APPROXIMATION PROPERTIES OF MODIFIED SCHURER OPERATORS BASED ON $\sigma \in [-1, 1]$

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Abstract. The goal of this manuscript is to introduce a new sequence of α-Stancu-Schurer operators. Further, basic estimates are calculated. In the next section, rapidity of convergence and order of approximation are studied in terms of first and second order modulus of continuity. Voronovskaja-type approximation result investigated to approximate the class of functions which has first and second order continuous derivatives. Moreover, local and global approximation properties are studied in several functional spaces. Lastly, A-statistical approximation results are presented.

1 Introduction

In 1962, Schurer [5] presented the following modification of Bernstein operators [1] is denoted as $B_{m,l} : C[0, 1 + l] \to C[0, 1]$ and defined by:

$$B_{m,l}(g; \mu) = \sum_{i=0}^{m+l} g\left(\frac{j}{m}\right) \binom{m+l}{j} \mu^k (1-\mu)^{m+l-j}, \mu \in [0, 1], \tag{1.1}$$

for $l \in \mathbb{N} \cup \{0\}$ and $g \in C[0, 1 + l]$.

In 2017, Chen et al. [6] presented a family of modified Bernstein operators which is called as σ -Bernstein sequence of operator as:

$$\mathcal{P}_{m,\sigma}(g; \mu) = \sum_{j=0}^{m} g\left(\frac{j}{m}\right) \mathcal{Q}_{m,j}^{(\sigma)}(\mu), \tag{1.2}$$

where σ-Bernstein basis $\mathcal{Q}_{m,j}^{(\sigma)}(\mu)$ of degree m is presented by $\mathcal{Q}_{1,0}^{(\sigma)}(\mu) = 1 - \mu$, $\mathcal{Q}_{1,1}^{(\sigma)}(\mu) = \mu$ and

$$\begin{aligned}
\mathcal{Q}_{m,j}^{(\sigma)}(\mu) &= \left[\binom{m-2}{j} (1-\sigma)\mu + \binom{m-2}{j-2} (1-\sigma)(1-\mu) \right. \\
&\quad + \left. \binom{m}{j} \sigma\mu(1-\mu) \right] \times \mu^{i-1}(1-\mu)^{m-j-1},
\end{aligned}$$

with $m \geq 2$, $\mu \in [0, 1]$. Further, Cai et al. [9] proved that the σ-Bernstein sequences of operators are linear and positive operators for $0 \leq \sigma \leq 1$. In [6], several local and global approximation properties are investgated in various functional spaces. Further, many researchers, i.e., Mursaleen et al. ([7]-[8], Çetin et al. ([10]-[11]), Kajla et al. [12], Mohiuddine et al. ([13]-[14], Aral et al. [15], Cai et al. [16], Acar et al.([17]-[18]), Acu et al. [21], Rao et al. ([22]-[24]),

Khan et al. ([25]. [26]), Ayman-Mursaleen [27] have studied intensively σ-Bernstein sequences of operators and their generalization for better convergence in approximation results. Recently, Çetin [19] presented a sequence of Schurer sequence of operators presented in (1.2) as:

$$\mathcal{L}_{m,\sigma,l}(g;\mu) = \sum_{j=0}^{m+l} g\left(\frac{j}{m}\right) \mathcal{C}_{m,j}^{(\sigma)}(\mu), \tag{1.3}$$

where σ-Bernstein-Schurer polynomial $\mathcal{C}_{m,j}^{(\sigma)}(\mu)$ of degree m is presented by $\mathcal{C}_{1,0}^{(\sigma)}(\mu) = 1 - \mu$, $\mathcal{C}_{1,1}^{(\sigma)}(\mu) = \mu$ and

$$\begin{aligned}
\mathcal{C}_{m,j}^{(\sigma)}(\mu) &= \left\{ \binom{m+l-2}{j}(1-\sigma)\mu + \binom{m+l-2}{j-2}(1-\sigma)(1-\mu) \right. \\
&\quad + \left. \binom{m+l}{j}\sigma\mu(1-\mu) \right\} \times \mu^{i-1}(1-\mu)^{n+p-i-1},
\end{aligned}$$

where $m+l \geq 2, \mu \in [0,1]$. Motivated by the above development, we construct σ-Stancu-Schurer operators as:

$$\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(g;\mu) = \sum_{j=0}^{m+l} \mathcal{C}_{m,j}^{(\sigma)}(\mu) g\left(\frac{j+\delta}{m+\gamma}\right), \tag{1.4}$$

whenever the above sum converges.

Remark 1.1. One can easily note the followings:
1. For $l = 0$, the sequences defined by (1.4) reduces to α-Bernstein-Stancu sequences [11].
2. For $\delta = \gamma = 0$, the operators presented in (1.4) turns in α-Bernstein Schurer operators constructed in (1.3).
3. For $l = 0$, and $\delta = \gamma = 0$, the operators given in (1.4) reduces to α-Baskakov operators given by (1.2).
4. For $l = 0$, $\delta = \gamma = 0$ and $\sigma = 0$, the sequences presented (1.4) reduces to classical Bernstein operators given by [1].

