-type configurations.

**Definition 1.6.** Let  $(G, +)$  be an abelian group. A sequence  $\langle v_\alpha \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})}$  in  $G$  is called a VIP system if there exists some non-negative integer  $d$  (the least such  $d$  is called the degree of the system) such that for every pairwise disjoint  $\alpha_0, \alpha_1, \dots, \alpha_d \in \mathcal{P}_f(\mathbb{N})$ , we have

$$\sum_{t=1}^{d+1} (-1)^t \sum_{\mathcal{B} \in [\{\alpha_0, \alpha_1, \dots, \alpha_d\}]^t} v_{\cup \mathcal{B}} = 0.$$

In their paper [6], the authors generalize this notion to partial semigroups. They defined VIP systems for partial semigroups in the following way.

**Definition 1.7.** Let  $(S, +)$  be a commutative partial semigroup. Let  $\langle v_\alpha \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})}$  be a sequence in  $S$ , then  $\langle v_\alpha \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})}$  is called a VIP system if there exists some  $d \in \mathbb{N}$  and a function from  $\mathcal{F}_d$  to  $S \cup \{0\}$ , written  $\gamma \rightarrow m_\gamma$ ,  $\gamma \in \mathcal{F}_d$ , such that

$$v_\alpha = \sum_{\gamma \subseteq \alpha, \gamma \in \mathcal{F}_d} m_\gamma \text{ for all } \alpha \in \mathcal{P}_f(\mathbb{N}). \text{ ( In particular, the sum is always defined.)}$$

The sequence  $\langle m_\gamma \rangle_{\gamma \in \mathcal{F}_d}$  is said to generate the VIP systems  $\langle v_\alpha \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})}$ .

Later, they proved the Central Sets Theorem for VIP systems in commutative adequate partial semigroups.

## 2 Algebraic background

Here, we briefly discuss about the Stone-Ćech compactification  $\beta S$  of a semigroup  $S$ .  $\beta S$  is the collection of all ultrafilters on  $S$ , i.e.,  $\beta S = \{p : p \text{ is an ultrafilter on } S\}$ , and we identify the principal ultrafilters with the points of  $S$ . For  $A \subseteq S$ ,  $\bar{A} = \{p \in \beta S : A \in p\}$ . The set  $\{\bar{A} : A \subseteq S\}$  forms a basis for a compact Hausdorff topology on  $\beta S$ . For more information about  $\beta S$  readers are requested to see [8]. For more knowledge on semigroups one can go through [10] and [14]. In this section, we discuss about partial semigroups.

**Definition 2.1.** A partial semigroup is a pair  $(S, *)$  where  $*$  maps a subset of  $S \times S$  to  $S$  and for all  $a, b, c \in S$ ,  $(a * b) * c = a * (b * c)$  in the sense that if either side is defined, then so is the other and they are equal.

For examples of partial semigroups readers are requested to go through [9].

**Definition 2.2.** Let  $(S, *)$  be a partial semigroup.

- (i) For  $s \in S$ ,  $\varphi(s) = \{t \in S : s * t \text{ is defined}\}$
- (ii) For  $H \in \mathcal{P}_f(S)$ ,  $\sigma(H) = \bigcap_{s \in H} \varphi(s)$
- (iii)  $\sigma(\emptyset) = S$
- (iv) For  $s \in S$  and  $A \subseteq S$ ,  $s^{-1}A = \{t \in \varphi(s) : s * t \in A\}$
- (v)  $(S, *)$  is *adequate* if and only if  $\sigma(H) \neq \emptyset$  for all  $H \in \mathcal{P}_f(S)$ .

**Lemma 2.3.** Let  $(S, *)$  be a partial semigroup, let  $A \subseteq S$  and let  $a, b, c \in S$ . Then  $c \in b^{-1}(a^{-1}A) \iff b \in \varphi(a)$  and  $c \in (a * b)^{-1}A$ . In particular, if  $b \in \varphi(a)$ , then  $b^{-1}(a^{-1}A) = (a * b)^{-1}A$ .

*Proof.* [6], Lemma 2.3. □

We are specifically interested in adequate partial semigroups, as they give rise to an interesting subsemigroup of  $\beta S$ . This subsemigroup is itself a compact right topological semigroup, which we define next.

**Definition 2.4.** Let  $(S, *)$  be a partial semigroup. Then

$$\delta S = \bigcap_{x \in S} \overline{\varphi(x)} = \bigcap_{H \in \mathcal{P}_f(S)} \overline{\sigma[H]}.$$

Note that  $\delta S \neq \emptyset$  when the partial semigroup  $S$  is adequate and for  $S$  being semigroup  $\delta S = \beta S$ .

For  $(S, \cdot)$  be a semigroup,  $A \subseteq S$ ,  $a \in S$ , and  $p, q \in \beta S$  then  $A \in a \cdot q \iff a^{-1}A \in q$  and

$$A \in p \cdot q \iff \{a \in S : a^{-1}A \in q\} \in p.$$

Now, we extend this notion for partial operation  $*$ .

Let  $(S, *)$  be an adequate partial semigroup. Then

(a) For  $a \in S$  and  $q \in \overline{\varphi(a)}$ ,  $a * q = \{A \subseteq S : a^{-1}A \in q\}$ .

(b) For  $p \in \beta S$  and  $q \in \delta S$ ,  $p * q = \{A \subseteq S : \{a \in S : a^{-1}A \in q\} \in p\}$ .

