

# Classical logic, Quantum logic and their algebraic structures

Sarvesh K. Mishra, Mukesh K. Shukla and Akhilesh K. Singh

Communicated by Manoj Kumar Patel

MSC 2010 Classifications: Primary 06A11, 28A12; Secondary 28E99, 06C15.

Keywords and phrases: classical logic, quantum logic, quantum structures, effect algebras.

*The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.*

**Abstract** The article presents a systematic survey of mathematical structures, which are the axiomatic model of classical logic and quantum logic (both sharp and unsharp). We compare and discuss the mathematical structures obtained from the classical and quantum theories. Finally the algebraic structure called effect algebras is discussed, which generalize and unify all the mathematical structures obtained.

## 1 Introduction

Quantum logics are logic-algebraic structures, and their abstractions emerge in the mathematical formulation of quantum theory (QT). The QT is different in fundamental aspects from classical theory. QT appears to be governed by quantum logic, while classical theory is governed by classical logic. From contemporary developments, it is evident that elementary particles show a non-classical behaviour. These particles have either as a particle nature or as a wave nature. When we study and measure the particle properties of the object it destroys the information that was obtained from an earlier study and measurement of the wave properties of that object. With the study of the behaviour of the elementary particles it is clear that that our nature follows a different logic than the classical logic. Here we are not going to study the quantum logic and the classical logic as the logic systems. We shall review the mathematical structures obtained from the axiomatization of (both sharp and unsharp) classical and quantum theories, see ([1]-[19]). We also compare their algebraic and some of the statistical properties (for derivation of the statistical properties, see [12]). Firstly sharp, classical and quantum theories are compared and it is seen that underlying mathematical structure of the classical logic is equivalent to an orthocomplemented distributive lattice (Boolean algebras (BAs)), while the mathematical structure of the Quantum logic is equivalent to an orthocomplemented weakly modular lattice called as orthomodular lattices (OMLs). Then the unsharp, classical and quantum theories are compared and it is observed that the underlying mathematical structures of the classical logic is congruent to MV-algebra while the mathematical structure of quantum logic is congruent to QMV-algebras ([4],[7],[8],[10]). Further, it has been observed in that effect algebras unify all the above mentioned mathematical structures [12].

The paper is arranged as follows: Section 1 is the brief introduction about the mathematical structures obtained after the axiomatization of classical logic and quantum logic (both sharp and unsharp). In section 2, we have discussed about sharp classical and quantum universes and their mathematical representatives. Section 3 is about the study of unsharp classical and quantum universes and their mathematical representatives. Section 4 contains the conclusion and its pictorial representation to increase the clarity about the mathematical representatives of both the logics.  $Q$  represents the set of rational numbers throughout the paper.

## 2 Sharp classical and quantum universes

In this section sharp, classical and quantum theories are compared and their underlying mathematical structure ([4], [7], [8], [9]) are discussed.

The study of measurements is the main objective in the study of quantum theory. A perfectly accurate measurement is called sharp or crisp. Practically, a sharp measurement is impossible to achieve and hence these measurements correspond to idealizations that can only be obtained as a limiting case. Hence a measurement is not precisely accurate but it is having fuzzy or unsharp behaviour. The description of the question of quantum logic can be considered as the question of preserving the classical duality if BAs are replaced by some other algebraic structures, typically weaker algebraic structures which arises from the mathematical formulation of QT. Before discussing the mathematical structures of sharp classical and quantum universes, it is necessary to recall the following mathematical interlude towards BAs:

**Definition 2.1.** A poset (partially ordered set) is a structure  $A = (A, \leq)$ , where  $A \neq \phi$  and  $\leq$  is a partial order relation on  $A$ .

**Definition 2.2.** A bounded poset is defined as a structure  $A = (A, \leq, 0, 1)$ , where

- $A = (A, \leq)$  is a poset;
- 0 and 1 are the special elements of  $A$ : the smallest element and the largest element with respect to the partial order relation  $\leq$ .