In the successive parts, we deduce some lemmas which help us to discuss the rate of convergence, order of approximation, pointwise and weighted approximation results with the help of Krokovin theorem, modulus of smoothness, second order modulus of smoothness, Peetre's K-functional. Further, pointwise and weighted approximation properties are discussed in various spaces of functions. In the last section, we present A-Statistical results of approximation.

2 Preliminary Results

Let $e_i(t) = t^i$, $\tau_i(t) = (t - \mu)^i$, $i \in \{0, 1, 2, 4\}$ be the test functions and central moments respectively. Then, we recall the following Lemma from [11] as:

Lemma 2.1. *For the operators given by ([1.3](#)), we have*

$$
\begin{aligned}
\mathcal{L}_{m,\sigma,l}(e_0;\mu) &= 1, \\
\mathcal{L}_{m,\sigma,l}(e_1;\mu) &= \left(1 + \frac{l}{m}\right)\mu, \\
\mathcal{L}_{m,\sigma,l}(e_2;\mu) &= \left(1 + \frac{l}{m}\right)^2 \mu^2 + \frac{[m+l+2(1-\sigma)]}{m^2}\mu(1-\mu), \\
\mathcal{L}_{m,\sigma,l}(e_3;\mu) &= \{\frac{1}{n^3}\left[(m+l)^3 - 3(m+l)^2 + 2(3\sigma-2)(m+l) + 12(1-\sigma)\right]\mu^3 \\
&\quad + 3\left[(m+l)^2 + (1-2\sigma)(m+l) + 6(\sigma-1)\right]\mu^2 \\
&\quad + \left[m+l+6(1-\sigma)\right]\mu\}, \\
\mathcal{L}_{m,\sigma,l}(e_4;\mu) &= \{\frac{1}{m^4}[(m+l)^4 - 6(m+l)^3 + (12\sigma-1)(m+l)^2 + 6(9-10\sigma)(m+l) \\
&\quad - 72(1-\sigma)]\mu^4 + 6\Big[(m+l)^3 - (1+2\sigma)(m+l)^2 + 2(8\sigma-7)(m+l) \\
&\quad + 24(1-\sigma)\Big]\mu^3 + \left[7(m+l)^2 + (29-36\sigma)(m+l) - 86(1-\sigma)\right]\mu^2 \\
&\quad + \left[m+l+14(1-\sigma)\right]\mu\}.
\end{aligned}
$$

Lemma 2.2. *For the operators given by ([1.4](#)), we have*

$$
\begin{aligned}
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_0;\mu) &= 1, \\
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_1;\mu) &= \left(\frac{m+l}{m+\gamma}\right)\mu + \frac{\delta}{m+\gamma}, \\
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_2;\mu) &= \left(\frac{m+l}{m+\gamma}\right)^2 \mu^2 + \left\{\frac{[m+l+2(1-\sigma)]}{(m+\gamma)^2}(1-\mu) + \frac{2\delta(m+l)}{(m+\gamma)^2}\right\}\mu \\
&\quad + \frac{\delta^2}{(m+\gamma)^2},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_4;\mu) &= \Big\{\frac{[(m+l)^4 - 6(m+l)^3 + (12\sigma-1)(m+l)^2 + 6(9-10\sigma)(m+l)]}{(m+\gamma)^4} \\
&\quad - \frac{72(1-\sigma)}{(m+\gamma)^4}\Big\}\mu^4 + o\left(\frac{1}{m+\gamma}\right).
\end{aligned}
$$

Proof. Lemma [2.2](#) can be prove with the aid of Lemma [2.1](#) as:

$$
\begin{aligned}
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_0;\mu) &= \sum_{j=0}^{m+l} \mathcal{C}_{m,j}^{(\sigma)}(\mu) \\
&= 1.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_1;\mu) &= \sum_{j=0}^{m+l} \mathcal{C}_{m,j}^{(\sigma)}(\mu)\frac{j+\delta}{m+\gamma} \\
&= \frac{m}{m+\gamma}\sum_{j=0}^{m+l} \mathcal{C}_{m,j}^{(\sigma)}(\mu)\left(\frac{j}{m}\right) + \frac{\delta}{m+\gamma}\sum_{j=0}^{m+l} \mathcal{C}_{m,j}^{(\sigma)}(\mu) \\
&= \frac{m}{m+\gamma}\mathcal{L}_{m,\sigma,l}(e_1;\mu) + \frac{\delta}{m+\gamma}\mathcal{L}_{m,\sigma,l}(e_0;\mu) \\
&= \left(\frac{m+l}{m+\gamma}\right)\mu + \frac{\delta}{n+\gamma}.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_2;\mu) &= \sum_{j=0}^{m+l} \mathcal{C}_{m,j}^{(\sigma)}(\mu)\left(\frac{j+\delta}{m+\gamma}\right)^2 \\
&= \frac{m^2}{(m+\gamma)^2}\mathcal{L}_{m,\sigma,l}(e_2;\mu) + \frac{2m\delta}{(m+\gamma)^2}\mathcal{L}_{m,\sigma,l}(e_1;\mu) \\
&\quad + \frac{\delta^2}{(m+\gamma)^2}\mathcal{L}_{m,\sigma,l}(e_0;\mu) \\
&= \left(\frac{m+l}{m+\gamma}\right)^2\mu^2 + \frac{[m+l+2(1-\sigma)]}{(m+\gamma)^2}\mu(1-\mu) \\
&\quad + \frac{2\delta(m+l)}{(m+\gamma)^2}\mu + \frac{\delta^2}{(m+\gamma)^2}.
\end{aligned}
$$