**Lemma 2.5.** Let  $(S, *)$  be an adequate partial semigroup.

- (i) If  $a \in S$  and  $q \in \overline{\varphi(a)}$ , then  $a * q \in \beta S$ .
- (ii) If  $p \in \beta S$  and  $q \in \delta S$ , then  $p * q \in \beta S$ .
- (iii) Let  $p \in \beta S$ ,  $q \in \delta S$ , and  $a \in S$ . Then  $\varphi(a) \in p * q$  if and only if  $\varphi(a) \in p$ .
- (iv) If  $p, q \in \delta S$ , then  $p * q \in \delta S$ .

*Proof.* [6], Lemma 2.7. □

**Lemma 2.6.** Let  $(S, *)$  be an adequate partial semigroup and let  $q \in \delta S$ . Then the function  $\rho_q : \beta S \rightarrow \beta S$  defined by  $\rho_q(p) = p * q$  is continuous.

*Proof.* [6], Lemma 2.8. □

**Theorem 2.7.** Let  $(S, *)$  be an adequate partial semigroup. Then  $(\delta S, *)$  is a compact Hausdorff right topological semigroup.

*Proof.* [6], Theorem 2.10. □

**Definition 2.8.** Let  $p = p * p \in \delta S$  and let  $A \in p$ . Then  $A^* = \{x \in A : x^{-1}A \in p\}$ .

For an idempotent  $p \in \delta S$  and  $A \in p$ , then  $A^* \in p$ .

**Lemma 2.9.** Let  $p = p * p \in \delta S$ , let  $A \in p$ , let  $x \in A^*$ . Then  $x^{-1}(A^*) \in p$ .

*Proof.* [6], Lemma 2.12. □

**Definition 2.10.** Let  $(S, *)$  be a partial semigroup and let  $A \subseteq S$ . Then  $A$  is syndetic if and only if there is some  $H \in \mathcal{P}_f(S)$  such that  $\sigma(H) \subseteq \bigcup_{t \in H} t^{-1}A$ .

**Lemma 2.11.** Let  $(S, *)$  be an adequate partial semigroup and let  $A \subseteq S$ . Then  $A$  is syndetic if and only if there exists  $H \in \mathcal{P}_f(S)$  such that  $\delta S \subseteq \bigcup_{t \in H} \overline{t^{-1}A}$ .

*Proof.* [6], Lemma 2.14. □

**Definition 2.12.**  $K(\delta S) = \{A : A \text{ is a minimal left ideal in } \delta S\}$ .

**Theorem 2.13.** Let  $(S, *)$  be an adequate partial semigroup and let  $p \in \delta S$ . The following statements are equivalent:

- (i)  $p \in K(\delta S)$ .
- (ii) for all  $A \in p$ ,  $\{x \in S : x^{-1}A \in p\}$  is syndetic.
- (iii) for all  $q \in \delta S$ ,  $p \in \delta S * q * p$ .

*Proof.* [6], Theorem 2.15. □

**Definition 2.14.** Let  $(S, *)$  be an adequate partial semigroup and let  $A \subseteq S$ . Then,

- (i) The set  $A$  is *piecewise syndetic* in  $S$  if and only if  $\overline{A} \cap K(\delta S) \neq \emptyset$ .
- (ii) The set  $A$  is *central* in  $S$  if and only if there is some idempotent  $p$  in  $K(\delta S)$  such that  $A \in p$ .
- (iii) A set  $A \subseteq S$  is a *J-set* if and only if for all  $F \in \mathcal{P}_f(\mathcal{F})$  and all  $L \in \mathcal{P}_f(S)$ , there exist  $m \in \mathbb{N}$ ,  $a \in S^{m+1}$ , and  $t \in \mathcal{J}_m$  such that for all  $f \in F$ ,

$$\left( \prod_{i=1}^m a(i) * f(t(i)) \right) * (a(m+1)) \in A \cap \sigma(L).$$

- (iv)  $J(S) = \{p \in \delta S : (\forall A \in p) (A \text{ is a J-set})\}$ .

**Lemma 2.15.** Let  $(S, *)$  be an adequate partial semigroup and let  $A \subseteq S$  be piecewise syndetic. There exists  $H \in \mathcal{P}_f(S)$  such that for every finite nonempty set  $T \subseteq \sigma(H)$ , there exists  $x \in \sigma(T)$  such that  $T * x \subseteq \bigcup_{t \in H} t^{-1}A$ .

*Proof.* [6], Lemma 2.17. □

We now introduce one of the crucial concepts: the adequate sequence for a partial semigroup.

**Definition 2.16.** Let  $(S, *)$  be an adequate partial semigroup and let  $f$  be a sequence in  $S$ . Then  $f$  is adequate if and only if

- (i) for each  $H \in \mathcal{P}_f(\mathbb{N})$ ,  $\prod_{t \in H} f(t)$  is defined and
- (ii) for each  $F \in \mathcal{P}_f(S)$ , there exists  $m \in \mathbb{N}$  such that

$$FP((f(t))_{t=m}^\infty) \subseteq \sigma(F).$$

**Definition 2.17.** Let  $(S, *)$  be an adequate partial semigroup. Then

$$\mathcal{F} = \{f : f \text{ is an adequate sequence in } S\}.$$

### 3 Phulara version of the Central Sets Theorem for adequate partial semigroups

In [9], Jillian McLeod establishes a version of Theorem 1.2 valid for commutative adequate partial semigroups. In [12], Kendra Pleasant and in [4], Arpita Ghosh, independently but later, prove a version of Theorem 1.4 for commutative adequate partial semigroups. In [11], Dev Phulara generalized the Central Sets Theorem for commutative semigroups. In this paper, we show that Theorem 1.5 remains valid for arbitrary adequate partial semigroups. To prove this, we need the following lemma.

