**Palestine Journal of Mathematics**

**Vol 14(3)(2025)** , 1364–1372 

# Fuzzy Uniform Topology on a Pseudo-UP Algebra

Alachew Amaneh*, Berhanu Assaye and Tilahun Mekonnen

Communicated by Madeleine Al-Tahan

MSC 2010 Classifications: Primary 03G25; Secondary 06D30.

Keywords and phrases: uniform space, extreme fuzzy pseudo-UP ideal, fuzzy uniform topological space.

*Correspondence should be addressed to Alachew Amaneh*.*

**Abstract** This paper introduces extreme fuzzy pseudo-UP ideals on pseudo-UP algebras and defines a fuzzy uniform topological pseudo-UP algebra. We explore its properties, demonstrating how this concept extends existing theories and leads to new insights into fuzzy topological pseudo-UP algebras.

## 1 Introduction

The notion of uniform space was first introduced by Weil (1938) in [6] as a tool which in contrast with the metrics was used for studying topological spaces with no countability assumptions. The concept of a uniform space can be considered either as axiomatization of some geometric notions, close to but quite independent of the concept of a topological space, or as convenient tools for an investigation of topological space. The theory of uniform spaces shows striking analogies with the theory of metric spaces contrast to the realm of its applicability which is much broader. For example, on every topological group, there are three natural uniformities which are useful in application of topological groups. In 2017, Mehrshad and Golzarpoor[7] investigated the characteristics of topological and uniform BE-algebras and uniform BE-algebras. In the same year, Iampan [8]presented a new class of algebras called UP-algebras which is an extension of KU-algebras [9] introduced by Prabpayak and Leerawat in 2009. In 2020, Romano [10] introduced another class of algebras called pseudo-UP algebras as an extension of UP- algebras. Also, he studied the concepts of pseudo-UP filers and pseudo-UP ideals of pseudo-UP algebras in [11].

In our daily lives, numerous models and traditional methods are employed to manage uncertainty and imprecision. Various fields, including medical, social, and physical sciences, rely heavily on these methods and tools to address imprecision and uncertainty in data recognition. To tackle such situations, different theories and ideas are continually being developed. One significant contribution is by Zadeh [5], who introduced the concept of fuzzy sets, which is applicable in many contexts. In 2023 Mechderso et al. [1, 2] applied fuzzy concepts to the pseudo-UP ideal and congruence relation of pseudo-UP algebra. In 2023 Iampan et al [13] applied the concept of L-fuzzy sets (LFSs) to UP (BCC)-algebras and introduced five types of LFSs in UP (BCC)-algebras: L-fuzzy UP (BCC)-subalgebras, L-fuzzy near UP (BCC)-filters, L-fuzzy UP (BCC)-filters, L-fuzzy UP (BCC)-ideals, and L-fuzzy strong UP (BCC)-ideals. In 2022 Muhiuddin et al [14] applied the concept of linear Diophantine fuzzy sets in BCK/BC I-algebras. In this respect, the notions of linear Diophantine fuzzy subalgebras and linear Diophantine fuzzy (commutative) ideals are introduced and some vital properties are discussed. In 2023 Al-Tahan et al [15] introduced the notion of linear Diophantine fuzzy $n$-fold weak subalgebra of a BE-algebra, studied its properties, and relate it to the concept of $n$-fold weak subalgebra. Numerous works have been published on this topic in the literature (for example see [17, 18, 19, 20]). In 2022, Jun et al.[16] have shown that the concept of UP-algebras (see [8]) and the concept of BCC-algebras.

Motivated by this previous research, we aim to answer the following question: How can we develop a fuzzified version of the uniform topology for pseudo-UP algebra based on ideal theory? We define the family of extreme fuzzy pseudo-UP ideals $\aleph$, on the larger class of pseudo-UP algebra, denoted by $X$, to construct a fuzzy uniform structure $(X, K)$. The component $K$ induces a fuzzy uniform topology $\tau_{\aleph}$ in $X$. We prove that the pair $(X, \tau_{\aleph})$ forms a fuzzy topological pseudo-UP algebra and investigate some properties of $(X, \tau_{\aleph})$. Finally, we provide characterizations of the fuzzy topological properties of $(X, \tau_{\aleph})$.

## 2 Preliminaries

In this section, we review some of the fundamental concepts and definitions that are essential for this paper.

**Definition 2.1.** [10] A pseudo-UP algebra is an algebra $(X, \cdot, \star, 0)$ of type $(2, 2, 0)$ that satisfies the following axioms for any $x, y, z \in X$:

1) $(y \cdot z) \cdot ((x \cdot y) \star (x \cdot z)) = 0$ and $(y \star z) \star ((x \star y) \cdot (x \star z)) = 0$,
2) $x \cdot y = 0 = y \cdot x$ and $x \star y = 0 = y \star x \Longrightarrow x = y$,
3) $(y \cdot 0) \star x = x$ and $(y \star 0) \cdot x = x$,
4) $x \leqslant y \Longleftrightarrow x \cdot y = 0$ and $x \leqslant y \Longleftrightarrow x \star y = 0$.

**Lemma 2.2.** *[10] In a pseudo-UP algebra $X$ the following holds, for each $x \in X$,*

*1)* $x \cdot 0 = 0$ *and* $x \star 0 = 0$,
*2)* $0 \cdot x = x$ *and* $0 \star x = x$, *and*

*3)* $x \cdot x = 0$ *and* $x \star x = 0$.

**Definition 2.3.** [1] A fuzzy relation $\psi$ on a set X is termed a fuzzy congruence relation on a pseudo-UP algebra of $X$ if it meets the following axioms for any $x, y, z \in X$.

1) $\psi(x, x) = \psi(0, 0)$,
2) $\psi(x, y) = \psi^{-1}(y, x)$,
3) $\psi(x, z) \geq \sup_{y \in X}\{min\{\psi(x, y), \psi(y, z)\}\}$,
4) $\psi(x \cdot z, y \cdot z) \geq \psi(x, y)$ and $\psi(x \star z, y \star z) \geq \psi(x, y)$, and
5) $\psi(z \cdot x, z \cdot y) \geq \psi(x, y)$ and $\psi(z \star x, z \star y) \geq \psi(x, y)$.

**Definition 2.4.** [12] A fuzzy set $\mu$ in a fuzzy topological space $(X, \tau)$ is a neighborhood of a point $x \in X$ if and only if there is $\eta \in \tau$ such that $\eta \subseteq \mu$ and $\mu(x) = \eta(x) > 0$.

**Theorem 2.5.** *[1] A fuzzy equivalence relation $\psi$ of $X$ is a fuzzy congruence relation pseudo-UP algebra of $X$ if and only if $\psi(x \cdot u, y \cdot v) \geq min\{\psi(x, y), \psi(u, v)\}$ and $\psi(x \star u, y \star v) \geq min\{\psi(x, y), \psi(u, v)\}$.*

**Theorem 2.6.** *[1] Let $\mu$ be a pseudo-UP ideal of $X$, and define a fuzzy relation $\psi_\mu(x, y) = min\{\mu(x \cdot y), \mu(x \star y), \mu(y \cdot x), \mu(y \star x)\}$. This is a fuzzy congruence relation on $X$.*

**Theorem 2.7.** *[2] Let $f : X \to Y$ be a homomorphism of a pseudo-UP algebra. If $\psi$ be a fuzzy pseudo-UP ideal of $Y$. Then $f^{-1}(\psi)$ is a fuzzy pseudo-UP ideal of $X$*

**Theorem 2.8.** *[2] Let $(X, \cdot, \star, 0)$ and $(Y, \cdot, \star, 0)$ be a pseudo-UP algebra. $f : X \to Y$ is surjective and $\psi$ is a fuzzy pseudo-UP ideals of $X$. Then $f(\psi)$ is a fuzzy pseudo-UP ideals of $Y$, provided that the sup property holds.*

# 3 Fuzzy Uniform Structure on a Pseudo-UP Algebra

A fuzzy set is simply an element of $I^X$ where $I = [0, 1]$ and if $\mu \in I^X$ then we can think of $\mu(x)$ as being the "degree to which $x$ belongs to $\mu$. A fuzzy subset $\mu$ of $X$ is a non-zero if there is $x \in X$ such that $\mu(x) \neq 0$. A fuzzy relation is, simply put, an element of $I^{X \times X}$. The notation $I^{X \times X}$ represents the set of all fuzzy relations on $X \times X$ which are functions from $X \times X$ to $I$.