In this direction, Lemma (2.2) can be proved easily.   □

Lemma 2.3. *Let $\tau_t^i(\mu) = (t-\mu)^i$ be the central moments. Then, we get*

$$
\begin{aligned}
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(\tau_t^0;\mu) &= 1, \\
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(\tau_t^1(t);\mu) &= \left(\frac{l-\gamma}{m+\gamma}\right)\mu + \frac{\delta}{m+\gamma}, \\
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(\tau_t^2(t);\mu) &= \left(\frac{l-\gamma}{m+\gamma}\right)^2\mu^2 + \left\{\frac{[m+l+2(1-\sigma)](1-\mu)}{m+\gamma^2} + \frac{2\delta(l-\gamma)}{(m+\gamma)}\right\} \\
&\quad + \frac{\delta^2}{(m+\gamma)^2}, \\
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(\tau_t^4(t);\mu) &= o\left(\frac{1}{m+\gamma}\right)^2.
\end{aligned}
$$

Proof. In the light of Lemma 2.2 and property of linearity, we have

$$
\begin{aligned}
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(\tau_t^1(t);\mu) &= \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(t;\mu) - \mu\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(1;0), \\
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(\tau_t^2(t);\mu) &= \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(t^2;\mu) - 2\mu\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(t;\mu) + \mu^2\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(1;0), \\
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(\tau_t^4(t);\mu) &= \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(t^4;\mu) - 4\mu\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(t^3;\mu) + 6\mu^2\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(t^2;\mu) \\
&\quad - 4\mu^3\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(t;\mu) + \mu^4\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(1;\mu),
\end{aligned}
$$

which completes the desired proof.   □

Now, we check the convergence of the sequence of operators introduced in (1.4) in terms of Korovkin theorem as:

Theorem 2.4. *Let $\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(.;.)$ be the operators given by* (1.4). *Then, $\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(g;\mu) \rightrightarrows g$ on $[0,1+l]$ where $\rightrightarrows$ denotes uniform convergence and $g \in C[0,1+l] \bigcap \left\{g : \frac{g(\mu)}{1+\mu^2} \text{ converges for } \mu \to \infty\right\}$.*

Proof. We prove the convergence of these sequences in terms of Korovkin's theorem, i.e., it is quite enough to prove that

$$
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_i;\mu) \to e_i(\mu) \text{ with } i = 0,1,2.
$$

With the aid of Lemma 2.2, it is obvious that $\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_i;\mu) \to e_i(\mu)$ as $n \to \infty$, $i = 0,1,2$ which complete the proof of Theorem 2.4.   □

3   Order of Convergence

Theorem 3.1. *Let $g \in C[0, 1 + l]$ and $\omega(.;.)$ presents the modulus of smoothness. Then*

$$\left| \mathcal{D}_{m,\sigma,l}^{\delta,\theta_{m,\gamma}}(g;\mu) - g(\mu) \right| \leq \left\{ 1 + \sqrt{\Gamma_m^{\delta,\gamma}(\mu)} \right\} \omega\left(g; \theta_{m,\gamma}\right),$$

where $\theta_{m,\gamma} = (m + \gamma)^{-\frac{1}{2}}$ and

$$\Gamma_m^{\delta,\gamma}(\mu) = \frac{(m-l)^2\mu^2 + [m + l + 2(1 - \sigma)]\,\mu(1 - \mu) + 2\delta(l - \gamma)\mu + \delta^2}{(m + \gamma)}.$$