**Definition 2.3.** A lattice is defined as a poset  $A = (A, \leq)$  wherein any two elements  $\alpha$  and  $\beta$  has a meet (infimum)  $\alpha \wedge \beta$  and join (supremum)  $\alpha \vee \beta$  in  $A$  such that

- $\alpha \wedge \beta \leq \alpha, \beta$  and  $\forall \gamma \in A : \gamma \leq \alpha, \beta$  implies  $\gamma \leq \alpha \wedge \beta$  ( $\wedge$  is inf);
- $\alpha, \beta \leq \alpha \vee \beta$  and  $\forall \gamma \in A ; \alpha, \beta \leq \gamma$  implies  $\alpha \vee \beta \leq \gamma$  ( $\vee$  is sup).

**Definition 2.4.** A bounded involution poset (lattice) is defined as a structure  $A = (A, \leq, ', 0, 1)$  where

- $A = (A, \leq, 0, 1)$  is bounded poset (lattice):  $\forall \beta \in A, 0 \leq \beta$  and  $\beta \leq 1$ ;
- $'$  is 1-ary operation (involution) such that  $\alpha'' = \alpha$  (double negation) and  $\alpha \leq \beta$  implies  $\beta' \leq \alpha'$  (contraposition).

**Definition 2.5.** An ortho-lattice is a bounded involution lattice  $A = (A, \wedge, \vee, \leq, ', 0, 1)$  such that  $\forall \alpha \in A$ , we have

- $\alpha \wedge \alpha' = 0$  (noncontradiction principle);
- $\alpha \vee \alpha' = 1$  (excluded middle principle).

**Definition 2.6.** A BA is an ortho-lattice  $A = (A, \wedge, \vee, ', 0, 1)$  such that  $\forall \alpha, \beta, \gamma \in A ; \alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$  (distributive).

**Definition 2.7.** An OML  $B$  is defined as an ortho-lattice  $B = (B, \wedge, \vee, ', 0, 1)$  such that  $\forall l, m \in B : l \leq m \Rightarrow l = (l \vee m') \wedge m$ .

Birkhoff-von Neumann (1936) [2]: made the first attempt to understand the logic of non-Boolean networks. However, since there are many non-Boolean networks, which network is said to be logically obtained? The Hilbert space  $K$  for QM plays the same role as in classical particle mechanics, the phase space. Now the question is why do not only subsets of  $H$  represent a perfect and complete mathematical representation (MR) of quantum properties, as in the case of phase space? The reason can be explained using the ‘‘superposition principle’’, which represents one of the fundamental dividing lines between QM and classical mechanics (CM). Unlike CM, in QM any unit vector, which can be represented as a linear combination (LC) of pure states, leads to a new pure state (called a superposition). Therefore, the MR of the properties of quantum physical systems must be closed for finite and infinite LC. The closed subspaces of  $K$  are just mathematical objects capable of performing that role. Therefore, the physical properties of the quantum system have been determined by the closed subspaces of  $K$ .

Why quantum properties generalize classical properties?

- CM: properties are (measurable) subsets of  $R^n$ . Subset  $Y$  are in 1 : 1 correspondence with characteristic functions  $f_Y : R^n \rightarrow \{0, 1\}$ .
- QM: properties are closed subspaces of  $K$ . Closed subspaces are in 1 : 1 correspondence with projection operators of  $K$ . The set of all eigen values of a projection operator is contained in  $\{0, 1\}$ .

**Theorem 2.8.** *If  $C(K)$  is the set of all closed subspaces of  $K$  and  $P(K)$  is the set of all projectors on  $K$ , then  $C(K) \cong P(K)$  and spectrum  $P \subset \{0, 1\}$ .*

We represent pure states as unit vectors  $\tau$  of  $K$ . Pure states correspond to maximal information about the physical system (pure states are particular examples of density operators, that is, positive, self-adjoint, trace-class linear operator of  $K$ , of trace 1). Any unit vector  $\tau$  determines a linear operator (the projection associated to the 1-dimensional subspace  $[\tau]$  corresponding to  $\tau$  and thus any pure state is represented by the projection  $P\tau$ ).