In this section, we construct the fuzzy uniform structures by the special family of extreme fuzzy pseudo-UP ideal, and then induce fuzzy uniform topologies. Moreover, we show that pseudo-UP algebras with fuzzy uniform topologies are fuzzy topological pseudo-UP algebras, and also some properties are investigated.

**Notation:** Let $X$ be a non empty pseudo-UP algebra, $\psi$ and $\phi$ be fuzzy relations on $X \times X$. Then we define the following:

1. $1_X(x) = 1$ and $0_X(x) = 0$, for all $x \in X$,
2. $\psi^{-1}(x, y) = \psi(y, x)$, for all $(x, y) \in X \times X$,
3. $\psi^{[x]}(y) = \psi(x, y)$, for all $(x, y) \in X \times X$,
4. $(\psi \circ \phi)(x, y) = \sup_{z \in X}\{min\{\psi(x, z), \phi(z, y)\}$, for all $(x, y) \in X \times X\}$.

**Definition 3.1.** A fuzzy pseudo-UP ideal $\mu$ of a pseudo-UP algebra $X$ is called an extreme fuzzy pseudo-UP ideal of $X$ if $\mu(0) = 1$. The family of an extreme fuzzy pseudo-UP ideals of $X$ is denoted by $\aleph$.

Interestingly, any fuzzy pseudo-UP ideal of a pseudo-UP algebra can be extended into an extreme fuzzy pseudo-UP ideal within the pseudo-UP algebra $X$.

**Theorem 3.2.** *Let $\zeta$ be a fuzzy pseudo-UP ideal of $X$. Then there exists an extreme fuzzy pseudo-UP ideal $\mu$ of $X$ such that $\zeta \subseteq \mu$.*

*Proof.* Let $\zeta$ be a fuzzy pseudo-UP ideal of $X$ and we define an extreme fuzzy pseudo-UP ideal $\mu$ by

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0 \\ \zeta(x), & \text{otherwise.} \end{cases}$$

Thus, $\zeta \subseteq \mu$. □

In general, the extension of a fuzzy pseudo-UP ideal into extreme fuzzy pseudo-UP ideal is not unique.

**Example 3.3.** Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation " $\cdot$ " and " $\star$ " defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 0 | 3 | 3 |
| 2 | 0 | 1 | 0 | 0 |
| 3 | 0 | 1 | 3 | 0 |

| $\star$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 0 | 3 | 3 |
| 2 | 0 | 1 | 0 | 0 |
| 3 | 0 | 2 | 3 | 0 |

See [3], $(X, \cdot, \star, 0)$ is a pseudo-UP algebra. We define a fuzzy set $\zeta : X \to [0, 1]$ as follows, $\zeta(0) = 0.8, \zeta(1) = 0.6, \zeta(2) = 0.4, \zeta(3) = 0.3$. It is easily checked that $\zeta$ is a fuzzy pseudo-UP ideal of $X$. We define a fuzzy set $\mu : X \to [0, 1]$ as follows, $\mu(0) = 1, \mu(1) = 0.6, \mu(2) = 0.4, \mu(3) = 0.3$, and $\lambda(0) = 1, \lambda(1) = 0.7, \lambda(2) = 0.4, \lambda(3) = 0.3$. Through standard calculation, $\mu$ and $\lambda$ can be expressed as extensions of $\zeta$, but $\mu \neq \lambda$.

**Definition 3.4.** Suppose that $\aleph$ is an arbitrary family of an extreme fuzzy pseudo-UP ideals of a pseudo-UP algebra $X$. A fuzzy relation $\psi \in I^{X\times X}$, then we define the following:

1. $\psi_\mu(x,y) = min\{\mu(x\cdot y), \mu(x\star y), \mu(y\cdot x), \mu(y\star x)\}$, for all $(x,y) \in X\times X$,
2. $\psi_\mu^{[x]}(y) = \psi_\mu(x,y)$, for $y\in X$, and $\psi_\mu^{[\lambda]}(x) = \sup_{y\in X}\{min\{\lambda(y), \psi_\mu(x,y)\}\}$, for $\lambda \in I^X$,
3. $K^\star = \{\psi_\mu : \mu \in \aleph\}$,
4. $\mathbb{K} = \{\psi : \psi_\mu \subseteq \psi, for\ some\ \psi_\mu \in K^\star\}$.

**Definition 3.5.** A fuzzy uniformity $K$ on a set $X$ is a collection of fuzzy relations on $X\times X$, ($i.e$ each element of $K$ is a fuzzy relation such that function from $X\times X \to I$), which fulfills the following properties:

$U_1$. For all $\psi \in K$, $\psi(x,x) = 1$ for all $x\in X$,

$U_2$. $\psi \in K \Rightarrow \psi^{-1} \in K$,

$U_3$. For all $\psi \in K$, there exist a fuzzy relation $\phi \in K$ such that $\phi\circ\phi \subseteq \psi$,

$U_4$. $\psi, \phi \in K$, then $\psi\cap\phi \in K$,

$U_5$. $\psi \in K$ and $\psi \subseteq \phi$, then $\phi \in K$.

Then the pair $(X, K)$ is a fuzzy uniform space.

**Theorem 3.6.** *Let $\aleph$ be family of an extreme fuzzy pseudo-UP ideals of a pseudo-UP algebra $X$, then $K^\star$ meets the criteria $(U_1)$ through $(U_4)$.*

*Proof.* Suppose $\mu \in \aleph$ and $\psi$ is a fuzzy relation.
$U_1$. Let $\psi_\mu \in K^\star$. Then $\psi_\mu(x,x) = min\{\mu(x\cdot x), \mu(x\star x), \mu(x\cdot x), \mu(x\star x)\} = \min\{\mu(0), \mu(0), \mu(0), \mu(0)\} = 1$. Thus $\psi_\mu(x,x) = 1$.
$U_2$. Let $\psi_\mu \in K^\star$, then we have

$$\psi_\mu^{-1}(x,y) = \psi_\mu(y,x) = min\{\mu(y\cdot x), \mu(y\star x), \mu(x\cdot y), \mu(x\star y)\} = min\{\mu(x\cdot y), \mu(x\star y), \mu(y\cdot x), \mu(y\star x)\} = \psi_\mu(x,y).$$

Hence, $\psi_\mu^{-1} \in K^\star$.
$U_3$. Let $\psi_\mu \in K^\star$ and $(\psi_\mu \circ \psi_\mu)(x,z)$, then there exist $y\in X$ such that $\psi_\mu\circ\psi_\mu(x,z) = \sup_{y\in X}\{min\{\psi_\mu(x,y), \psi_\mu(y,z)\}\} \le \psi_\mu(x,z)$, by Theorem 2.6 $\psi_\mu$ is a fuzzy congruence relation on $X$. Thus $\psi_\mu\circ\psi_\mu \subseteq \psi_\mu$.
$U_4$. Let $\mu,\ \lambda \in \aleph$ and $\psi_\mu, \psi_\lambda \in K^\star$. We need to show that $\psi_\mu \cap \psi_\lambda \in K^\star$.