Proof. For $g \in C[0, 1 + l]$, $\mu \in [0, 1 + l]$ and in terms of monotonicity and linearity property of the sequence of operators given by (1.4), we can easily find

$$\left| \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(g;\mu) - g(\mu) \right| \leq \left\{ 1 + \theta_{m,\gamma}^{-1} \sqrt{\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}\left(\tau_t^2; \mu\right)} \right\} \omega\left(g; \theta_{m,\theta}\right)$$

$$\leq \left\{ 1 + \sqrt{\frac{(m-l)^2\mu^2 + [m + l + 2(1 - \sigma)]\,\mu(1 - \mu) + 2\delta(l - \gamma)\mu + \delta^2}{(m + \gamma)}} \right\} \omega\left(g; \theta_{m,\gamma}\right),$$

where $\theta_{m,\gamma} > 0$ and which obtained by using Lemma 2.2 and choosing $\theta_{m,\gamma} = (m + \gamma)^{-\frac{1}{2}}$. Thus, we complete the desired result. $\square$

Now, we give the order of convergence for the sequence of operators given by (1.4) in terms of modulus of smoothness of functions which has first order continuous derivatives , i.e., $\omega\left(f'; \theta_{m,\gamma}\right) = \omega_1\left(f; \theta_{m,\gamma}\right)$.

Theorem 3.2. *Let the sequence of operators be given by (1.4) and $0 \leq \theta \leq \gamma$. Then, one has*

$$\left| \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(g;\mu) - g(\mu) \right| \leq \omega_1\left((m + \gamma)^{-1}\right) \sqrt{\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}\left(\tau_t^2(t); \mu\right)}$$

$$\left\{ 1 + \sqrt{(m + \gamma)} \sqrt{\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}\left(\tau_t^2(t); \mu\right)} \right\}.$$

Proof. For any $a \leq \mu_1, \mu_2 \leq b$, we know that

$$\begin{aligned} g\left(\mu_1\right) - g\left(\mu_2\right) &= \left(\mu_1 - \mu_2\right) g'(\xi) \\ &= \left(\mu_1 - \mu_2\right) g'\left(\mu_1\right) + \left(\mu_1 - \mu_2\right)\left[g'(\xi) - g'\left(\mu_1\right)\right], \end{aligned} \tag{3.1}$$

where $\xi \in \left(\mu_1, \mu_2\right)$. Further, we have

$$\left|\left(\mu_1 - \mu_2\right)\left[g'(\xi) - g'\left(\mu_1\right)\right]\right| \leq \left|\mu_1 - \mu_2\right|(\lambda + 1)\omega_1(\theta_{m,\gamma}), \quad \lambda = \lambda\left(\mu_1, \mu_2; \theta_{m,\gamma}\right). \tag{3.2}$$

Next, we obtain

$$\left| \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(g;\mu) - g(\mu) \right| = \left| \sum_{j=0}^{\infty} C_{m,j}^{(\sigma)}(\mu) \left\{ g\left(\frac{j + \delta}{m + \gamma}\right) - g(\mu) \right\} \right|. \tag{3.3}$$

In view of (3.1), (3.2) and (3.3), we obtain

$$
\begin{aligned}
\left| \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(g;\mu) - g(\mu) \right| &\leq \left| \sum_{j=0}^{\infty} C_{m,j}^{(\sigma)}(\mu) \left(\frac{j+\delta}{m+\gamma} - \mu \right) g'(\mu) \right| \\
&+ \omega_1\left(\theta_{m,\gamma}\right) \sum_{j=0}^{\infty} \left| \frac{j+\delta}{m+\gamma} - \mu \right| (\lambda+1) C_{m,j}^{(\sigma)}(\mu) \\
&\leq \omega_1\left(\theta_{m,\gamma}\right) \left\{ \sum_{j=0}^{\infty} \left| \frac{j+\delta}{m+\gamma} - \mu \right| C_{m,j}^{(\sigma)}(\mu) \right. \\
&+ \left. \sum_{\lambda \geq 1} \left| \frac{j+\delta}{m+\gamma} - \mu \right| \lambda \left(\mu_1, \frac{j+\delta}{m+\gamma}; \theta \right) C_{m,j}^{(\sigma)}(\mu) \right\} \\
&\leq \omega_1\left(\theta_{m\gamma}\right) \left\{ \sum_{j=0}^{\infty} \left| \frac{j+\delta}{m+\gamma} - \mu \right| C_{m,j}^{(\sigma)}(\mu) \right. \\
&+ \left. \theta^{-1} \sum_{j=0}^{\infty} \left(\frac{j+\delta}{m+\gamma} - \mu \right)^2 C_{m,l}^{(\sigma)}(\mu) \right\} \\
&\leq \omega_1\left(\theta_{m,\gamma}\right) \sqrt{\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}\left(\tau_t^2(t);\mu\right)} \left\{ 1 + \theta_{m,\gamma}^{-1} \sqrt{\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}\left(\tau_t^2(t);\mu\right)} \right\}.
\end{aligned}
$$

Choosing $\theta_{m,\gamma} = (m+\gamma)^{-1}$, we get

$$
\begin{aligned}
\left| \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(g;\mu) - g(\mu) \right| &\leq \omega_1\left((m+\gamma)^{-1}\right) \sqrt{\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}\left(\tau_t^2(t);\mu\right)} \\
&\left\{ 1 + \sqrt{(m+\gamma)} \sqrt{\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}\left(\tau_t^2(t);\mu\right)} \right\}.
\end{aligned}
$$