**Lemma 3.1.** Let  $(S, *)$  be an adequate partial semigroup and let  $A$  be a J-set in  $S$ . Let  $r \in \mathbb{N}$ , let  $F \in \mathcal{P}_f(\mathcal{F})$ , and let  $L \in \mathcal{P}_f(S)$ . There exist  $m \in \mathbb{N}$ ,  $a \in S^{m+1}$ , and  $t \in \mathcal{J}_m$  such that  $t(1) > r$  and for all  $f \in F$ ,

$$\prod_{i=1}^m a(i) * f(t(i)) * a(m+1) \in A \cap \sigma(L).$$

*Proof.* [7], Lemma 3.5. □

We now state the main theorem of this section. We deeply appreciate the help of the anonymous reviewer in making the proof of Theorem 3.2 more rigorous.

**Theorem 3.2.** Let  $(S, *)$  be an adequate partial semigroup and let  $r$  be an idempotent in  $J(S)$ , and let  $\langle C_n \rangle_{n=1}^\infty$  be a sequence of members of  $r$ . Then there exist functions  $m^* : \mathcal{P}_f(\mathcal{F}) \rightarrow \mathbb{N}$ ,  $\alpha \in \times_{F \in \mathcal{P}_f(\mathcal{F})} S^{m^*(F)+1}$  and  $\tau \in \times_{F \in \mathcal{P}_f(\mathcal{F})} \mathcal{J}_{m^*(F)}$  such that

(1) if  $F, G \in \mathcal{P}_f(\mathcal{F})$  and  $G \subsetneq F$ , then  $\tau(G)(m^*(G)) < \tau(F)(1)$  and

(2) if  $s, m \in \mathbb{N}$ ,  $G_1, G_2, \dots, G_s \in \mathcal{P}_f(\mathcal{F})$ ,  $G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_s$ ,  $|G_1| = m$ , and for each  $i \in \{1, 2, \dots, s\}$ ,  $f_i \in G_i$ , then

$$\prod_{i=1}^s \left( \left( \prod_{j=1}^{m^*(G_i)} \alpha(G_i)(j) * f_i(\tau(G_i)(j)) \right) * \alpha(G_i)(m^*(G_i) + 1) \right) \in C_m.$$

*Proof.* We may assume that  $C_{n+1} \subseteq C_n$  for each  $n \in \mathbb{N}$ . For  $n \in \mathbb{N}$ , let  $C_n^* = \{x \in C_n : x^{-1}C_n \in r\}$ . Then  $C_n^* \in r$  and by Lemma 2.9, for each  $x \in C_n^*$ ,  $x^{-1}C_n^* \in r$ .

We define  $m^*(F) \in \mathbb{N}$ ,  $\alpha(F) \in S^{m^*(F)+1}$ ,  $\tau(F) \in \mathcal{J}_{m^*(F)}$  for  $F \in \mathcal{P}_f(\mathcal{F})$  by induction on  $|F|$  satisfying the following induction hypothesis:

(1) if  $\emptyset \neq G \subsetneq F$ , then  $\tau(G)(m^*(G)) < \tau(F)(1)$  and

(2) if  $s, m \in \mathbb{N}$ ,  $G_1, G_2, \dots, G_s \in \mathcal{P}_f(\mathcal{F})$ ,  $G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_s = F$ ,  $|G_1| = m$ , and for each  $i \in \{1, 2, \dots, s\}$ ,  $f_i \in G_i$ , then

$$\prod_{i=1}^s \left( \left( \prod_{j=1}^{m^*(G_i)} \alpha(G_i)(j) * f_i(\tau(G_i)(j)) \right) * \alpha(G_i)(m^*(G_i) + 1) \right) \in C_m^*.$$

Assume first that  $F = \{f\}$ . Since  $C_1^*$  is a  $J$ -set, pick  $k \in \mathbb{N}$ ,  $a \in S^{k+1}$ , and  $t \in \mathcal{J}_k$  such that  $\prod_{j=1}^k a(j) * f(t(j)) * a(k+1) \in C_1^*$  (The partial operation  $*$  is defined via the definition of  $J$ -set). Let  $m^* = k$ ,  $\alpha(F) = a$ , and  $\tau(F) = t$ . Hypothesis (1) holds vacuously. To verify hypothesis (2), let  $s, n \in \mathbb{N}$  and assume that  $G_1, G_2, \dots, G_s \in \mathcal{P}_f(\mathcal{F})$ ,  $G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_s = F$ ,  $|G_1| = m$ , and for each  $i \in \{1, 2, \dots, s\}$ ,  $f_i \in G_i$ . Then  $s = 1$ ,  $G_1 = F = \{f\}$ , and  $f_1 = f$ . So

$$\prod_{i=1}^s \left( \left( \prod_{j=1}^{m^*(G_i)} \alpha(G_i)(j) * f_i(\tau(G_i)(j)) \right) * \alpha(G_i)(m^*(G_i) + 1) \right) = \prod_{j=1}^k a(j) * f(t(j)) * a(k+1) \in C_1^*.$$

Now let  $n > 1$ , let  $F \in \mathcal{P}_f(\mathcal{F})$  with  $|F| = n$ , and assume that for each proper (nonempty) subset  $G$  of  $F$ ,  $m^*(G)$ ,  $\alpha(G)$ , and  $\tau(G)$  have been defined satisfying hypothesis (1) and (2). Let  $K = \{\tau(G)(m^*(G) + 1) : \emptyset \neq G \subsetneq F\}$  and let  $d = \max K$ .