$$\begin{aligned}(\psi_\mu\cap\psi_\lambda)(x,y) &= min\{\psi_\mu(x,y), \psi_\lambda(x,y)\}\\ &= min\{min\{\mu(x\cdot y), \mu(x\star y), \mu(y\cdot x), \mu(y\star x)\}, min\{\lambda(x\cdot y), \lambda(x\star y)\lambda(y\cdot x), \lambda(y\star x)\}\}\\ &= min\{min\{\mu(x\cdot y), \mu(x\star y), \lambda(x\cdot y), \lambda(x\star y)\}, min\{\mu(y\cdot x), \mu(y\star x)\lambda(y\cdot x), \lambda(y\star x)\}\}\\ &= min\{(\mu\cap\lambda)(x\cdot y), (\mu\cap\lambda)(x\star y), (\mu\cap\lambda)(y\cdot x), (\mu\cap\lambda)(y\star x)\}\}\\ &= \psi_{\mu\cap\lambda}(x,y).\end{aligned}$$

Therefore, $\psi_\mu\cap\psi_\lambda = \psi_{\mu\cap\lambda}$, then we have $\psi_\mu\cap\psi_\lambda \in K^\star$, since $\psi_{\mu\cap\lambda} \in K^\star$.
Hence, $K^\star$ satisfies the conditions $(U_1)$ through $(U_4)$. □

**Remark 3.7.** $K^\star$ is not a fuzzy uniform structure on $X$.

**Example 3.8.** Let $X = \{0, a, b, c\}$ and two binary operation " $\cdot$ " and " $\star$ " defined by the following cayley table:

| · | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | 0 | 0 | 0 | 0 |
| b | 0 | a | 0 | c |
| c | 0 | a | b | 0 |

| ⋆ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | 0 | 0 | 0 | 0 |
| b | 0 | b | 0 | 0 |
| c | 0 | b | b | 0 |

See [3] $(X,\cdot,\star)$ is a pseudo-UP algebra. We define a fuzzy set $\mu$ as follows:
$\mu(0) = 1, \mu(a) = 0.5 = \mu(b) = \mu(c)$. Then $\mu$ is an extreme fuzzy pseudo-UP ideal of a pseudo-UP algebra. And $\psi_\mu(0,0) = \psi_\mu(a,a) = \psi_\mu(b,b) = \psi_\mu(c,c) = 1$, $\psi_\mu(0,a) = \psi_\mu(a,0) = \psi_\mu(a,b) = \psi_\mu(b,a) = \psi_\mu(c,a) = (b,c) = \psi_\mu(c,b) = \psi_\mu(a,c) = 0.5$. Again we define a fuzzy set $\lambda$ as follows $\lambda(0) = 1$, $\lambda(a) = 0.7 = \lambda(b)$ and $\lambda(c) = 0.9$ and then we have, $\psi_\lambda(0,0) = \psi_\lambda(a,a) = \psi_\lambda(b,b) = \psi_\lambda(c,c) = 1, \psi_\lambda(a,b) = \psi_\lambda(a,0) = \psi_\lambda(0,b) = \psi_\lambda(a,c) = \psi_\lambda(b,c) = 0.7$ and $\psi_\lambda(c,0) = \psi_\lambda(0,c) = 0.9$. It follows that $\psi_\mu \subseteq \psi_\lambda$. However, $\lambda$ does not qualify as a fuzzy pseudo-UP ideal of $X$, which means

$$\begin{aligned}&\Rightarrow \lambda(b\cdot a) \ge min\{\lambda(b\cdot(c\star a), \lambda(c))\}\\ &\Rightarrow \lambda(a) \ge min\{\lambda(b\cdot b), \lambda(c)\}\\ &\Rightarrow \lambda(a) \ge min\{\lambda(0), \lambda(c)\}\\ &\Rightarrow 0.7 \ge min\{1, 0.9\}.\end{aligned}$$

This is inconsistent with the definition of a fuzzy pseudo-UP ideal. Which implies that $\psi_\lambda \notin K^\star$. Hence, $K^\star$ does not satisfy condition $U_5$ from Definition 3.5.

**Corollary 3.9.** *Let $X$ be a pseudo-UP algebra, and define $\mathbb{K} = \{\psi : \exists\psi_\mu \in K^\star$ such that $\psi_\mu \subseteq \psi\}$. Then $\mathbb{K}$ is a fuzzy uniform structure on X.*

*Proof.* From the above Theorm 3.6, we find that $\mathbb{K}$ satisfies conditions $(U_1)$ through $(U_4)$. It is sufficient to demonstrate that $\mathbb{K}$ satisfies $(U_5)$. Let $\psi \in \mathbb{K}$ and $\psi \subseteq \phi$, then there exist $\psi_\mu \subseteq \psi \subseteq \phi$ implies that $\psi_\mu \subseteq \phi$, which means that $\phi \in \mathbb{K}$. Hence, the pair $(X, \mathbb{K})$ forms a fuzzy uniform structure on the pseudo-UP algebra $X$. □

**Definition 3.10.** Let $(X, K)$ be a fuzzy uniform space. A subfamily of fuzzy relation $\Phi$ of $K$ is called a base for $K$ if $\psi \in K$ there exist $\beta \in \Phi$ such that $\beta \subseteq \psi$.

**Theorem 3.11.** *A non-void family of fuzzy relations $\Phi$ is a base for some fuzzy uniformity on $X$ if and only if the following conditions holds:*

1. *For all $\beta \in \Phi,\ \beta(x, x) = 1, \forall x \in X$*
2. *For all $\beta \in \Phi$ implies $\beta^{-1}$ contanis some elements of $\Phi$,*
3. *For all $\beta, \beta' \in \Phi$ implies $\beta \cap \beta'$ contians some elements of $\Phi$.*
4. *For all $\beta \in \Phi$ there exist $\beta^\star \in \Phi$ such that $\beta^\star \circ \beta^\star \subseteq \beta$,*

*Proof.* Let $X$ be a non empty pseudo-UP algebra, and define $K = \{\psi : \beta \subseteq \psi, \text{for some } \beta \in \Phi\}$, where $\Phi$ is a non-void family of fuzzy relation. Suppose $\Phi$ is a base for a fuzzy uniformity $K$. We need to show that conditions (1) through (4) are satisfied.

1. Let $\beta \in \Phi$, then $\beta \in K$, since $\Phi$ is a base for fuzzy uniformity $K$. Which implies that $\beta(x, x) = 1$, by Definition 3.5 $(U_1)$. Therefore, for all $\beta \in K$, $\beta(x, x) = 1$.

2. Let $\beta \in \Phi$, then $\beta \in K$, since $\Phi$ is a base for a fuzzy uniformity.

$$\Rightarrow \exists \beta^{\backprime} \in \Phi \text{ such that } \beta^{\backprime}(x, y) \leq \beta(x, y), \text{ (by the definition of base)}$$

$$\Rightarrow {\beta^{\backprime}}^{-1}(y, x) = \beta^{\backprime}(x, y) \leq \beta(x, y) = \beta^{-1}(y, x)$$

$$\Rightarrow {\beta^{\backprime}}^{-1}(y, x) \leq \beta^{-1}(y, x)$$

$$\Rightarrow {\beta^{\backprime}}^{-1} \subseteq \beta^{-1}$$

$$\Rightarrow {\beta^{\backprime}}^{-1} \in K.$$

Which implies that there exist $\beta^\star \in \Phi$ such that $\beta^\star(y, x) \leq {\beta^{\backprime}}^{-1}(y, x) \leq \beta^{-1}(y, x)$ implies $\beta^{-1}$ contains some elements of $\Phi$ such that $\beta^\star \subseteq \beta^{-1}$.