- Let  $Y$  be a closed subspace of  $K$ . By projection theorem, every vector  $\tau \in K$  can be uniquely expressed as a linear combination  $\tau_1 + \tau_2$ , where  $\tau_1 \in Y$  and  $\tau_2$  is an orthogonal to any vector of  $Y$ . Then  $\forall \tau \in K : P_Y(\tau) = \tau_1$  is a projection operator on  $K$ .
- Let  $P$  be a projection operator on  $K$ . Then  $Y_p = \{\tau : \exists \phi(P\phi = \tau)\}$  is a closed subspace of  $K$ .

The structure  $C(K)$  turns out to simulate “a quasi-Boolean behaviour” however, it is **not** a Boolean algebra. Something very essential is missing: conjunction and disjunction are not distributive. Generally,  $l \wedge (m \vee n) \neq (l \wedge m) \vee (l \wedge n)$  (the reason is that unlike the classical bivalent semantics, a quantum disjunction may be true, even if neither member is true).

- Modularity is just weakening of distributivity:  $\forall l, m, n \in B : l \leq n \Rightarrow (l \vee m) \wedge n = l \vee (m \wedge n)$ ;
- Orthomodularity is weakening of modularity:  $\forall l, m \in B : l \leq m \Rightarrow l = (l \vee m') \wedge m$ .

**Theorem 2.9.**  *$C(K) = (C(K), \wedge, \vee, ', 0, 1)$  is (non-distributive) orthomodular lattice, i.e. an orthomodular ortho-lattice where  $C(K)$  is the lattice of closed subspaces;  $\wedge$  is the set-theoretic intersection;  $\vee$  is the closure of the set-theoretic union;  $'$  is the orthogonal complement; 0 and 1 represent, respectively, the null subspace and the total space  $K$ .*

Because of the 1 : 1 correspondence between closed subspaces and projectors, the above Theorem can be rephrased in terms of projectors.

**Theorem 2.10.**  *$P(K) = (P(K), \wedge, \vee, \perp, I, o)$  is (non-distributive) orthomodular lattice, i.e. an orthomodular ortho-lattice where  $I$  and  $o$  are the identity operator and the null operator, respectively;  $P \wedge Q$  is the projector onto the closed subspace associated to the intersection of the closed subspaces uniquely associated to the projectors  $P$  and  $Q$ ;  $P \vee Q$  is the projector onto the smallest closed subspace associated to the union of the closed subspaces uniquely associated to the projectors  $P$  and  $Q$  and  $P^\perp = I - P$ .*

When we compare the algebraic structures of classical and quantum theories, we find that there are close analogies between the properties of mathematical structures of sharp, classical and quantum universes. The following table 1 (cf. [4], [9]) gives the mathematical structures of all the properties while table 2 (cf. [12]) gives statistical properties of the mathematical structures in sharp, classical and quantum universes:

Unsharp Universe		
	Classical	Quantum
Structure of all properties	MV-Algebra: $N = (N, \oplus, ', 0, 1)$ with $l \odot m = (l' \oplus m)'$ satisfying MV 1 to MV 7.	QMV-Algebra: $N = (N, \oplus, ', 0, 1)$ with $\bar{n}, \bar{m}$ satisfying $(l \oplus ((l' \bar{n} m) \cap (n \bar{m} l'))) = (l \oplus m) \bar{n} (l \oplus n)$ .
Summary	Universe $\Rightarrow$ (reduced to) $[0,1]$ Algebra $\Rightarrow$ MV (multivalued) algebra Logic $\Rightarrow$ Łukasiewicz (fuzzy) Logic $L_\infty$	Universe $\Rightarrow$ (reduced to) effects Algebra $\Rightarrow$ QMV-algebra Logic $\Rightarrow$ Łukasiewicz (quantum) Logic