For  $m \in \{1, 2, \dots, n - 1\}$ , let

$$M_m = \left\{ \begin{array}{l} \prod_{i=1}^s \left( \left( \prod_{j=1}^{m^*(G_i)} \alpha(G_i)(j) * f_i(\tau(G_i)(j)) \right) * \alpha(G_i)(m^*(G_i) + 1) \right) : \\ s \in \mathbb{N}, G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_s \subsetneq F, |G_1| = m, \\ \text{and for each } i \in \{1, 2, \dots, s\}, f_i \in G_i \end{array} \right\}.$$

Then by hypothesis (2),  $M_m \subseteq C_m^*$ .

Let  $A = C_n^* \cap \bigcap_{m=1}^{n-1} \bigcap_{x \in M_m} x^{-1}C_m^*$ . Then  $A \in r$ . Let  $L = \bigcup_{m=1}^{n-1} M_m$ . By Lemma 3.1, pick  $k \in \mathbb{N}$ ,  $a \in S^{k+1}$ , and  $t \in \mathcal{J}_k$  such that  $t(1) > d$  and for all  $f \in F$ ,  $\left( \prod_{j=1}^k a(j) * f(t(j)) \right) * a(k+1) \in A \cap \sigma(L)$ . Let  $m^*(F) = k$ ,  $\alpha(F) = a$ , and  $\tau(F) = t$ .

To verify hypothesis (1), assume  $\emptyset \neq G \subsetneq F$ . Then  $\tau(G)(m^*(G) + 1) \leq d < \tau(F)(1)$ . To verify hypothesis (2), let  $s, m \in \mathbb{N}$  and assume that  $G_1, G_2, \dots, G_s \in \mathcal{P}_f(\mathcal{F})$ ,  $G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_s = F$ ,  $|G| = m$ , and for each  $i \in \{1, 2, \dots, s\}$ ,  $f_i \in G_i$ .

Assume first that  $s = 1$ . Then

$$\prod_{i=1}^s \left( \left( \prod_{j=1}^{m^*(G_i)} \alpha(G_i)(j) * f_i(\tau(G_i)(j)) \right) * \alpha(G_i)(m^*(G_i) + 1) \right) = \left( \prod_{j=1}^k a(j) * f_1(t(j)) \right) * a(k+1) \in A \subseteq C_n^* \text{ and } n = |F| = |G_1|.$$

Now assume that  $s > 1$ . Let

$$x = \prod_{i=1}^{s-1} \left( \left( \prod_{j=1}^{m^*(G_i)} \alpha(G_i)(j) * f_i(\tau(G_i)(j)) \right) * \alpha(G_i)(m^*(G_i) + 1) \right) \text{ and let}$$

$y = \left( \left( \prod_{j=1}^{m^*(G_s)} \alpha(G_s)(j) * f_s(\tau(G_s)) \right) * \alpha(G_s)(m^*(G_s) + 1) \right)$ . Then  $x \in M_m$  and  
 $y = \left( \prod_{j=1}^k a(j) * f_s(t(j)) \right) * a(k + 1) \in A \cap \sigma(L) \subseteq x^{-1}C_m^*$ . Since  $x \in L$  and  $y \in \sigma(L)$ ,  
 $x * y$  is defined and  
 $\prod_{i=1}^s \left( \left( \prod_{j=1}^{m^*(G_i)} \alpha(G_i)(j) * f_i(\tau(G_i)) \right) * \alpha(G_i)(m^*(G_i) + 1) \right) = x * y \in C_m^*$ .

The inductive construction is complete. Conclusions (1) and (2) of the theorem follow from hypothesis (1) and (2) for  $G_s = F$ . Thus, the proof is complete.  $\square$

**Corollary 3.3.** *Let  $(S, *)$  be a commutative adequate partial semigroup and let  $r$  be an idempotent in  $J(S)$  and let  $\langle C_n \rangle_{n=1}^\infty$  be a sequence of members of  $r$ . Then there exist functions  $\gamma : \mathcal{P}_f(\mathcal{F}) \rightarrow S$  and  $H : \mathcal{P}_f(\mathcal{F}) \rightarrow \mathcal{P}_f(\mathbb{N})$  such that*

- (i) *if  $F, G \in \mathcal{P}_f(\mathcal{F})$  and  $G \subsetneq F$ , then  $\max H(G) < \min H(F)$  and*
- (ii) *if  $n, m \in \mathbb{N}$ ,  $G_1, G_2, \dots, G_n \in \mathcal{P}_f(\mathcal{F})$ ,  $G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_n$ ,  $|G_1| = m$  and for each  $i \in \{1, 2, \dots, n\}$ ,  $f_i \in G_i$ , then  $\prod_{i=1}^n \left( \gamma(G_i) * \prod_{t \in H(G_i)} f_i(t) \right) \in C_m^*$ .*

*Proof.* Let  $m^*$ ,  $\alpha$  and  $\tau$  be as guaranteed by previous theorem. For  $F \in \mathcal{P}_f(\mathcal{F})$ , let  $\gamma(F) = \prod_{j=1}^{m^*(F)+1} \alpha(F)(j)$  and let  $H(F) = \{\tau(F)(j) : j \in \{1, 2, \dots, m^*(F)\}\}$ .  $\square$

