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Pointwise Bi-Slant Submanifolds in Locally Conformal Kähler Manifolds Immersed as Warped Products

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1 Introduction

Vaisman introduced locally conformal Kähler (lcK) manifolds as a generalisation of Kähler manifolds [33, 34, 35, 36, 21, 37, 38]. An lcK manifold is a Hermitian manifold that can be written as the union of Kähler manifolds such that the lcK metric is locally conformal to these Kähler metrics. LcK manifolds are characterised by the existence of a globally defined closed 1-form ω, called the *Lee form*, such that the fundamental 2-form of the lcK metric satisfies $d\Omega = \Omega \wedge \omega$. The Lee form and its associated Lee Vector field play an important part in the geometry of lcK manifolds.

From an extrinsic geometric standpoint, holomorphic and totally real submanifolds are important objects of study in the setting of almost Hermitian manifolds. Bejancu [5, 6] defined CR submanifolds as a generalisation of holomorphic and totally real submanifolds which were further studied by Chen [11, 12]. Later, Chen [13, 14] extended the class of holomorphic and totally real submanifolds by introducing the notion of slant submanifolds. The concept was further generalised to pointwise slant submanifolds [20] by the same author. The study of CR submanifolds and slant submanifolds was later generalised by several authors to semi-slant submanifolds, hemi-slant submanifolds(also called pseudo-slant submanifolds) and bi-slant submanifolds, in various ambient manifolds.

Semi-slant submanifolds in almost Hermitian manifolds were studied by Papaghiuc [28]. Cabrerizo et al. [9, 10] studied semi-slant submanifolds in Sasakian manifolds. Slant and semi-slant submanifolds in almost product Riemannian manifolds were studied in [2, 24, 29]. Hemi-slant submanifolds were also studied in nearly Kenmotsu manifolds [4], LCS-manifolds [3] and locally product Riemannian manifolds [31].

Bishop and O'Neill [7] while studying examples of manifolds with negative sectional curvature, defined warped product manifolds by homothetically warping the product metric on a product manifold. Warped products are a natural generalisation of Riemannian products and they have found extensive applications in relativity. Most notably the Schwarzschild metric describing the gravitational field outside a spherical mass under certain assumptions and the Robertsen Walker metric (FLRW metric) are examples of warped product metrics. A natural example of warped product manifolds are surfaces of revolution. Hiepko [22] gave a characterisation for a Riemannian manifold to be the warped product of its submanifolds, generalising the deRham decomposition theorem for product manifolds. Later on Nölker [27] and Chen [15, 16, 19] initiated the study of extrinsic geometry of warped product manifolds.

Chen [17, 18] initiated the study of CR submanifolds immersed as warped products in Kähler manifolds. He proved that given any holomorphic submanifold M_T and totally real submanifold $M_\perp$ of a Kähler manifold, every warped product of the form $M_T \times_\lambda M_\perp$ in a Kähler manifold satisfies the inequality

$$||h||^2 \geq 2n_2||\text{grad}(\ln \lambda)||^2 \tag{1.1}$$

where λ is the warping function, n_2 is the dimension of $M_\perp$, $||h||^2$ is the squared norm of the second fundamental form and grad$(\ln \lambda)$ is the gradient of $\ln \lambda$. Bonanzinga and Matsumoto [8, 26, 25] continued the study in the setting of lcK manifolds. Nargis Jamal et al. [23] studied generic warped products in lcK manifolds. Further studies of semi-slant and hemi-slant submanifolds of lcK manifolds were carried out in [1, 30, 32]. Generic submanifolds, CR-submanifolds and pointwise semi-slant submanifolds immersed as warped products in lcK manifolds were studied by [23, 1].

We continue the study by considering pointwise bi-slant submanifolds in an lcK manifold. In particular we give characterisation theorems and establish Chen's inequality for the squared norm of the second fundamental form of pointwise bi-slant submanifolds immersed as warped products in an lcK manifold.