Table 3

Unsharp Universe		
Statistical Properties	Classical	Quantum
Physical system	Measurable space $(\Omega, D)$	Hilbert space $K$
Event	Measurable functions $f \in E(\Omega, D)$ the set of all random variables in $[0,1]^\Omega$ (Classical effects)	$P \in E(K)$ : the set of all bounded self-adjoint operator $P$ with spectrum in $[0,1]$ (Quantum effects)
State	Probability measure $\mu$	Density operator $\tau \in K, \ \tau\  = 1$
Probability of an event	$E(f) = \int f d\mu$	$\langle D\tau, \tau \rangle$
Measurement	$\sigma$ -homomorphism $h: A(R) \rightarrow E(\Omega, D)$	$\sigma$ -homomorphism $h: A(R) \rightarrow E(K)$ such that $h(R) = 1$ and if $D_i \in A(R)$ are mutually orthogonal then $h(\cup D_i) = \sum_i h(D_i)$ (normalized positive operator-valued POV measure)

### 3 Unsharp classical and quantum universes

In this section unsharp, classical and quantum theories are compared and their underlying mathematical structure ([4], [7], [8], [9]) are discussed.

Łukasiewicz (1920) [14]: “On three-valued logic” presents a semantic description of the logic that was later named Ł3 (the three-valued logic of Łukasiewicz). All the important points regarding the semantics of Ł3 are already present and can be naturally generalized to the case of  $n$  general truth values as well as infinite values.

$$U_3 = \{0, \frac{1}{2}, 1\} \Rightarrow L^3 \text{ (3-valued logic)}$$

$$U_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, 1\} \Rightarrow L^n \text{ (n-valued logic)}$$

$$U_\infty = [0, 1] \subset Q \Rightarrow L^\infty \text{ (infinitely many valued logic)}$$

**Definition 3.1.** Let  $U = [0, 1]$ . We can define a negation and two kinds of conjunction and disjunction as following:

$$AND_1(l, m) = \min(l, m); OR_1(l, m) = \max(l, m)$$

$$AND_2(l, m) = \max(0, l + m - 1); OR_2(l, m) = \min(1, l + m)$$

(Łukasiewicz conjunction  $\odot$  and Łukasiewicz disjunction  $\oplus$ )

$$Not(l) = 1 - l: \text{ (Łukasiewicz complement)}$$

**Remark 3.2.** (i)  $U = \{0, 1\}$  then  $AND_1 = AND_2$  and  $OR_1 = OR_2$ .

(ii) The non-contradiction principle for the conjunction is generally violated!

$$AND_1(l, NOT(l)) \neq 0.$$

(iii) The excluded middle law for the disjunction is generally violated!

$$OR_1(l, NOT(l)) \neq 1.$$

**Definition 3.3.** An MV algebra is a structure  $N = (N, \oplus, ', 0, 1)$  such that once defined  $l \odot m = (l' \oplus m)'$  satisfies the following conditions for all  $l, m, n \in N$  :

- MV1:  $l \oplus m = m \oplus l$ ;
- MV2:  $l \oplus (m \oplus n) = (l \oplus m) \oplus n$ ;
- MV3:  $l \oplus l' = 1$ ;
- MV4:  $l \oplus 0 = l$ ;
- MV5:  $l \oplus 1 = 1$ ;
- MV6:  $l'' = l$ ;
- MV7:  $(l \oplus m') \odot m = (m \oplus l') \odot l$ ; (Łukasiewicz axiom).

**Theorem 3.4.** Let us consider the structure  $N_{[0,1]} = ([0, 1], \oplus, ', 0, 1)$ , where  $[0, 1] \subset Q$  and  $\forall l, m \in [0, 1] : l \oplus m = \min(l + m, 1)$  (truncated sum). Then  $N_{[0,1]} = ([0, 1], \oplus, ', 0, 1)$  is an MV algebra.