Hence, Theorem 3.2 is proved. $\square$

4   Direct Results

Now, we recall the definition of given by Modulus of smoothness [20] as:

$$
\begin{aligned}
\omega_{\varphi^\lambda}^2(g) &= \sup_{0 < g^* \leq \theta} \left\| \theta_{g^*\varphi(\mu)}^2 g(\mu) \right\| \\
&= \sup_{0 < g \leq \theta} \sup_{\mu \pm g^*\varphi^\lambda \in [0,1+l]} \left| g\left(\mu - g^*\varphi^\lambda(\mu)\right) - 2g(\mu) + g\left(\mu + g^*\varphi^\lambda(\mu)\right) \right|,
\end{aligned}
$$
(4.1)

with $\varphi^2(\mu) = \mu(1+\mu)$ and

$$
K_{\varphi^{\lambda m,\gamma}}\left(g,\theta^2\right) = \inf_g \left(\|g-f\|_{C[0,1+l]} + \theta^2 \left\| \varphi^2 \lambda g'' \right\|_{C[0,1+l]} \right), \quad g, g' \in AC_{loc},
$$
(4.2)

where $K_{\varphi^{\lambda m,\gamma}}\left(h,\theta^2\right)$ denotes Peetre's K-functional.
Here, we recall the relation between K-functional and the modulus of continuity

$$
C^{-1} K_{\varphi^\lambda}\left(h,\theta^2\right) \leq \omega_{\varphi^\lambda}^2(h,\theta) \leq C K_{\varphi^\lambda}\left(h,\theta^2\right).
$$
(4.3)

Theorem 4.1. *For* $0 \leq \theta \leq \gamma$, *we have*

$$
\left| \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h;\mu) - h(\mu) \right| \leq C \omega_{\varphi^\lambda}^2 \left(h; (m+\gamma)^{-\frac{1}{2}} \varphi(\mu)^{1-\lambda} \right) \textit{ for the large value of } m.
$$

Proof. In the light of (4.1), (4.2) and (4.3) also by choosing $h_m^* \equiv h_{m,\mu,\lambda}^*$, we have μ and $\lambda + 1$ such that

$$\|h - h^*\|_{C[0,1+l]} \leq A\omega_{\varphi^\lambda}^2 \left(h; \theta_{m,\gamma}^{-\frac{1}{2}}\varphi(\mu)^{1-\lambda}\right), \tag{4.4}$$

$$m^{-1}\varphi(\mu)^{2-2\lambda}\left\|\varphi^{2\lambda}h^{*''}\right\|_{C[0,1+l]} \leq B\omega_{\varphi^\lambda}^2\left(h; \theta_{m,\gamma}^{-\frac{1}{2}}\varphi(\mu)^{1-\lambda}\right). \tag{4.5}$$

Next

$$
\begin{aligned}
\left|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h;\mu) - h(\mu)\right| &\leq \left|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h - h_m^*;\mu) - (h - h_n^*)(\mu)\right| \\
&+ \left|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h_m^*;\mu) - h_m^*(\mu)\right| \\
&\leq 2\|h - h_m^*\|_{C[0,1+l]} + \left|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h_m^*;\mu) - h_m^*(\mu)\right|.
\end{aligned}
$$

From (4.4), we get

$$\left|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h;\mu) - h(\mu)\right| \leq 2A\omega_{\varphi^\lambda}^2\left(h, \theta^{-\frac{1}{2}}\varphi(\mu)^{1-\lambda}\right) + \left|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h_m^*;\mu) - h_m^*(\mu)\right|.$$