2 Preliminaries

Definition 2.1. A Hermitian Manifold $(\widetilde{M}^{2n}, J, g)$ is said to be a *locally conformal Kähler* (lcK) manifold if there exists an open cover $\{U_i\}_{i\in I}$ of $\widetilde{M}^{2n}$ and a family $\{f_i\}_{i\in I}$ of C^∞ functions $f_i : U_i \to \mathbb{R}$ such that for each $i \in I$, the metric

$$g_i = e^{-f_i} g|_{U_i} \tag{2.1}$$

on U_i is a Kähler metric.

Given an lcK manifold $(\widetilde{M}^{2n}, J, g)$, let U, V denote smooth sections of $T\widetilde{M}^{2n}$, then the local 1-forms df_i glue up to a globally defined closed 1-form ω, called the *Lee form*, and it satisfies the following equation

$$d\Omega = \Omega \wedge \omega \tag{2.2}$$

where $\Omega(U, V) = g(JU, V)$ is the fundamental 2-form associated to (J, g).

Denote by B the vector field equivalent to ω with respect to g, i.e. $\omega(U) = g(B, U)$. B is called the *Lee vector field*.

Let $\overline{\nabla}$ denote the Levi-Civita connection of $(\widetilde{M}^{2n}, g)$ and $\widetilde{\nabla}_i$ denote the Levi-Civita connection of the local metrics g_i for all $i \in I$. Then $\widetilde{\nabla}_i$ glue up to a globally defined torsion-free linear connection $\widetilde{\nabla}$ on $\widetilde{M}^{2n}$ given by

$$\widetilde{\nabla}_U V = \overline{\nabla}_U V - \frac{1}{2}\{\omega(U)V + \omega(V)U - g(U, V)B\} \tag{2.3}$$

where $U, V \in T\widetilde{M}^{2n}$ and satisfying

$$\widetilde{\nabla} g = \omega \otimes g \tag{2.4}$$

$\widetilde{\nabla}$ is called the *Weyl connection* of the lcK manifold $(\widetilde{M}^{2n}, J, g)$. As g_i are Kähler metrics, the almost complex structure J is parallel with respect to the Weyl connection, i.e. $\widetilde{\nabla} J = 0$. This gives

$$\overline{\nabla}_U JV = J\overline{\nabla}_U V + \frac{1}{2}\{\Theta(V)U - \omega(V)JU - g(U, V)A + \Omega(U, V)B\} \tag{2.5}$$

Now as ω is a closed form on $\widetilde{M}^{2n}$, we have

$$(\overline{\nabla}_U \omega)V = (\overline{\nabla}_V \omega)U \tag{2.6}$$

Let M^m be a Riemannian manifold isometrically immersed in an lcK manifold $(\widetilde{M}^{2n}, J, g)$. Let U, V, W denote smooth sections of TM^m and ξ, η denote smooth sections of $T^\perp M^m$.

The Gauss and Weingarten formulae with respect to the Riemannian connection of $\widetilde{M}^{2n}$ are given as

$$\overline{\nabla}_U V = \nabla_U V + h(U, V) \tag{2.7}$$

$$\overline{\nabla}_U \xi = -\mathfrak{A}_\xi U + \nabla^\perp_U \xi \tag{2.8}$$

where h is the second fundamental form, $\mathfrak{A}$ is the shape operator and $\nabla, \nabla^\perp$ are respectively the induced connections in the tangent bundle and the normal bundle of M^m with respect to $\overline{\nabla}$.

Let H denote the trace of h, then H is called the mean curvature vector of M^m in $(\widetilde{M}^{2n}, J, g)$ and is a smooth section of $T^\perp M^m$. We say M^m is a totally umbilic submanifold of $(\widetilde{M}^{2n}, J, g)$, if

$h(U,V) = g(U,V)H$. We say M^m is a totally geodesic submanifold of $(\widetilde{M}^{2n}, J, g)$, if $h(U,V) = 0$.