**Definition 3.5.** Let  $(N, \oplus, ', 0, 1)$  be an MV-algebra. We can define:

- $l \overline{\cap} m = (l \oplus m') \odot m$  ( $\odot$  : Łukasiewicz conjunction)
- $l \overline{\cup} m = (l \odot m') \oplus m$  ( $\oplus$  : Łukasiewicz disjunction)
- $l \rightarrow_L m = l \oplus m$  ( $\rightarrow_L$  : Łukasiewicz implication)
- $l \leq m$  iff  $l \overline{\cap} m = l$ .
- The Łukasiewicz axiom is  $l \overline{\cup} m = m \overline{\cup} l$  (or  $l \overline{\cap} m = m \overline{\cap} l$ )

In the SHARP (orthodox) approach to QM:

Any physical property P can be considered a kind of “apparent” property. Whenever a State assigns to P a probability value that is not equal to 1 or 0, the semantic uncertainty associated with such a situation depends entirely on the ambiguity of the State (first degree of ambiguity).

In the UNSHARP approach to QM:

- Any physical property P is a “genuine ambiguous” property (second degree of fuzziness). Such an ambiguity may be considered as depending on:

- (1) The accuracy of the measurement (which tests the property);
- (2) The accuracy involved in the operational definition for the physical quantities which the property refers to.

**Definition 3.6.** An effect of  $K$  is a bounded linear operator  $E$  that satisfies the following condition, for any state (=density operator)  $\rho : tr(\rho E) \in [0, 1] \Rightarrow spectrum(E) \subseteq [0, 1]$

The set of all effects  $E(K)$  of  $K$ , can be partially ordered as:  $E \leq F$  iff for all density operators  $\rho : tr(\rho E) \leq tr(\rho F)$ ;  $E(K)$  is bounded by the null operator  $o$  and the identity operator  $I$ ; The operation  $' : E(K) \rightarrow E(K)$  such that  $E' = I - E$  is an involution. Any projection satisfies the definition of effect but not conversely.

**Theorem 3.7.** *The structure  $(E(K), \leq, ', o, I)$  is a bounded involution poset, which is not a lattice. The effect poset  $E(K)$  is in fact properly fuzzy (unsharp): the non-contradiction principle is violated. e.g. the semitransparent effect  $\frac{1}{2}I$  ( $I$  is identity operator) satisfies the following condition:  $\frac{1}{2}I \wedge (\frac{1}{2}I)' = \frac{1}{2}I \wedge \frac{1}{2}I \neq 0$ .*

**Definition 3.8.** A Supplement algebra (S-algebra) is a structure  $N = (N, \oplus, ', 0, 1)$  such that  $\forall l, m, n \in N :$

- S1:  $l \oplus m = m \oplus l$ ;
- S2:  $l \oplus (m \oplus n) = (l \oplus m) \oplus n$ ;
- S3:  $l'' = l$ ;
- S4:  $l \oplus 0 = l$ ;
- S5:  $l \oplus 1 = 1$ ;

**Definition 3.9.** Let  $N = (N, \oplus, ', 0, 1)$  be an S-algebra: Define

- $l \odot m = (l' \oplus m')'$ ; (pseudo inf; generalized Sasaki projection)
- $l \bar{\cap} m = (l \oplus m') \odot m$ ; (pseudo sup)
- $l \bar{\cup} m = (l \odot m') \oplus m$ ;
- $l \leq m$  iff  $l \bar{\cap} m$ .

**Definition 3.10.** A Quantum MV algebras (QMV-algebra) is a S-algebra  $N = (N, \oplus, ', 0, 1)$  that satisfies the axiom:  $(l \oplus (l' \bar{\cap} m) \cap (n \bar{\cap} l')) = ((l \oplus m) \bar{\cap} (l \oplus n))$ . MV-algebra can also be defined as a S-algebra satisfying Łukasiewicz axiom ( $l \bar{\cup} m = m \bar{\cap} l$ ).

- QMV algebras can be thought of as non-commutative (w.r.t.  $\bar{\cap}, \bar{\cup}$ ) generalizations of MV algebras since  $\mathbf{MV}=\mathbf{QMV} + (l \bar{\cap} m = m \bar{\cap} l)$ ; (as  $\mathbf{BA}=\mathbf{OML} + (l \bar{\cap} m = m \bar{\cap} l)$ ).