3. Let $\beta,\ \beta^{\backprime} \in \Phi$, then $\beta,\ \beta^{\backprime} \in K$. Which implies $\beta \cap \beta^{\backprime} \in K$, then there exist $\beta^\star \in \Phi$ such that $\beta^\star(x, y) \leq min\{\beta(x, y), \beta^{\backprime}(x, y)\} \Rightarrow \beta^\star \subseteq \beta \cap \beta^{\backprime}$. Therefore, $\beta \cap \beta^{\backprime}$ contains some memebers of $\Phi$.

4. Let $\beta \in \Phi$, then $\beta \in K$. And there exist $\beta^{\backprime} \in \Phi$ such that $\beta^{\backprime}(x, y) \leq \beta(x, y) \Rightarrow \beta^{\backprime} \subseteq \beta$, then $\beta^{\backprime} \in K$. Now, by Definition 3.5$(U_3)$, we have $\beta^{\backprime} \circ \beta^{\backprime} \subseteq \beta$

$$\Rightarrow \beta^{\backprime} \circ \beta^{\backprime}(x, y) \leq \beta(x, y)$$
$$\Rightarrow \sup_{z \in X}\{min\{\beta^{\backprime}(x, z), \beta^{\backprime}(z, y)\}\} \leq \beta(x, y).$$

Then there exist $\beta^\star \in \Phi$ such that $\beta^\star(x, y) \leq \beta^{\backprime}(x, y) \Rightarrow \beta^\star \subseteq \beta^{\backprime}$.

Which implies that $\beta^\star \circ \beta^\star(x, y) = \sup_{z \in X}\{min\{\beta^\star(x, z), \beta^\star(z, y)\}\} \leq \sup_{z \in X}\{min\{\beta^{\backprime}(x, z), \beta^{\backprime}(z, y)\}\} \leq \beta(x, y) \Rightarrow \beta^\star \circ \beta^\star \subseteq \beta$.

Hence, for $\beta \in \Phi$ there exist $\beta^\star \in \Phi$ such that $\beta^\star \circ \beta^\star \subseteq \beta$.

Conversely, suppose $\Phi$ is a base for some fuzzy uniformity that satisfies conditions (1) through (4), and define $K = \{\psi : \beta \subseteq \psi,\ \text{for some } \beta \in \Phi\}$. Then we need to show that $K$ is a fuzzy uniformity on $X$.

$U_1$. Let $\psi \in K$, then there exist $\beta \in \Phi$ such that $\beta(x, y) \leq \psi(x, y)$. From condition (1), we have $1 = \beta(x, x) \leq \psi(x, x) \Rightarrow \psi(x, x) = 1$.

$U_2$. Let $\psi \in K$, then there exist $\beta \in \Phi$ such that $\beta(x, y) \leq \psi(x, y)$

$$\Rightarrow \beta^{-1}(y, x) = \beta(x, y) \leq \psi(x, y) = \psi^{-1}(y, x)$$

$$\Rightarrow \beta^{-1}(y, x) \leq \psi^{-1}(y, x)$$

$$\Rightarrow \beta^{-1} \subseteq \psi^{-1}.$$

From condition (2), there exist $\beta^\star \in \Phi$ such that $\beta^\star(y, x) \leq \beta^{-1}(y, x)$ implies $\beta^\star(y, x) \leq \beta^{-1}(y, x) \leq \psi^{-1}(y, x) \Rightarrow \beta^\star \subseteq \psi^{-1} \Rightarrow \psi^{-1} \in K$.

$U_3$. Let $\psi, \phi \in K$, then there exist $\beta_1, \beta_2 \in \Phi$ such that $\beta_1(x, y) \leq \psi(x, y)$ and $\beta_2(x, y) \leq \phi(x, y)$.
$\Rightarrow min\{\beta_1(x, y), \beta_2(x, y)\} \leq min\{\psi(x, y), \phi(x, y)\} \Rightarrow \beta_1 \cap \beta_2 \subseteq \psi \cap \phi$.
From condition (4), there exist $\beta \in \Phi$ such that

$$\begin{aligned}
\beta(x, y) &\leq min\{\beta_1(x, y), \beta_2(x, y)\} \\
&\leq min\{\psi(x, y), \phi(x, y)\} \\
\Rightarrow \beta(x, y) &\leq min\{\psi(x, y), \phi(x, y)\} \\
&\Rightarrow \beta \subseteq \psi \cap \phi \\
&\Rightarrow \psi \cap \phi \in K.
\end{aligned}$$

$U_4$. Let $\psi \in K$, then there exist $\beta \in \Phi$ such that $\beta(x, y) \leq \psi(x, y)$. From condition (3), there exist $\beta^{\backprime} \in \Phi$ such that $\beta^{\backprime} \circ \beta^{\backprime}(x, y) = \sup_{z \in X}\{min\{\beta^{\backprime}(x, z), \beta^{\backprime}(z, y)\}\} \leq \beta(x, y) \leq \psi(x, y) \Rightarrow \beta^{\backprime} \circ \beta^{\backprime}(x, y) \leq \psi(x, y) \Rightarrow \beta^{\backprime} \circ \beta^{\backprime} \subseteq \psi$, such that there exist $\beta^{\backprime} \in \Phi \Rightarrow \beta^{\backprime} \in K$. Therefore, $\psi \in K$, there exist $\beta^{\backprime} \in K$ such that $\beta^{\backprime} \circ \beta^{\backprime} \subseteq \psi$.

$U_5$. Let $\psi \in K$ and $\psi \subseteq \phi$. There exist $\beta \in \Phi$ such that $\beta \subseteq \psi \subseteq \phi \Rightarrow \beta \subseteq \phi \Rightarrow \phi \in K$. By definition of fuzzy uniformity, $K = \{\psi : \beta \subseteq \psi,\ \text{for some } \beta \in \Phi\}$ is a fuzzy uniform space on $X$. □

**Definition 3.12.** Let $(X, K)$ be a fuzzy uniform space. A sub family of fuzzy relation $\mathfrak{S}$ of $K$ is called a subbase for $K$ if all finite intersection member of $\mathfrak{S}$ forms a base for a fuzzy uniformity $K$.

**Proposition 3.13.** *Let $X$ be a non-empty pseudo-UP algebra. Then a non-empty class fuzzy relation $\mathfrak{S}$ on $X \times X$ is a subbase for some fuzzy uniformity on $X$ if the following conditions hold:*

1. *For all $S \in \mathfrak{S}, S(x, x) = 1,$ for all $x \in X$*
2. *For all $S \in \mathfrak{S} \Rightarrow S^{-1}$ contains some elements of $\mathfrak{S}$,*
3. *For all $S \in \mathfrak{S}$, there exist $S^\star \in \mathfrak{S}$ such that $S^\star \circ S^\star \subseteq S$.*