**Corollary 3.4.** *If  $(S, *)$  be a nontrivial commutative adequate partial semigroup with  $|\mathcal{F}|$  infinite, and  $r$  be an idempotent in  $J(S)$ . Let  $\langle C_n \rangle_{n=1}^\infty$  be a sequence of members of  $r$ , let  $k \in \mathbb{N}$  and for each  $l \in \{1, 2, \dots, k\}$ , let  $\langle y_{l,n} \rangle_{n=1}^\infty$  be an adequate sequence in  $S$ . Then there exist a sequence  $\langle a_n \rangle_{n=1}^\infty$  in  $S$  and a sequence  $\langle H_n \rangle_{n=1}^\infty$  in  $\mathcal{P}_f(\mathbb{N})$  with  $\max H_n < \min H_{n+1}$  for each  $n$  such that for  $l \in \{1, 2, \dots, k\}$  and for each  $F \in \mathcal{P}_f(\mathbb{N})$  with  $m = \min F$  one has*

$$\prod_{n \in F} \left( a_n * \prod_{t \in H_n} y_{l,t} \right) \in C_m.$$

*Proof.* We may assume that  $C_{n+1} \subseteq C_n$  for each  $n \in \mathbb{N}$ . Pick  $\gamma$  and  $H$  as guaranteed by previous corollary. For  $u \in \mathbb{N}$ , choose  $\gamma_u \in \mathcal{F} \setminus \{\langle y_{1,n} \rangle_{n=1}^\infty, \langle y_{2,n} \rangle_{n=1}^\infty, \dots, \langle y_{k,n} \rangle_{n=1}^\infty\}$  with  $\gamma_u \neq \gamma_v$  if  $u \neq v$ . For  $u \in \mathbb{N}$ , let

$$G_u = \{\langle y_{1,n} \rangle_{n=1}^\infty, \langle y_{2,n} \rangle_{n=1}^\infty, \dots, \langle y_{k,n} \rangle_{n=1}^\infty\} \cup \{\gamma_1, \gamma_2, \dots, \gamma_u\}.$$

Let  $a_u = \gamma(G_u)$  and  $H_u = H(G_u)$ . Let  $l \in \{1, 2, \dots, k\}$  and let  $F \in \mathcal{P}_f(\mathbb{N})$  be enumerated in order as  $\{n_1, n_2, \dots, n_s\}$ , so that  $m = n_1$  then  $G_m = G_{n_1} \subsetneq G_{n_2} \subsetneq \dots \subsetneq G_{n_s}$ . Also for each  $i \in \{1, 2, \dots, s\}$ ,  $\langle y_{l,t} \rangle_{t=1}^\infty \in G_{n_i}$  and  $|G_{n_1}| = m + k$ , so

$$\prod_{n \in F} \left( a_n * \prod_{t \in H_n} y_{l,t} \right) = \prod_{i=1}^s \left( \gamma(G_{n_i}) * \prod_{t \in H(G_{n_i})} y_{l,t} \right) \in C_{m+k} \subseteq C_m.$$

$\square$

### 4 Phulara’s version of Central Sets Theorem for VIP systems in partial semigroups

In this section, we focus on a special class of finite families of VIP systems and aim to further generalize the Central Sets Theorem. The definitions of VIP systems for commutative semigroups and adequate partial semigroups have already been provided in the Introduction (Definitions 1.6 and 1.7).

**Definition 4.1.** Let  $(S, +)$  be commutative adequate partial semigroup. A finite set

$$\left\{ \langle v_\alpha^{(i)} \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})} : 1 \leq i \leq k \right\}$$

of VIP systems is said to be *adequate* if there exist  $d, t \in \mathbb{N}$ , a set  $\{\langle m_\gamma \rangle_{\gamma \in \mathcal{F}_d} : i \in \{1, 2, \dots, k\}\}$ , a set of VIP systems

$$\left\{ \langle u_\alpha^{(i)} = \sum_{\gamma \subseteq \alpha, \gamma \in \mathcal{F}_d} n_\gamma^{(i)} \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})} : i \in \{1, 2, \dots, t\} \right\},$$

and sets  $E_1, E_2, \dots, E_k \subseteq \{1, 2, \dots, t\}$  such that

- (i) For each  $i \in \{1, 2, \dots, k\}$ ,  $\langle m_\gamma \rangle_{\gamma \in \mathcal{F}_d}$  generates  $\langle v_\alpha^{(i)} \rangle_{\alpha \in \mathcal{F}}$ .
- (ii) For every  $H \in \mathcal{P}_f(S)$ , there exists  $m \in \mathbb{N}$  such that for every  $l \in \mathbb{N}$  and pairwise distinct  $\gamma_1, \gamma_2, \dots, \gamma_l \in \mathcal{F}_d$  with each

$$\gamma_i \not\subseteq \{1, 2, \dots, m\}, \sum_{i=1}^t \sum_{j=1}^l n_{\gamma_j}^{(i)} \in \sigma(H) \cup \{0\}.$$

( In particular, the sum is defined.)

- (iii)  $m_\gamma^{(i)} = \sum_{t \in E_i} n_\gamma^{(t)}$  for all  $i \in \{1, 2, \dots, k\}$  and all  $\gamma \in \mathcal{F}_d$ .

**Definition 4.2.** Let  $(S, +)$  be a commutative adequate partial semigroup and let  $\mathcal{A} \subseteq \mathcal{P}_f(S)$ .  $\mathcal{A}$  is said to be *adequately partition regular family* if for every finite subset  $H$  of  $S$  and every  $r \in \mathbb{N}$ , there exists a finite set  $F \subseteq \sigma(H)$  having the property that if  $F = \cup_{i=1}^r C_i$  then for some  $j \in \{1, 2, \dots, r\}$ ,  $C_j$  contains a member of  $\mathcal{A}$ .  $\mathcal{A}$  is said to be *shift invariant* if for all  $A \in \mathcal{A}$  and all  $x \in \sigma(A)$ ,  $A + x = \{a + x : a \in A\} \in \mathcal{A}$ .