**Theorem 3.11.** A QMV algebra is an OML iff  $\forall m \in M : l \oplus l = l$ .

- QMV algebras can be thought of as non-idempotent (w.r.t.  $\oplus$ ) generalizations of OMLs since  $\mathbf{OML}=\mathbf{QMV} + (l \oplus l = l)$ ; (as  $\mathbf{BA}=\mathbf{MV} + (l \oplus l = l)$ ).

Thus, a QMV-algebra generalizes both MV-algebras (classical sharp) and Orthomodular lattices (Quantum unsharp).

**Theorem 3.12.** *Let for any  $E, F \in E(K) : E \oplus F = E + F$  if  $E + F \in E(K)$ ,  $I$  otherwise, where  $+$  is usual operator sum  $E' = I - E$ . Then  $E(K) = (E(K), \oplus, ', o, I)$  is a QMV-algebra that is not a MV-algebra. The poset  $(E(K), \leq)$  is not a lattice.*

The following table 3 (cf. [4], [9]) gives the mathematical structures of all the properties of unsharp, classical and quantum universes while table 4 (cf. [12]) gives statistical properties of the mathematical structures:

<p>Distribution</p>	$\mu_h(A) = \int h(A) d\mu$ <p>(<math>A \in A(R)</math> is Borel set and <math>\mu_h: A(R) \rightarrow [0,1]</math> is the probability measure)</p>	$\tau_h(A) = \langle h(A)\tau, \tau \rangle$ <p>(<math>A \in A(R)</math> is Borel set and <math>\mu_h: A(R) \rightarrow [0,1]</math> is the probability measure)</p>
<p>Expectation</p>	$\int \lambda \mu_h(d\lambda)$	$\int \lambda \langle h(d\lambda)\tau, \tau \rangle$

Table 4

Distribution	$\mu_h(A) = \int h(A)d\mu$ <p>(<math>A \in A(R)</math> is Borel set and <math>\mu_h: A(R) \rightarrow [0,1]</math> is the probability measure)</p>	$\tau_h(A) = \langle h(A)\tau, \tau \rangle$ <p>(<math>A \in A(R)</math> is Borel set and <math>\mu_h: A(R) \rightarrow [0,1]</math> is the probability measure)</p>
Expectation	$\int \lambda \mu_h(d\lambda)$	$\int \lambda \langle h(d\lambda)\tau, \tau \rangle$

Table 4

Thus in comparison of sharp and unsharp theories, the four kinds of algebraic structures emerged as the mathematical axiomatization of the theories: Boolean Algebras, Orthomodular Lattices, MV-Algebras and QMV-Algebras. We now give an algebraic structure called, an effect algebra that generalizes and unifies all the above four algebraic structures obtained [12]. A unifying operation that appears in each of these structures is the orthogonal sum  $\oplus$ . Note that  $\oplus$  is only a partial binary operation because it is only defined for orthogonal pairs.

**Definition 3.13.** Bennett and Foulis (1994) [7]. An effect algebra is a partial structure  $B = (B, \oplus, 0, 1)$  consisting of a set  $B$ , two special elements 0 and 1, and a partially defined binary operation  $\oplus$  on  $B$  satisfying the following axioms for every  $\alpha, \beta, \gamma \in B$ ;

- L1:  $\alpha \oplus \beta = \beta \oplus \alpha$  if both sides are defined;
- L2:  $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$  if both sides are defined;
- L3: For every  $\alpha \in B$ , there exists a unique  $a \in B$  such that  $\alpha \oplus a = 1$  ( $a = \alpha^\perp$ );
- L4:  $\alpha \oplus 1$  is defined then  $\alpha = 0$ .