*Proof.* Let $(X, K)$ be a fuzzy uniform space. Let $\mathfrak{S}$ be a non-empty class of fuzzy relation on $X \times X$ that satisfies conditions (1)through (3).
Define: $\Phi = \inf\{S_i, S_i \in \mathfrak{S}$, for each i=1,2,3,$\cdot\cdot\cdot$,n$\}$. Now $\mathfrak{S}$ to be a subbase for fuzzy uniformity, $\Phi$ must be a base for a fuzzy uniformity. Now we need to prove that $\Phi$ is a base for a fuzzy uniform space $(X, K)$.
1. Let $\psi \in \Phi$. Define $\psi(x, y) = \inf\{S_i(x, y)$, where $S_i \in \mathfrak{S}$ and for each $i = 1, 2, 3, \cdots, n\}$. From condition (1), we have $S_i(x, x) = 1$ which implies that $\inf\{S_i(x, x)\} = 1 \Rightarrow \psi(x, x) = 1$. Thus $\psi \in \Phi, \psi(x, x) = 1$.
2. Let $\psi \in \Phi$. Define $\psi(x, y) = \inf\{S_i(x, y)$ where $S_i \in \mathfrak{S}$, for each $i = 1, 2, 3, \cdots, n\}$. From condtion (2), we have $S_i \in \mathfrak{S}$ implies that $S_i^{-1}$ contains some elements of $\mathfrak{S}$. Then there exist $\lambda_i \in \mathfrak{S}$ such that $\lambda_i \subseteq S_i^{-1}$, for each $i = 1, 2, 3, \cdots, n$.. It is clear that $\inf\{\lambda_i(x, y)\} \leq \inf\{S_i^{-1}(x, y)\}$. Now we need to show that $\psi^{-1}(x, y) = \inf\{S_i^{-1}(x, y)\}$, for each $i$. Now

$$\begin{aligned}
\psi^{-1}(x, y) &= \{\psi(y, x) : (x, y) \in X \times X\} \\
&= \{\inf\{S_i(y, x)\} : (x, y) \in X \times X, \text{ for each } i\} \\
&= \{\inf\{S_i^{-1}(x, y)\} : (x, y) \in X \times X\} \\
\Rightarrow \psi^{-1}(x, y) &= \inf\{S_i^{-1}(x, y)\}.
\end{aligned}$$

Let $\psi^\star \in \Phi$ and let $\inf\{\lambda_i(x, y)\} = \psi^\star(x, y)$ such that $\psi^\star(x, y) = \inf\{\lambda_i(x, y)\} \leq \inf\{S_i^{-1}(x, y)\} = \psi^{-1}(x, y)$. Which implies that $\psi^\star \subseteq \psi^{-1}$.
Hence, $\psi^{-1}$ contains some elements of $\Phi$.
3. Let $\psi \in \Phi$. Define $\psi(x, y) = \inf\{S_i(x, y), S_i \in \mathfrak{S}\}$, for some $i$, from condition (3), then there exist $\lambda_i \in \mathfrak{S}$ such that $\lambda_i \circ \lambda_i \subseteq S_i$, for each $i = 1, 2, 3, \cdots, n$ implies that $\inf\{(\lambda_i \circ \lambda_i)(x, y)\} \leq \inf\{\lambda_i(x, y)\} \circ \inf\{\lambda_i(x, y)\} \leq \inf\{S_i(x, y)\}$, for some $i$. Let $\psi^\star \in \Phi$ and let $\inf\{\lambda_i(x, y)\} = \psi^\star(x, y)$. Hence, $\psi^\star \circ \psi^\star \subseteq \psi$. For all $\psi \in \Phi$, there exist $\psi^\star$ such that $\psi^\star \circ \psi^\star \subseteq \psi$.
4. Let $\psi, \psi^* \in \Phi$. Define $\psi(x, y) = \inf\{S_i(x, y)$, for each $i = 1, 2, 3, \cdot\cdot\cdot, n\}$ and $\psi^*(x, y) = \inf\{S_i(x, y)$, for each $i = 1, 2, 3, \cdot\cdot\cdot, k\}$. Now, $min\{\psi(x, y), \psi^*(x, y)\} = min\{\inf\{S_i(x, y)\}, \inf\{S_i(x, y)\}\} = \inf\{S_i(x, y)$, for each $i = 1, 2, 3, \cdots, t\}$, where $t = min\{k, n\}$. Which implies that $\psi \cap \psi^*$ contains it self where $\psi \cap \psi^* \in \Phi$. Thus by Theorem 3.11 $\Phi$ is a base for a fuzzy uniform space $(X, K)$. Therefore, $\mathfrak{S}$ is a subbase for a fuzzy uniform space. □

**Theorem 3.14.** *Let $\omega = \{\beta_\mu : \mu \in \aleph\}$, then $\omega$ is a subbase for a fuzzy uniformity.*

*Proof.* $\omega$ satisfies all conditoins of Proposition 3.13 becuase $\psi_\mu$ is a fuzzy equvalance relation and by Theorem 3.6. So it is easy to show $\omega$ is a subabse for a fuzzy uniformity. □

**Lemma 3.15.** *Let $\psi$ and $\phi$ be fuzzy relations. If $\psi \subseteq \phi$, then $\psi^{[x]} \subseteq \phi^{[x]}$, for $x \in X$.*

*Proof.* Suppose that $\psi$, $\phi$ are fuzzy relations, and $\psi \subseteq \phi$. We claim that $\psi^{[x]} \subseteq \phi^{[x]}$. Now, there exist $b \in X$ such that $\psi^{[x]}(b) = \psi(x, b) \leq \phi(x, b) \Rightarrow \psi^{[x]}(b) \leq \phi^{[x]}(b)$. Which implies that $\psi^{[x]} \subseteq \phi^{[x]}$. □

**Lemma 3.16.** *Let $\mu, \lambda \in \aleph$. If $\lambda \subseteq \mu$, then $\psi_\lambda \subseteq \psi_\mu$.*

*Proof.* Suppose $\lambda \subseteq \mu$ that is $\lambda(x) \leq \mu(x)$.
Now, $\psi_\lambda(x, y) = min\{\lambda(x \cdot y), \lambda(y \cdot x), \lambda(x \star y), \lambda(y \star x)\} \leq min\{\mu(x \cdot y), \mu(y \cdot x), \mu(x \star y), \mu(y \star x)\} = \psi_\mu(x, y)$.
Hence, $\psi_\lambda \subseteq \psi_\mu$. □

**Theorem 3.17.** *Let $(X, K)$ be a fuzzy uniform structure, then $\tau = \{G \subseteq I^X : G(x) = 1,$ there exist $\psi \in K, \psi^{[x]} \subseteq G\}$ is a fuzzy topology on a pseudo-UP algebra $X$.*

*Proof.* Define $\tau = \{G \subseteq I^X : G(x) = 1$, there exist $\psi \in K, \psi^{[x]} \subseteq G\}$. We need to show that $\tau$ is a fuzzy topology on $X$. Let $(X, K)$ be a fuzzy uniform stucture, there exist $\psi \in K, 1_X(x) = 1$, forall $x \in X$ then $\psi^{[x]} \subseteq 1_X$. Thus $1_X \in \tau$.

Suppose $0 \in K$ and we define $0^{[x]}(y) = 0(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$

A fuzzy subset in $X$ is empty iff its membership function is identically zero on $X$ and it is denoted by $0_X$. And we define the fuzzy sets $0_X(x) = \begin{cases} 0, & \text{if } x \in X \\ 1, & \text{if } x \notin X. \end{cases}$. Then which implies that $0^{[x]} \subseteq 0_X$ for all $x \notin X$. Thus, $0_X \in \tau$.

Let $\{G_{i,\ i \in I}\} \subseteq \tau$. We need to show that $\cup_{i \in I} G_i \in \tau$. By the definition of $\tau$, for each $G_i \in \tau$, there exist $\psi_i \in K$ such that $\psi_i^{[x]} \subseteq G_i$. This implies that $\psi_i^{[x]} \subseteq \cup_{i \in I} G_i$. Hence, $\cup_{i \in I} G_i \in \tau$.
Let $G_1, G_2 \in \tau$. By definition there exist $\psi_1, \psi_2 \in K$ such that $\psi_1^{[x]} \subseteq G_1$ and $\psi_2^{[x]} \subseteq G_2$, for all $x \in X$. Consider $G = G_1 \cap G_2$, then $G(x) = min\{G_1(x), G_2(x)\}$. We need to show that there exist $\psi \in K$ such that $\psi^{[x]} \subseteq G$. Since $K$ is a fuzzy uniformity, it is closed under finite intersections. Therefore, $\psi = \psi_1 \cap \psi_2$. For each $x \in X, \psi^{[x]} \subseteq \psi_1^{[x]} \cap \psi_2^{[x]} \subseteq G_1 \cap G_2 = G \Rightarrow G \in \tau$. Therefore, $\tau$ is closed under finite intersection. Hence, $\tau$ is a fuzzy topology on $X$. □