Let us now mention some useful theorems from [6] for the proof of our main theorem.

**Theorem 4.3.** Let  $(S, +)$  be a commutative adequate partial semigroup and let  $k \in \mathbb{N}$ . If  $\{\langle v_\alpha^{(i)} \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})} : 1 \leq i \leq k\}$  is an adequate set of VIP systems in  $S$ , and  $\beta \in \mathcal{P}_f(\mathbb{N})$ , then the family

$$\mathcal{A} = \left\{ \begin{array}{l} \{a, a + v_\alpha^{(1)}, a + v_\alpha^{(2)}, \dots, a + v_\alpha^{(k)}\} : \\ \alpha \in \mathcal{P}_f(\mathbb{N}), a \in \sigma(\{v_\alpha^{(1)}, v_\alpha^{(2)}, \dots, v_\alpha^{(k)}\}) \text{ and } \alpha > \beta \end{array} \right\}$$

is *adequately partition regular*.

*Proof.* [6], Theorem 3.7. □

**Theorem 4.4.** Let  $(S, +)$  be a commutative adequate partial semigroup and let  $\mathcal{A}$  be a shift invariant, adequately partition regular family of finite subsets of  $S$ . Let  $E \subseteq S$  be piecewise syndetic. Then  $E$  contains a member of  $\mathcal{A}$ .

*Proof.* [6], Theorem 3.8. □

**Theorem 4.5.** Let  $\{\langle v_\alpha^{(i)} \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})} : 1 \leq i \leq k\}$  be an adequate set of VIP systems and pick  $d, t \in \mathbb{N}$ , a set  $\{\langle m_\gamma^{(i)} \rangle_{\gamma \in \mathcal{F}_d} : 1 \leq i \leq k\}$ , a set of VIP systems

$$\left\{ \langle u_\alpha^{(i)} = \sum_{\gamma \subseteq \alpha, \gamma \in \mathcal{F}_d} n_\gamma^{(i)} \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})} : 1 \leq i \leq t \right\},$$

and sets  $E_1, E_2, \dots, E_k \subseteq \{1, 2, \dots, t\}$  satisfying conditions (1), (2), and (3) of Definition 4.1. Let  $\alpha_1, \alpha_2, \dots, \alpha_s \in \mathcal{P}_f(\mathbb{N})$  with  $\alpha_1 < \alpha_2 < \dots < \alpha_s$ . For  $F \subseteq \{1, 2, \dots, s\}$ ,  $i \in \{1, 2, \dots, k\}$  and  $\varphi \in \mathcal{F}_d$  with  $\varphi > \alpha_s$ , and  $1 \leq i \leq k$ , let

$$b_\varphi^{(i,F)} = \sum_{\psi \subseteq \cup_{j \in F} \alpha_j, |\psi| \leq d - |\varphi|} m_{\varphi \cup \psi}^{(i)}.$$

For  $F \subseteq \{1, 2, \dots, s\}$ ,  $i \in \{1, 2, \dots, k\}$ , and  $\beta \in \mathcal{F}_d$  with  $\beta > \alpha_s$ , let

$$q_\beta^{(i,F)} = \sum_{\varphi \subseteq \beta, \varphi \in \mathcal{F}_d} b_\varphi^{(i,F)}.$$

Then  $\{\langle q_\beta^{(i,F)} \rangle_{\beta \in \mathcal{P}_f(\mathbb{N}), \beta > \alpha_s} : i \in \{1, 2, \dots, k\}, F \subseteq \{1, 2, \dots, s\}\}$  is an adequate set of VIP systems.

*Proof.* [6], Theorem 3.10. □

Here is our main theorem of this section.

**Theorem 4.6.** *Let  $(S, +)$  be commutative adequate partial semigroup, let  $p$  be an idempotent in  $K(\delta S)$ , and let  $\langle C_n \rangle_{n=1}^\infty$  be a sequence of members of  $p$  and let*

$$\left\{ \langle v_\alpha^{(i)} \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})} : 1 \leq i \leq k \right\}$$

*be  $k$ -many adequate set of VIP systems. Then there exist sequences  $\langle a_n \rangle_{n=1}^\infty$  in  $S$  and  $\langle \alpha_n \rangle_{n=1}^\infty$  in  $\mathcal{P}_f(\mathbb{N})$  such that  $\alpha_n < \alpha_{n+1}$  for every  $n$  and for every  $F \in \mathcal{P}_f(\mathbb{N})$ ,  $\gamma = \bigcup_{t \in F} \alpha_t$  such that for  $m = \min F$*

$$\left\{ \sum_{t \in F} a_t \right\} \cup \left\{ \sum_{t \in F} a_t + v_\gamma^{(i)} : 1 \leq i \leq k \right\} \subseteq C_m.$$

*Proof.* We may assume that  $C_{n+1} \subseteq C_n$  for each  $n \in \mathbb{N}$  ( If not, consider  $B_n = \bigcap_{i=1}^n C_i$ , so  $B_{n+1} \subseteq B_n$ ). For each  $n \in \mathbb{N}$ , let

$$C_n^* = \{x \in C_n : -x + C_n \in p\}.$$

Then for each  $x \in C_n^*$ ,  $-x + C_n^* \in p$  by Lemma 2.9. Let

$$\mathcal{A} = \left\{ \left\{ a, a + v_\alpha^{(1)}, a + v_\alpha^{(2)}, \dots, a + v_\alpha^{(k)} \right\} : \alpha \in \mathcal{P}_f(\mathbb{N}), a \in \sigma \left( \left\{ v_\alpha^{(1)}, v_\alpha^{(2)}, \dots, v_\alpha^{(k)} \right\} \right) \right\}.$$