Now, we apply Taylor's formula

$$
\begin{aligned}
\left|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h_m^*(t) - h_m^*(\mu);\mu)\right| &\leq \left|h_m^{*'}(\mu)\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}((t-\mu);\mu)\right| \\
&+ \left|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}\left(\int_t^\mu (\mu - u)h_m^{*''}(u)du;\mu\right)\right| \\
&\leq \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}\left(\frac{\left|\mu - \frac{j}{m}\right|}{\varphi^{2\lambda}(\mu)}\int_{\frac{j}{m}}^\mu \varphi^{2\lambda}(u)\left|h_m^{*''}(u)du\right|;\mu\right) \\
&\leq \left|\left|\varphi^{2\lambda}h_m^{*''}\right|\right|_{[0,1+l]}\frac{1}{\varphi^{2\lambda}(\mu)}\left[\left(\frac{m+l}{m+\gamma} - 1\right)^2\mu^2\right. \\
&+ \frac{[m+l+2(1-\sigma)]\mu(1-\mu)}{(m+\gamma)^2} + \frac{2\delta(l-\gamma)\mu}{(m+\gamma)^2} \\
&+ \left.\frac{\delta^2}{(m+\gamma)^2}\right] \\
&\leq \left|\left|\varphi^{2\lambda}h_m^{*''}\right|\right|_{[0,1+l]}\frac{\mu(1+\mu)(m+\gamma)^{-1}}{\varphi^{2\lambda}(\mu)}\left[\frac{(l-\gamma)^2}{(m+\gamma)}\frac{\mu}{1+\mu}\right. \\
&+ \frac{[m+l+2(1-\sigma)]}{(m+\gamma)}\frac{1-\mu}{1+\mu} + \frac{2\delta(l-\gamma)}{(m+\gamma)}\frac{1}{(1+\mu)} \\
&+ \left.\frac{\delta^2}{(m+\gamma)}\frac{1}{\mu(1+\mu)}\right] \\
&\leq \left|\left|\varphi^{2\lambda}h_m^{*''}\right|\right|_{[0,1+l]}\varphi^{2-2\lambda}(\mu)(m+\gamma)^{-1}\left[\frac{(l-\gamma)^2}{(m+\gamma)}\frac{\mu}{1+\mu}\right. \\
&+ \frac{[m+l+2(1-\sigma)]}{(m+\gamma)}\frac{1-\mu}{1+\mu} + \frac{2\delta(l-\gamma)}{(m+\gamma)}\frac{1}{(1+\mu)} \\
&+ \left.\frac{\delta^2}{(m+\gamma)}\frac{1}{\mu(1+\mu)}\right].
\end{aligned}
$$

From (4.3), we have

$$
\begin{aligned}
\left|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h_m^*(t) - h_m^*(\mu);\mu)\right| &\leq B\omega_{\varphi^\lambda}^2\left(h, (m+\gamma)^{-\frac{1}{2}}\right)\varphi(\mu)^{1-\lambda}\left[\frac{(l-\gamma)^2}{(m+\gamma)}\frac{\mu}{1+\mu}\right. \\
&+ \frac{[m+l+2(1-\sigma)]}{(m+\gamma)}\frac{1-\mu}{1+\mu} + \frac{2\delta(l-\gamma)}{(m+\gamma)}\frac{1}{(1+\mu)} \\
&+ \left.\frac{\delta^2}{(m+\gamma)}\frac{1}{\mu(1+\mu)}\right].
\end{aligned}
$$

Using (4.2) and (4.3), we get

$$
\begin{aligned}
\left| \mathcal{D}_{m,\sigma,l}^{\delta,\gamma} \left(h_m^*(t) - h_m^*(\mu); \mu \right) \right| \leq\ & M \omega_{\varphi^\lambda}^2 \left(h, (m+\gamma)^{-\frac{1}{2}} \right) \varphi(\mu)^{1-\lambda} \Bigg[1 + \frac{(l-\gamma)^2}{(m+\gamma)} \frac{\mu}{1+\mu} \\
& + \frac{[m+l+2(1-\sigma)]}{(m+\gamma)} \frac{1-\mu}{1+\mu} + \frac{2\delta(l-\gamma)}{(m+\gamma)} \frac{1}{(1+\mu)} \\
& + \frac{\delta^2}{(m+\gamma)} \frac{1}{\mu(1+\mu)} \Bigg],
\end{aligned}
$$

where $M = max(2A, B)$. For large value of m

$$
\left| \mathcal{D}_{m,\sigma,l}^{\delta,\gamma} \left(h_m^*(t) - h_m^*(\mu); \mu \right) \right| \leq M \omega_{\varphi^\lambda}^2 \left(h, (m+\gamma)^{-\frac{1}{2}} \right) \varphi(\mu)^{1-\lambda}.
$$

$\square$

Theorem 4.2. *For $0 \leq \delta \leq \gamma$ and $h \in C^2[0, 1+l]$, one has*

$$
\lim_{m \to \infty} m \left\{ \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h; \mu) - h(\mu) \right\} = [(l-\gamma)\mu + \delta] \, h'(\mu) + \frac{\mu(1-\mu)}{2} h''(\mu).
$$

Proof. For $\mu, t \in [0, 1], h \in C^2[0, 1+l]$ and in terms of Taylor's Theorem, we get

$$
h(t) = h(\mu) + (t-\mu)h'(\mu) + \frac{(t-\mu)^2}{2} h''(\mu) + \eta(t, \mu)(t-\mu)^2,
$$

where $\eta(t, \mu)$ presents the continuous function which belongs to $C[0, 1+l]$ with $\lim_{t \to \mu} \eta(t, \mu) = 0$. Operating $C_{m,j}^\sigma$ both the sides and summing over j, one get

$$
\begin{aligned}
\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h; \mu) =\ & h(\mu) \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(1; \mu) \\
& + h'(\mu) \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(t-\mu; \mu) + \frac{h''(\mu)}{2} \mathcal{D}_{m,\sigma,l}^{\delta,\gamma} \left((t-\mu)^2; \mu \right) \\
& + \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(\eta(t, \mu)(t-\mu); \mu).
\end{aligned}
$$