**Theorem 3.18.** *Let $\Phi$ be a base for the fuzzy uniformity $K$ and $x \in X$. Then $\{\phi^{[x]} : \phi \in \Phi\}$ is a base for the fuzzy topology $\tau$.*

*Proof.* Let $(X, K)$ be a fuzzy uniform space. Let $\tau = \{G \subseteq I^X : G(x) = 1,$ there exist $\psi \in K, \psi^{[x]} \subseteq G\}$ be a fuzzy topology on $X$. Let $\Phi$ be a base for fuzzy uniformity $K$.
Consider $G_B = \{\phi^{[x]} : \phi \in \Phi\}$. We need to show that $G_B$ is a base for a fuzzy topological space. By Theorem 3.17 $G(x) = 1$, then there exist $\psi \in K$ such that $\psi^{[x]} \subseteq G$. Since $\Phi$ is a base for fuzzy uniform space, then there exist $\phi \in \Phi$ such that $\phi \subseteq \psi$. By Lemma 3.15, we have, $\phi^{[x]} \subseteq \psi^{[x]} \Rightarrow \phi^{[x]} \subseteq \psi^{[x]} \subseteq G$. Which implies that $\phi^{[x]} \subseteq G$. For all $G \in \tau$, there exist $\phi^{[x]} \in G_B$ such that $\phi^{[x]} \subseteq G$. By Definition 3.10, $G_B$ is a base for a fuzzy topology $\tau$ on $X$. □

# 4 Fuzzy Uniform Topological Properties of the space $(X, \tau_\aleph)$

Note that from Theorem 3.17 the family $\aleph$ of extreme fuzzy pseudo-UP ideals of the pseudo-UP algebra $X$ is closed under intersection. This allows us to induce a fuzzy uniform topology $\tau_\aleph$ on $X$. In this section, we explore the fuzzy topological properties on $(X, \tau_\aleph)$. Let $X$ be a pseudo-UP algebra and suppose $\psi_\mu^{[x]}$, $\psi_\lambda^{[y]}$ are belongs to $\tau_\aleph$, then we define

$$(\psi_\mu^{[x]} \cdot \psi_\lambda^{[y]})(r) = \sup_{r=a\cdot b}\{min\{\psi_\mu(x,a), \psi_\lambda(y,b)\}\} \text{ and } (\psi_\mu^{[x]} \star \psi_\lambda^{[y]})(t) = \sup_{t=a\star b}\{min\{\psi_\mu(x,a), \psi_\lambda(y,b)\}\}.$$

**Definition 4.1.** A pseudo-UP algebra $X$ equipped with a fuzzy topology $\tau$ is called a fuzzy topological pseudo-UP algebra if for each an open fuzzy set $G$ neighborhood of $x \cdot y$, and open fuzzy set $H$ neighborhood of $x \star y$, then there exist two open fuzzy sets $U, V$ neighborhoods of $x$ and $y$, respectively such that $U \cdot V \subseteq G$, $U \star V \subseteq H$.

**Definition 4.2.** If $(X, K)$ is a fuzzy uniform space, then the fuzzy topology $\tau$ is called fuzzy uniform topology on $X$ induced by $K$.

**Theorem 4.3.** *Let $(X, \tau_\aleph)$ be a fuzzy uniform topological space. Then $\mathfrak{B}_\aleph = \{\psi_\mu^{[x]} : x \in X\}$ is a base for $\tau_\aleph$.*

*Proof.* Let $(X, \tau_\aleph)$ be a fuzzy uniform topological space. For $\psi \in K$, and for all $x \in X$, we get $\psi_\mu^{[x]} \subseteq \psi^{[x]}$. Then $\tau_\aleph = \{G \subseteq I^X : G(x) = 1 \ \exists \mu \in \aleph$ such that $\psi_\mu^{[x]} \subseteq G\}$. Then it is easy show that $\mathfrak{B}_\aleph$ is a base for $\tau_\aleph$. That is, for all $G \in \tau_\aleph$, there exist $\psi_\mu^{[x]} \in \mathfrak{B}_\aleph$ such that $\psi_\mu^{[x]} \subseteq G$. By Definition 3.10, $\mathfrak{B}_\aleph$ is a base for a fuzzy uniform topology on $X$. □

**Theorem 4.4.** *In a pseudo-UP algebra $X$, let $\tau_\aleph$ be a fuzzy uniform topology induced by $K$. Then $(X, \tau_\aleph)$ is a fuzzy topological pseudo-UP algebra.*

*Proof.* Define $K = \{\psi : \psi_\mu \subseteq \psi$, for some $\psi_\mu \in K^\star\}$. For all $\psi \in K, \mu \in \aleph$, and for all $x \in X$ we get $\psi_\mu^{[x]} \subseteq \psi^{[x]}$ for some $\psi_\mu \in K^\star$. Then $\tau_\aleph = \{G \subseteq I^X : G(x) = 1, \exists \mu \in \aleph$ such that $\psi_\mu^{[x]} \subseteq G\}$ be a fuzzy uniform topology on $X$, and $G$ and $H$ are fuzzy open set of $\tau_\aleph$ such that $G(x \cdot y) = 1$ and $H(x \star y) = 1$. Which implies that there exist $\psi \in K$ such that $\psi^{[x\cdot y]} \subseteq G$ and $\psi^{[x\star y]} \subseteq H$. Since $\mu$ is an extreme fuzzy pseudo-UP ideal such that $\psi_\mu \subseteq \psi$.
We claim that $\psi_\mu^{[x]} \cdot \psi_\mu^{[y]} \subseteq G$ and $\psi_\mu^{[x]} \star \psi_\mu^{[y]} \subseteq H$. Now

$$\begin{aligned}
(\psi_\mu^{[x]} \cdot \psi_\mu^{[y]})(r) &= \sup_{r=a\cdot b}\{\min\{\psi_\mu^{[x]}(a), \psi_\mu^{[y]}(b)\}\} \\
&= \sup_{r=a\cdot b}\{min\{\psi_\mu(x,a), \psi_\mu(y,b)\}\} \\
&\leq \sup_{r=a\cdot b}\{min\{\psi(x,a), \psi(y,b)\}\}, \text{since } \psi_\mu \subseteq \psi \\
&\leq \sup_{r=a\cdot b}\{\psi(x\cdot y, a\cdot b)\}, \text{ by Theorem 2.5} \\
&= \sup_{r=a\cdot b}\{\psi^{[x\cdot y]}(a\cdot b)\}
\end{aligned}$$

$$\Rightarrow \sup_{r=a\cdot b}\{\psi^{[x\cdot y]}(a\cdot b)\} \leq G(a\cdot b).$$

Thus, $\psi_\mu^{[x]} \cdot \psi_\mu^{[y]} \subseteq G$.

$$\begin{aligned}
(\psi_\mu^{[x]} \star \psi_\mu^{[y]})(t) &= \sup_{t=a\star b}\{\min\{\psi_\mu^{[x]}(a), \psi_\mu^{[y]}(b)\}\} \\
&= \sup_{t=a\star b}\{min\{\psi_\mu(x,a), \psi_\mu(y,b)\}\} \\
&\leq \sup_{t=a\star b}\{\min\{\psi_\mu(x\star y, a\star b)\}\}, \text{ since } \psi_\mu \subseteq \psi \\
&\leq \sup_{t=a\star b}\{\psi(x\star y, a\star b)\}, \text{by Theorem 2.5} \\
&= \sup_{t=a\star b}\{\psi^{[x\star y]}(a\star b)\}
\end{aligned}$$

$$\Rightarrow \sup_{t=a\star b}\{\psi^{[x\star y]}(a\star b)\} \leq H(a\star b).$$