Now we proceed the proof by induction on  $n$ , let  $n \in \mathbb{N}$  and assume that we have chosen  $\langle a_t \rangle_{t=1}^n$  in  $S$  and  $\langle \alpha_t \rangle_{t=1}^n$  in  $\mathcal{P}_f(\mathbb{N})$  such that

(1) for  $t \in \{1, 2, \dots, n-1\}$ , if any,  $\alpha_t < \alpha_{t+1}$  and

(2) for  $\emptyset \neq F \subseteq \{1, 2, \dots, n\}$ ,  $\min F = m$ , if  $\gamma = \bigcup_{t \in F} \alpha_t$ , then  $\sum_{t \in F} a_t \in C_m^*$  and for each  $i \in \{1, 2, \dots, k\}$ ,  $\sum_{t \in F} a_t + v_\gamma^{(i)} \in C_m^*$ .

Then by Theorem 4.3,  $\mathcal{A}$  is adequately partition regular and  $\mathcal{A}$  is trivially shift invariant. Since for each  $n \in \mathbb{N}$ ,  $C_n^* \in p$  and  $p \in K(\delta S)$ ,  $C_n^*$  is piecewise syndetic. So by Theorem 4.4, for some  $a_1 \in S$  and  $\alpha_1 \in \mathcal{P}_f(\mathbb{N})$  such that

$$\left\{ a_1, a_1 + v_{\alpha_1}^{(1)}, a_1 + v_{\alpha_1}^{(2)}, \dots, a_1 + v_{\alpha_1}^{(k)} \right\} \subseteq C_1^*.$$

For each  $\gamma \in FU(\langle \alpha_t \rangle_{t=1}^n)$  and each  $i \in \{1, 2, \dots, k\}$ , let

$$\langle q_\beta^{(i, \gamma)} \rangle_{\beta \in \mathcal{P}_f(\mathbb{N})} = \langle v_{\gamma \cup \beta}^{(i)} - v_\gamma^{(i)} \rangle_{\beta \in \mathcal{P}_f(\mathbb{N}), \beta > \alpha_n}.$$

By Theorem 4.5, the family,

$$\left\{ \langle q_\beta^{(i, \gamma)} \rangle_{\beta \in \mathcal{P}_f(\mathbb{N}), \beta > \alpha_n} : 1 \leq i \leq k, \gamma \in FU(\langle \alpha_t \rangle_{t=1}^n) \right\} \cup \left\{ \langle v_\beta^{(i)} \rangle_{\beta \in \mathcal{P}_f(\mathbb{N})} : 1 \leq i \leq k \right\}$$

is an adequate set of VIP systems. Let

$$\mathcal{B} = \left\{ \begin{aligned} & \{a\} \cup \left\{ a + v_\alpha^{(i)} : 1 \leq i \leq k \right\} \cup \\ & \bigcup_{\gamma \in FU(\langle \alpha_t \rangle_{t=1}^n)} \left\{ a + q_\alpha^{(i, \gamma)} : 1 \leq i \leq k \right\} : \alpha \in \mathcal{P}_f(\mathbb{N}), \alpha > \alpha_n, \text{ and} \\ & a \in \sigma \left( \left\{ v_\alpha^{(i)} : 1 \leq i \leq k \right\} \cup \left\{ q_\alpha^{(i, \gamma)} : 1 \leq i \leq k, \gamma \in FU(\langle \alpha_t \rangle_{t=1}^n) \right\} \right) \end{aligned} \right\}.$$

Then by Theorem 4.3,  $\mathcal{B}$  is adequately partition regular. Let

$$D = C_{n+1}^* \cap \bigcap_{m=1}^n \left[ \bigcap \left\{ -\sum_{t \in H} a_t + C_m^* : \emptyset \neq H \subseteq \{1, 2, \dots, n\}, m = \min H \right\} \cap \bigcap \left\{ -\left( \sum_{t \in H} a_t + v_\gamma^{(i)} \right) + C_m^* : 1 \leq i \leq k, \emptyset \neq H \subseteq \{1, 2, \dots, n\}, m = \min H, \text{ and } \gamma = \bigcup_{t \in H} \alpha_t \right\} \right].$$

Then  $D \in p$  and  $D$  is piecewise syndetic. So, by Theorem 4.4, for some  $\alpha_{n+1} \in \mathcal{P}_f(\mathbb{N})$  such that  $\alpha_{n+1} > \alpha_n$  and some

$$a_{n+1} \in \sigma \left( \left\{ v_{\alpha_{n+1}}^{(i)} : 1 \leq i \leq k \right\} \cup \left\{ q_{\alpha_{n+1}}^{(i,\gamma)} : 1 \leq i \leq k \text{ and } \gamma \in FU(\langle \alpha_t \rangle_{t=1}^n) \right\} \right)$$

such that

$$\{a_{n+1}\} \cup \left\{ a_{n+1} + v_{\alpha_{n+1}}^{(i)} : i \in \{1, 2, \dots, k\} \right\} \cup \bigcup_{\gamma \in FU(\langle \alpha_t \rangle_{t=1}^n)} \left\{ a_{n+1} + q_{\alpha_{n+1}}^{(i,\gamma)} : i \in \{1, 2, \dots, k\} \right\} \subseteq D.$$