From Lemma (2.2), we get

$$
\begin{aligned}
\lim_{m \to \infty} m \left\{ \mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h; \mu) - h(\mu) \right\} =\ & [(l-\gamma)\mu + \delta] \, h(\mu)' + \frac{\mu(1-\mu)}{2} h''(\mu) \\
& + \lim_{m \to \infty} m \mathcal{D}_{m,\sigma,l}^{\delta,\gamma} \left(\eta(t; \mu)(t-\mu)^2; \mu \right). \quad (4.6)
\end{aligned}
$$

To obtain last term, we use Lemma (2.3) and Hölder's inequality as:

$$
m \mathcal{D}_{m,\sigma,l}^{\delta,\gamma} \left(\eta(t; \mu)(t-\mu)^2; \mu \right) \leq m^2 \mathcal{D}_{m,\sigma,l}^{\delta,\gamma} \left((t-\mu)^4; \mu \right) \mathcal{D}_{m,\sigma,l}^{\delta,\gamma} \left(\eta(t; \mu)^2; \mu \right).
$$

Let $\varphi(t; \mu) = \eta^2(t; \mu)$. Then, $\lim_{t \to \mu} \varphi(t; \mu) = 0$. Therefore,

$$
\lim_{m \to \infty} m \mathcal{D}_{m,\sigma,l}^{\delta,\gamma} \left(\eta(t; \mu)(t-\mu)^2; \mu \right) = 0
$$

In the light of (4.6), we reach the required result. $\square$

5 Uniform Approximation Results

From [29], Let $1 + \mu^2$, $0 \leq \mu < \infty$, denotes the weight function and $B_{1+\mu^2}[0, 1+l] = \{g(\mu) : |g(\mu)| \leq N_g(1+\mu^2), N_g$ is real fixed value depending on $g\}$, $C_{1+\mu^2}[0, 1+l] \subset B_{1+\mu^2}[0, 1+l]$, equipped with the norm $\|g\|_{1+\mu^2} = \sup\limits_{\mu \in [0,1+l]} \frac{|g(\mu)|}{1+\mu^2}$ and

$$
C_{1+\mu^2}^k[0, 1+l] = \left\{ g \in C_{1+\mu^2}[0, 1+l] : \lim_{\mu \to \infty} \frac{g(\mu)}{1+\mu^2} = K, \text{ where } k \text{ is a constant} \right\}.
$$

Theorem 5.1. *Let $\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(.;.)$ be the operators given by (1.4) and $\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(.;.) : C_{1+\mu^2}^k[0,\infty) \to B_{1+\mu^2}[0,\infty)$. Then, we have*

$$\lim_{m\to\infty} \|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(h;\mu) - h\|_{1+\mu^2} = 0,$$

where $h \in C_{1+\mu^2}^k[0,\infty)$.

Proof. It is enough to prove this result

$$\lim_{m\to\infty} \|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_i;\mu) - \mu^i\|_{1+\mu^2} = 0, \ i = 0, 1, 2.$$

Using Lemma 2.2, we get

$$\|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_0;\mu) - \mu^0\|_{1+\mu^2} = \sup_{\mu\in[0,\infty)} \frac{|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(1;\mu) - 1|}{1+\mu^2} = 0 \text{ for } i = 0.$$

For $i = 1$, we have

$$\|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_1;\mu) - \mu^1\|_{1+\mu^2} = \sup_{\mu\in[0,\infty)} \frac{\left(\frac{m+l}{m+\gamma} - 1\right)\mu + \frac{\delta}{m+\gamma}}{1+\mu^2}$$

$$= \left(\frac{m+l}{m+\gamma} - 1\right) \sup_{\mu\in[0,\infty)} \frac{\mu}{1+\mu^2}$$

$$+ \frac{\delta}{m+\gamma} \sup_{\mu\in[0,\infty)} \frac{1}{1+\mu^2},$$

which implies that $\|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_1;\mu) - \mu^1\|_{1+\mu^2} \to 0$ an $m \to \infty$.
Finally, for $i = 2$, one yield the following

$$\|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_2;\mu) - \mu^2\|_{1+\mu^2} = \sup_{\mu\in[0,\infty)} \frac{\left|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_2;\mu) - \mu^2\right|}{1+\mu^2}$$

$$= \sup_{\mu\in[0,\infty)} \left(\frac{m+l}{m+\gamma}\right)^2 \mu^2 + \left\{ \frac{[m+l+2(1-\sigma)]}{(m+\gamma)^2}(1-\mu) \right.$$

$$\left. + \frac{2\delta(m+l)}{(m+\gamma)^2} \right\}\mu + \frac{\delta^2}{(m+\gamma)^2},$$

$$\leq \left(\left(\frac{m+l}{m+\gamma}\right)^2 - \left(\frac{[m+l+2(1-\sigma)]}{(m+\gamma)^2}\right) - 1 \right) \sup_{\mu\in[0,\infty)} \frac{\mu^2}{1+\mu^2}$$