Thus, $\psi_\mu^{[x]} \star \psi_\mu^{[y]} \subseteq H$.
Hence, $(X, \tau_\aleph)$ is a fuzzy topological pseudo-UP algebra. □

**Theorem 4.5.** *Let $\mu,\ \lambda \in \aleph$. If $\lambda \subseteq \mu$, then $\tau_\mu \subseteq \tau_\lambda$.*

*Proof.* Suppose $\aleph = \{\mu\}$, $K^\star = \{\psi_\mu\}$ and $K = \{\psi : \psi_\mu \subseteq \psi$, for some $\psi_\mu \in K^\star\}$. Let $O \in \tau_\mu$. Then for all $x \in X$ such that $O(x) = 1$, there exist $\psi \in K$ such that $\psi^{[x]} \subseteq O$ and so $\psi_\mu^{[x]} \subseteq \psi^{[x]} \subseteq O$. By Lemma 3.16 $\psi_\lambda \subseteq \psi_\mu$. It follows that $\psi_\lambda^{[x]} \subseteq \psi_\mu^{[x]} \subseteq O \Rightarrow O \in \tau_\lambda$. Hence, $\tau_\mu \subseteq \tau_\lambda$. □

**Theorem 4.6.** *Let $\aleph$ be a family of extreme fuzzy pseudo-UP ideals of a pseudo-UP algebra $X$ which is closed under intersection. If $\lambda = \cap\{\mu : \mu \in \aleph\}$, then $\tau_\aleph = \tau_\lambda$.*

*Proof.* Let $K$ and $K^\star$ defined as in Definition 3.4. Now consider $\aleph_1 = \{\lambda\}$ and define $K_1 = \{\psi : \psi_\lambda \subseteq \psi\}$ and $K_1^\star = \{\psi_\lambda\}$. Suppose that $O \in \tau_\aleph$. Then $O(x) = 1$ there exist a fuzzy relation $\psi \in K$ such that $\psi^{[x]} \subseteq O$. Since $\psi \in K$ there exist $\mu \in \aleph$ such that $\psi_\mu^{[x]} \subseteq \psi$. Since $\lambda \subseteq \mu$ by Lemma 3.16 $\psi_\lambda \subseteq \psi_\mu$. Hence, $\psi_\lambda^{[x]} \subseteq \psi_\mu^{[x]} \subseteq O$, which implies that $O \in \tau_\lambda$. Therefore, $\tau_\aleph \subseteq \tau_\lambda$.
Conversely, let $O \in \tau_\lambda$. Then for all $x \in X$ such that $O(x) = 1$, there exist $\psi \in K_1$ such that $\psi^{[x]} \subseteq O$. So, $\psi_\lambda^{[x]} \subseteq \psi^{[x]}$. Since $\aleph$ is closed under intersection, then $\lambda \in \aleph$. Then we obtain $\psi_\lambda \in K$ and $O \in \tau_\aleph \Rightarrow \tau_\lambda \subseteq \tau_\aleph$. Hence, $\tau_\lambda = \tau_\aleph$. □

**Definition 4.7.** Let $(X, K)$ be a fuzzy uniform structure is called fuzzy compact set if every fuzzy open cover has a finite fuzzy sub cover.

**Theorem 4.8.** *Let $\mu$ be an extreme fuzzy pseudo-UP ideal of $X$, then $\mu$ is fuzzy compact in $(X, \tau_\mu)$.*

*Proof.* For $\psi \in K, \mu \in \aleph$, for all $x \in X$ such that $\psi_\mu^{[x]} \subseteq \psi^{[x]}$, then we get $\tau_\mu = \{O : O(x) = 1, \psi_\mu^{[x]} \subseteq O\}$. Suppose that $\psi_\mu^{[x]} \subseteq \cup_{i\in\Lambda} O_i$, where $O_i$ is fuzzy open set in $\tau_\mu$, for each $i \in \Lambda$. Since $\mu(0) = 1$ and there exist $i \in \Lambda$ such that $O_i(0) = 1$. Then an extreme fuzzy pseudo-UP ideal $\mu(0) = \psi_\mu^{[0]}(0)$. Which implies that $\mu = \psi_\mu^{[0]} \subseteq O_i$. Since $\mu$ is entirely contained in one of the fuzzy open set $O_i$. Hence, $\mu$ is a fuzzy compact of $(X, \tau_\mu)$. □

**Theorem 4.9.** *Let $\aleph$ be a family of extreme fuzzy pseudo-UP ideals of a pseudo-UP algebra $X$ which is closed under intersection. If $\lambda = \cap\{\mu : \mu \in \aleph\}$ then for all $x \in X$, $\psi_\lambda^{[x]}$ is fuzzy compact in $(X, \tau_\aleph)$.*

*Proof.* Suppose $\lambda = \cap\{\mu : \mu \in \aleph\}$. Let $\psi_\lambda^{[x]} \subseteq \cup_{i\in\Lambda} O_i$, where $O_i$ is a fuzzy open set of $\tau_\aleph$ and $\Lambda$ is any index set. For $x \in X$ such that $O_i(x) = 1$. Then $\psi_\lambda^{[x]} \subseteq O_i$. Hence, $\psi_\lambda^{[x]}$ is a fuzzy compact in $(X, \tau_\aleph)$. □

**Definition 4.10.** Let $(X, \tau_\aleph)$ be a fuzzy topological space where $\aleph$ is a family of an extreme fuzzy pseudo-UP ideals of $X$ and $\mu \in \aleph$. Then for any fuzzy subset $\lambda$ of $X$, $cl(\lambda)(x) = \inf\{\psi_\mu^{[\lambda]}(x) : \psi_\mu \in K^\star\}$.

**Theorem 4.11.** *Let $f : X \to Y$ be a pseudo-UP homomorphism between two pseudo-UP algebras of $X$ and $Y$ and let $\mu$ be an extreme fuzzy pseudo-UP ideal of $Y$, then for $x_1, x_2$*

$$\psi_{f^{-1}(\mu)}(x_1, x_2) = \psi_\mu(f(x_1), f(x_2)).$$

*Proof.* For all $x_1, x_2 \in X$, we have

$$\begin{aligned}\psi_{f^{-1}(\mu)}(x_1, x_2) &= min\{f^{-1}(\mu)(x_1 \cdot x_2), f^{-1}(\mu)(x_2 \cdot x_1), f^{-1}(\mu)(x_1 \star x_2), f^{-1}(\mu)(x_2 \star x_1)\}\\ &= min\{\mu(f(x_1 \cdot x_2)), \mu(f(x_2 \cdot x_1)), \mu(f(x_1 \star x_2)), \mu(f(x_2 \star x_1))\}\\ &= min\{\mu(f(x_1) \cdot f(x_2)), \mu(f(x_2) \cdot f(x_1)), \mu(f(x_1) \star f(x_2)), \mu(f(x_2) \star f(x_1))\}\\ &= \psi_\mu(f(x_1), f(x_2)).\end{aligned}$$

Hence, $\psi_{f^{-1}(\mu)}(x_1, x_2) = \psi_\mu(f(x_1), f(x_2))$. □

**Theorem 4.12.** *Let $f : X \to Y$ be a pseudo-UP isomorphism between two pseudo-UP algebras $X, Y$ and let $\mu$ be an extreme fuzzy pseudo-UP ideal in $Y$. Then the following statements hold, for all $x \in Y$ and $y \in Y$.*

1) $f(\psi_{f^{-1}(\mu)}^{[x]}) \geq \psi_\mu^{[f(x)]}$.
2) $f^{-1}(\psi_\mu^{[y]}) = \psi_{f^{-1}(\mu)}^{[f^{-1}(y)]}$.