Induction hypothesis (1) trivially holds. To verify hypothesis (2), let  $\emptyset \neq F \subseteq \{1, 2, \dots, n + 1\}$  and let  $\gamma = \cup_{t \in F} \alpha_t$ . If  $n + 1 \notin F$ , the conclusion holds by assumption. If  $F = \{n + 1\}$ , then we have

$$\{a_{n+1}\} \cup \left\{ a_{n+1} + v_{\alpha_{n+1}}^{(i)} : i \in \{1, 2, \dots, k\} \right\} \subseteq D \subseteq C_{n+1}^*.$$

So, let us assume that  $\{n + 1\} \subsetneq F$ , let  $H = F \setminus \{n + 1\}$ , and let  $\mu = \cup_{t \in F} \alpha_t$ . Then  $a_{n+1} \in D \subseteq -\sum_{t \in H} a_t + C_m^*$ , where  $m = \min H$  so  $\sum_{t \in H} a_t \in C_m^*$ .

Let  $\gamma = \cup_{t \in H} \alpha_t$ , and let  $i \in \{1, 2, \dots, k\}$ . Then

$$a_{n+1} + q_{\alpha_{n+1}}^{(i,\gamma)} \in D \subseteq -\left(\sum_{t \in H} a_t + v_{\gamma}^{(i)}\right) + C_m^*,$$

where  $m = \min H$

and so  $\left(\sum_{t \in H} a_t + v_{\gamma}^{(i)}\right) + \left(a_{n+1} + q_{\alpha_{n+1}}^{(i,\gamma)}\right) \in C_m^*$ . That is,

$$\sum_{t \in F} a_t + v_{\mu}^{(i)} = \left(\sum_{t \in H} a_t + a_{n+1}\right) + \left(v_{\gamma}^{(i)} + q_{\alpha_{n+1}}^{(i,\gamma)}\right) \in C_m^* \subseteq C_m.$$

This completes the proof. □

We conclude Section 4 with the remark that similar minor adjustments in the proofs establish the natural analogues of Theorems 3.12, 3.13, Corollary 3.14, and Theorem 3.15 of [6].

### 5 APPLICATION

A brief discussion on the application of Theorem 4.6 is presented in this section:

**Definition 5.1.** Let  $l \in \mathbb{N}$ , a set-monomial (over  $\mathbb{N}^l$ ) in the variable  $X$  is an expression  $m(X) = S_1 \times S_2 \times \dots \times S_l$ , where for each  $i \in \{1, 2, \dots, l\}$ ,  $S_i$  is either the symbol  $X$  or a nonempty singleton subset of  $\mathbb{N}$  (these are called coordinate coefficients). The degree of the monomial is the number of times the symbol  $X$  appears in the list  $S_1, \dots, S_l$ . For example, taking  $l = 3$ ,  $m(X) = \{5\} \times X \times X$  is a set-monomial of degree 2, while  $m(X) = X \times \{17\} \times \{2\}$  is a set-monomial of degree 1. A *set-polynomial* is an expression of the form  $p(X) = m_1(X) \cup m_2(X) \cup \dots \cup m_k(X)$ , where  $k \in \mathbb{N}$  and  $m_1(X) \cup m_2(X) \cup \dots \cup m_k(X)$  are set-monomials. The degree of a set-polynomial is the largest degree of its set-monomial ‘‘summands’’, and its constant term consists of the ‘‘sum’’ of those  $m_i$  that are constant, i.e., of degree zero.

**Lemma 5.2.** Let  $l \in \mathbb{N}$  and let  $\mathcal{P}$  be a finite family of set polynomial over  $(\mathcal{P}_f(\mathbb{N}^l), +)$  whose constant terms are empty. Then there exists  $q \in \mathbb{N}$  and an IP ring

$$\mathcal{F}^{(1)} = \{\alpha \in \mathcal{P}_f(\mathbb{N}) : \min \alpha > q\}$$

such that  $\{\langle P(\alpha) \rangle_{\alpha \in \mathcal{F}^{(1)}} : P(X) \in \mathcal{P}\}$  is an adequate set of VIP systems.

*Proof.* [6], Lemma 4.3. □

**Theorem 5.3.** Let  $l \in \mathbb{N}$  and let  $\mathcal{P}$  be a finite family of set polynomial over  $(\mathcal{P}_f(\mathbb{N}^l), +)$  whose constant terms are empty, and let  $p$  be a minimal idempotent in  $\delta(\mathcal{P}_f(\mathbb{N}^l))$ , and if  $\langle C_n \rangle_{n=1}^\infty$  be a sequence of central sets in  $p$ . Then, there exists sequences  $\langle A_n \rangle_{n=1}^\infty$  in  $\mathcal{P}_f(\mathbb{N}^l)$  and  $\langle \alpha_n \rangle_{n=1}^\infty$  in  $\mathcal{P}_f(\mathbb{N})$  with  $\alpha_n < \alpha_{n+1}$  for each  $n$  and for every  $F \in \mathcal{P}_f(\mathbb{N})$ , we have  $\{A_\gamma\} \cup \{A_\gamma + P(\gamma) : P \in \mathcal{P}\} \subseteq C_m$ , where  $m = \min F$ ,  $\gamma = \cup_{t \in F} \alpha_t$  and  $A_\gamma = \sum_{t \in F} A_t$ .

*Proof.* By Lemma 5.2, there is an IP ring  $\mathcal{F}^{(1)}$  such that  $\{\langle P(\alpha) \rangle_{\alpha \in \mathcal{F}^{(1)}} : P(X) \in \mathcal{P}\}$  is an adequate set of VIP systems. Thus Theorem 4.6 applies.  $\square$

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