For  $\alpha, \beta \in B$  define a partial order relation  $\leq$  on  $B : \alpha \leq \beta \Leftrightarrow \gamma \in B : \alpha \perp \gamma, \alpha \oplus \gamma = \beta$ . Also for  $\alpha, \beta \in B, \alpha \leq \beta^\perp \Leftrightarrow \alpha \perp \beta$ ; and  $\forall \alpha \in B, 0 \leq \alpha \leq 1$ . The category of all effect algebras are turns out to be categorically equivalent to Difference Posets (Kopka and Chovanec, 1994) [13]. Every effect algebra can be uniquely transformed into a (quasi-linear) QMV algebra (whose total operation  $\oplus$ , extends to partial operation  $\oplus$  of effect algebras) and vice versa:

**Theorem 3.14.** (Gudder, 1995, Giuntini, 1995) ([10], [11]). Let  $(B, \oplus, 0, 1)$  be an effect algebra and let  $(N, \oplus, ', 0, 1)$  be a QMV algebra. Define if  $\alpha \oplus \beta = \alpha \oplus \beta$ , if  $\alpha \leq \beta^\perp$  then;

- (i)  $B$  is a (quasi-linear) QMV algebra.
- (ii)  $N$  is an effect algebra.

It is easy to check that each of the four structures that we have discussed above is an effect algebra. Strictly speaking each of these structures is closed under countable orthogonal sums  $\oplus$ . Hence effect algebras can be considered as the unifying structure.

### 4 Conclusion

Following Figure 1 (cf. [4], [9]) summarizes the relation between the mathematical structures obtained after the axiomatization of the classical logic and quantum logic (both sharp and unsharp). Boolean algebras are found to be the representative of sharp and classical universe, OMLs are found to be representative of sharp and quantum and MV-algebras are found to be representative of unsharp and classical universe while QMV-algebras are representative of unsharp and quan-

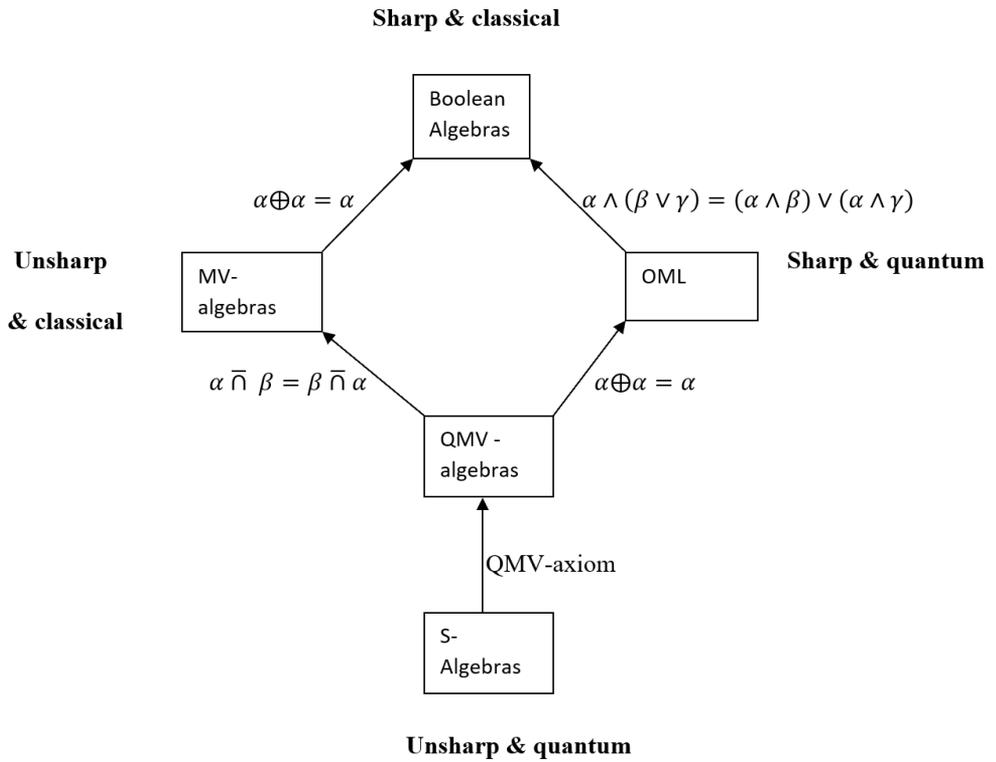


Figure 1

tum universe.