*Proof.* 1) Let $y \in Y$, then there exist $x_1 \in X$ such that $f(x_1) = y$. Now

$$\begin{aligned}f(\psi_{f^{-1}(\mu)}^{[x]})(y) &= \sup_{x_1\in f(y)} \{\psi_{f^{-1}(\mu)}^{[x]}(x_1), f^{-1}(y) \neq \emptyset\}\\ &= \sup_{x_1\in f^{-1}(y)} \{\psi_{f^{-1}(\mu)}(x, x_1)\}\\ &= \sup_{x_1\in f^{-1}(y)} \{min\{f^{-1}(\mu)(x \cdot x_1), f^{-1}(\mu)(x \star x_1), f^{-1}(\mu)(x_1 \cdot x), f^{-1}(\mu)(x_1 \star x)\}\}\\ &= \sup_{x_1\in f^{-1}(y)} \{min\{\mu(f(x \cdot x_1)), \mu(f(x \star x_1)), \mu(f(x_1 \cdot x)), \mu(f(x_1 \star x))\}\}\\ &= \sup_{x_1\in f^{-1}(y)} \{min\{\mu(f(x) \cdot f(x_1)), \mu(f(x) \star f(x_1)), \mu(f(x_1) \cdot f(x)), \mu(f(x_1) \star f(x))\}\}\\ &= \sup_{x_1\in f^{-1}(y)} \{min\{\mu(f(x) \cdot y), \mu(f(x) \star y), \mu(y \cdot f(x)), \mu(y \star f(x))\}\}\\ &= \sup\{\psi_\mu^{[f(x)]}(y)\}\\ &\geq \psi_\mu^{[f(x)]}(y).\end{aligned}$$

$$\Rightarrow f(\psi_{f^{-1}(\mu)}^{[x]})(y) \geq \psi_\mu^{[f(x)]}(y)$$

$\Rightarrow \psi_{\mu}^{[f(x)]} \subseteq f(\psi_{f^{-1}(\mu)}^{[x]})$.

2) We need to show that $f^{-1}(\psi_{\mu}^{[y]}) = \psi_{f^{-1}(\mu)}^{[f^{-1}(y)]}$.

$$\begin{aligned} f^{-1}(\psi_{\mu})^{[y]}(x) &= \psi_{\mu}^{[y]}(f(x)) \\ &= \psi_{\mu}(f(x), y) \\ &= \psi_{f^{-1}(\mu)}(f^{-1}(f(x), f^{-1}(y))), \text{ by Theorem 4.11} \\ &= \psi_{f^{-1}(\mu)}(x, f^{-1}(y)) \\ &= \psi_{f^{-1}(\mu)}^{[f^{-1}(y)]}(x). \end{aligned}$$

Therefore, $f^{-1}(\psi_{\mu}^{[y]}) = \psi_{f^{-1}(\mu)}^{[f^{-1}(y)]}$. □

# 5 Conclusion remarks

In this work, we introduced the concept of a fuzzy uniform space on pseudo-UP algebras and developed the notion of a fuzzy uniform topological pseudo-UP algebra. We explored the conditions under which a pseudo-UP algebra, equipped with a specific fuzzy topology, ensures the topological fuzzy continuity of its binary operations. This work extends the theory of fuzzy uniform topological pseudo-UP algebras, leading to the derivation of several important properties of fuzzy topological pseudo-UP algebras. Future research could focus on further generalizations of fuzzy topological pseudo-UP algebras, exploring additional properties and applications in various fields.

# References


[1] A.A. Mechdesro, B.A. Alaba, T.M. Munie and T.G. Alemayehu, *Fuzzy congruence relation on pseudo-UP algebra*, Research in Mathematics, **10(1)**, 2235843, (2023).

[2] A.A. Mechdesro, B.A. Alaba, T.M. Munie and T.G. Alemayehu, *Fuzzy pseudo-UP ideal of pseudo-UP algebra*, Research in Mathematics, **10(1)**, 2234217, (2023).

[3] M.A. Yousef, A.B. Khalaf, *Topological pseudo-UP algebras*, JMCS, **26(1)**, 80–89, (2022).

[4] R. Lowen, *Fuzzy uniform spaces*, J. Math. Anal. Appl., **82(2)**, 370–385, (1981).

[5] L.A. Zadeh, *Fuzzy sets*, Information and control, **8(3)**, 338–353, (1965).

[6] A. Weil, *Sur les espaces à structure uniforme et sur la topologie générale*, (1938)

[7] S. Mehrshad, J. Golzarpoor, *On topological BE-algebras*, Mathematica Moravica, **21(2)**, 1–13, (2017).

[8] A. Iampan, *A new branch of the logical algebra: UP-algebras*, JART, **5(1)**, 35–54, (2017).

[9] C. Prabpayak, U. Leerawat, *On ideals and congruences in KU-algebras*, Sci. Magna., **5(1)**, 54–57, (2009).

[10] D.A. Romano, *Pseudo-UP algebras*, An introduction. Bull. Int. Math. Virtual Inst, **10(2)**, 349–355, (2020).

[11] D.A. Romano, *Pseudo-UP ideals and pseudo-UP filters in pseudo-UP algebras*, Mathematical Sciences and Applications E-Notes, **8(1)**, 155-158, (2020).

[12] R.H. Warren, *Continuity of mappings of fuzzy topological spaces*, Notices Amer. Math. Soc, **21**, (A-451), (1974).

[13] A. Iampan, A. Satirad, R. Chinram, R. Prasertpong and P. Julatha, *Lattice Valued Fuzzy Sets in UP (BCC)-Algebras*, Int. J. Anal. Appl., **21**, 59–59, (2023).

[14] G. Muhiuddin, M. Al-Tahan, A. Mahboob, S. Hoskova-Mayerova and S. Al-Kaseasbeh, *Linear Diophantine fuzzy set theory applied to BCK/BCI-Algebras*, Mathematics, **10(12)**, 2138, (2022).

[15] M. Al-Tahan, A. Rezaei, S. Al-Kaseasbeh, B. Davvaz, and M. Riaz,*Linear Diophantine fuzzy $n$-fold weak subalgebras of a BE-algebra*, MJMS, **35(2)**, 136–148, (2023).

[16] Y.B. Jun, B. Brundha, N. Rajesh and R.K. Bandaru, *(3, 2)-Fuzzy UP (BCC)-Subalgebras and (3, 2)-Fuzzy UP (BCC)- Filters,*, J. Mahani Math. Res. **11**, 1—14, (2022).

[17] A.K.Rath and A.S. Ranadive, *On $(\epsilon, \epsilon\upsilon q)$-fuzzy prime and $(\epsilon, \epsilon, \upsilon q)$-fuzzy primary submodules*. PJM, **13(3)** (2024).

[18] V.L.Fotea, B. Davvaz and A. Sonea, *A SURVEY ON THE FUZZY DEGREE OF A HYPERGROUP*.

[19] M. Balamurugan, G. Muhiuddin, K. Dhilipkumar, *GENERALIZATIONS OF $(\epsilon, \epsilon\upsilon\ (k^{\star}, qk))$-ANTI-FUZZY B-IDEALS OF BCI-ALGEBRAS.* PJM **13**(2024)

[20] M.M. Jansirani and S.Sujatha, *LUKASIEWICZ FUZZY IDEALS APPLIED TO BCK/BCI-ALGEBRAS*, PJM **13**, (2024)

**Author information**

Alachew Amaneh*, Department of Mathematics, Bahir Dar University and Kebri Dehar University, Ethiopia.
E-mail: alachewamaneh@gmail.com

Berhanu Assaye, Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia.
E-mail: birhanu.assaye290113@gmail.com

Tilahun Mekonnen, Bahir Dar University, Bahir Dar, Ethiopia.
E-mail: tilametekek@yahoo.com

Received: 2024-11-22.
Accepted: 2024-12-29.