$$+ \left(\left(\frac{[m+l+2(1-\sigma)]}{(m+\gamma)^2}\right) + \frac{2\delta(m+l)}{(m+\gamma)^2} \right) \sup_{\mu\in[0,\infty)} \frac{\mu}{1+\mu^2}$$

$$+ \frac{\delta^2}{(m+\gamma)^2} \sup_{\mu\in[0,\infty)} \frac{1}{1+\mu^2},$$

which implies that $\|\mathcal{D}_{m,\sigma,l}^{\delta,\gamma}(e_2;\mu) - \mu^2\|_{1+\mu^2} \to 0$ as $m \to \infty$ and this completes the proof of this Theorem. □

6 A-statistical approximation

Gadjiev *et al.* [30] discussed statistical approximation theorems. Here some notation from [30]. Let $A = (a_{nk})$ denotes the non-negative infinite summability matrix. Let a given sequence $\mu := (\mu_k)$, the A-transform of μ holds by $A\mu : ((A\mu)_m)$ as:

$$(A\mu)_m = \sum_{k=1}^{\infty} a_{mk}\mu_k,$$

gave the series converges for every m. A is termed to be regular if $\lim(A\mu)_m = L$ where $\lim \mu = L$. Then $\mu = (\mu_m)$ is called to be a A-statistically convergent to L i.e. $st_A - \lim \mu = L$ if for every $\epsilon > 0$, $\lim_m \sum_{k:|\mu_k - L| \geq \epsilon} a_{mk} = 0$.
We prove the following theorem.

Theorem 6.1. *For $A = (a_{nk})$ a non-negative regular suitability matrix, we have*

$$st_A - \lim_m \|\mathcal{D}^{\delta,\gamma}_{m,\sigma,l}(h;\mu) - h\|_{1+\mu^2} = 0, \text{ for all } h \in C^k_{1+\mu^2}[0,\infty).$$

Proof. From ([31], p. 191, Th. 3), it is adequate to prove that for $\lambda_1 = 0$,

$$st_A - \lim_m \|\mathcal{D}^{\delta,\gamma}_{m,\sigma,l}(e_i;\mu) - e_i\|_{1+\mu^2} = 0, \text{ for } i \in \{0,1,2\}.$$

From Lemma 2.2, we have

$$\|\mathcal{D}^{\delta,\gamma}_{m,\sigma,l}(e_1;\mu) - \mu\|_{1+\mu^2} = \sup_{\mu \in [0,\infty)} \frac{1}{1+\mu^2} \left| \left(\frac{m+l}{m+\gamma}\right)\mu + \frac{\delta}{m+\gamma} - \mu \right|$$

$$\leq \left| \left(\frac{m+l}{m+\gamma}\right) - 1 \right| \sup_{\mu \in [0,\infty)} \frac{\mu}{1+\mu^2}$$

$$+ \left| \frac{\delta}{m+\gamma} \right| \sup_{\mu \in [0,\infty)} \frac{1}{1+\mu^2}$$

$$\leq \frac{1}{2} \left| \left(\frac{m+l}{m+\gamma}\right) - 1 \right| + \left| \frac{\delta}{m+\gamma} \right|.$$

Then, we get

$$st_A - \lim_m \frac{1}{2} \left| \left(\frac{m+l}{m+\gamma}\right) - 1 \right| = st_A - \lim_m \left| \frac{\delta}{m+\gamma} \right| = 0. \tag{6.1}$$

Now, for a given $\epsilon > 0$, we define the following sets

$$J_1 := \left\{ m : \|\mathcal{D}^{\delta,\gamma}_{m,\sigma,l}(e_1;\mu) - \mu\| \geq \epsilon \right\},$$

$$J_2 := \left\{ m : \frac{1}{2} \left| \left(\frac{m+l}{m+\gamma}\right) - 1 \right| \geq \frac{\epsilon}{2} \right\},$$

$$J_3 := \left\{ m : \left| \frac{\delta}{m+\gamma} \right| \geq \frac{\epsilon}{2} \right\}.$$

This implies that $J_1 \subseteq J_2 \cup J_3$, which shows that $\sum_{k_1 \in J_1} a_{mk_1} \leq \sum_{k_1 \in J_2} a_{mk} + \sum_{k_1 \in J_3} a_{mk}$. Hence, from (6.1) we get

$$st_A - \lim_m \|\mathcal{D}^{\delta,\gamma}_{m,\sigma,l}(e_1;\mu) - \mu\|_{1+\mu^2} = 0.$$

Similarly, we can show that

$$st_A - \lim_m \|\mathcal{D}^{\delta,\gamma}_{m,\sigma,l}(e_2;\mu) - \mu^2\|_{1+\mu^2} = 0.$$

Hence Theorem 6.1 is proved. $\square$

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