References

[1] L. Beran, *Orthomodular Lattices*, Reidel (1985).  
 [2] G. Birkhoff and von Neumann, The Logic of Quantum Mechanics, *Annals of Mathematics*, **37**, 823-843 (1936).  
 [3] C.C. Chang, Algebraic Analysis of Many Valued Logic, *Trans. American Math. Soc.*, **88**, 74-80 (1958).  
 [4] M.L.D. Chira, R. Guintini and R. Greechie: *Reasoning in Quantum Theory: Sharp and Unsharp Quantum logics*, Springer 2004.  
 [5] C.C. Chang, A New Proof of the Completeness of Lukasiewicz Axiom, *Trans. American Math. Soc.*, **93**, 467-490 (1958).  
 [6] I. Chajda and H. Länger, Algebraic Structures Formalizing the Logic of Quantum Mechanics Incorporating Time Dimension, *Studia Logica*, 1-19 (2024).  
 [7] D.J. Foulis and M.K. Bennett, Effect Algebras and Unsharp Quantum Logics, *Foundations Phys.*, **24**, 1331-1351 (1994).  
 [8] D.J. Foulis, R.J. Greechie, M.L. Dalla Chiara and R. Guintini, Quantum logic, *Mathematical Tools for Physicists*, 439-473 (2005).  
 [9] R. Guintini, F. Paoli, *From sharp to unsharp logic*, Seminar Series Quantumness: Quantum logic as a logic at University of Cagliari, IAS-TUM (2024).  
 [10] R. Guintini, Quasilinear QMV algebras, *Inter. J. Theoretical Phys.*, **34**, 1397-1407 (1995).  
 [11] S.P. Gudder, Total Extensions of Effect Algebras, *Foundations of Physics Letters* **8**, 243-252 (1995).  
 [12] S.P. Gudder, Sharp and Unsharp quantum effects, *Advances in Applied Mathematics* **20**, 169-187 (1998).  
 [13] F. Kopka and F. Choanec, D-Posets, *Math. Slovaca*, **44**, 21-34(1994).  
 [14] J. Łukasiewicz, On three valued logic, *Ruch filozoficzny* **5**, 170-171 (1920).  
 [15] G.W. Mackey, *Mathematical Foundations of Quantum Mechanics*, Mathematical Physics Monograph Series (1963).  
 [16] N. Sorin, From Classical to Fuzzy Logic and Quantum Logic: A General View, *Inter. J. of Computers, Communications and Control*, **16**(1) 4125 (2021).  
 [17] D. Lehmann, *Projection-algebras and quantum logic*. arXiv preprint arXiv:2402.07042 (2024).

- [18] A. Oldofredi, G. Carcassi and C.A. Aidala, On the Common Logical Structure of Classical and Quantum Mechanics, *Erkenntnis* **89(4)** 1507-1533 (2024).
- [19] H. Zhou, Y. Wang, B. Wang and X. Wang, *Intuitionistic Quantum Logic Perspective: A New Approach to Natural Revision Theory*, arXiv preprint arXiv, 2404.13608 (2024).

### Author information

Sarvesh K. Mishra, Department of Mathematics, United University, Prayagraj-211012, U.P., India.  
E-mail: sarveshkumarmishra.phdmt21@uniteduniversity.edu.in, mishrasarvesh.math@gmail.com

Mukesh K. Shukla, Department of Mathematics, United University-211012, U.P., India.  
E-mail: mukesh.shukla@united.edu.in, mkshuklaau@gmail.com

Akhilesh K. Singh, Department of APSH, Kamla Nehru Institute of Technology Sultanpur-228118, U.P., India.  
E-mail: akhilesh.singh@knit.ac.in, au.akhilesh@gmail.com

Received: 2024-01-22.

Accepted: 2024-03-27.