Let B^T, B^N denote the tangential and normal components of the Lee vector field B. Define

$$JU = PU + FU \qquad\qquad J\xi = t\xi + f\xi \tag{2.9}$$

where $PU, t\xi$ and $FU, f\xi$ are respectively the tangential and normal parts. Then, we have

$$\begin{aligned} P^2 + tF &= -I \qquad & f^2 + Ft &= -I \\ FP + fF &= 0 \qquad & tf + Pt &= 0 \end{aligned} \tag{2.10}$$

Bishop and O'Neill [7] defined warped product as

Definition 2.2. Let $(M_1^{n_1}, g_1)$ and $(M_2^{n_2}, g_2)$ be Riemmanian manifolds and let

$$\pi_1 : M_1 \times M_2 \to M_1 \text{ and } \pi_2 : M_1 \times M_2 \to M_2$$

be the canonical projections. Let $\lambda : M_1 \to (0, \infty)$ be a smooth function. Then the warped product manifold $(M, g) = M_1 \times {}_\lambda M_2$ is defined as the manifold $M_1 \times M_2$ equipped with the Riemannian metric

$$g = \pi_1^\star g_1 + \lambda^2 \pi_2^\star g_2 \tag{2.11}$$

Warped product manifolds are a generalization of the usual product of two Riemannian manifolds. In fact we have the following characterisation theorem

Theorem 2.3 ([22]). *Let (M^m, g) be a connected Riemannian manifold equipped with orthogonal, complementary, involutive distributions $\mathcal{D}_1$ and $\mathcal{D}_2$. Further let the leaves of $\mathcal{D}_1$ be totally geodesic and the leaves of $\mathcal{D}_2$ be extrinsic spheres in M^m, where by extrinsic spheres we mean totally umbilic submanifolds such that the mean curvature vector is parallel in the normal bundle. Then (M^m, g) is locally a warped product $(M, g) = M_1 \times {}_\lambda M_2$, where M_1 and M_2 respectively denote the leaves of $\mathcal{D}_1$ and $\mathcal{D}_2$ and $\lambda : M_1 \to (0, \infty)$ is a smooth function such that grad$(\ln \lambda)$ is the mean curvature vector of M_2 in M.*

Further, if (M^m, g) is simply connected and complete, then (M^m, g) is globally a warped product.

For $(M_1^{n_1}, g_1)$, $(M_2^{n_2}, g_2)$ and (M, g) denote respectively the Levi-Civita connections by ∇^1, ∇^2 and ∇. Given any smooth function $\lambda : M_1 \to \mathbb{R}$, let grad$(\lambda)$ denote the lift of the gradient vector field of λ to (M, g).

Theorem 2.4 ([22]). *Given a warped product manifold $(M, g) = M_1 \times {}_\lambda M_2$ of Riemannian manifolds $(M_1^{n_1}, g_1)$ and $(M_2^{n_2}, g_2)$, we have for all $X, Y \in \mathcal{L}(M_1)$ and $Z, W \in \mathcal{L}(M_2)$,*

$$\nabla_X Y = \nabla^1_X Y \tag{2.12}$$

$$\nabla_X Z = \nabla_Z X = X(\ln \lambda) Z \tag{2.13}$$

$$\nabla_Z W = \nabla^2_Z W - g(Z, W) grad(\ln \lambda) \tag{2.14}$$

It follows from Lemma 2.4 that $\mathcal{H} = -\text{grad}(\ln \lambda)$ is the mean curvature vector of M_2 in M.

3 Pointwise Bi-Slant Submanifolds of lcK manifolds

Let M^m be a Riemannian manifold isometrically immersed in an lcK manifold $(\widetilde{M}^{2n}, J, g)$. M^m is said to be a *pointwise bi-slant submanifold* if it admits two orthogonal complementary distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$, such that $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ are pointwise slant with slant functions $\theta_1, \theta_2 : M \to \left(0, \frac{\pi}{2}\right)$ and $\theta_1 \neq \theta_2$, i.e. $P^2 X = -\cos^2 \theta_1 X$, for every smooth vector field $X \in \mathcal{D}^{\theta_1}$ and $P^2 Z = -\cos^2 \theta_2 Z$, for every smooth vector field $Z \in \mathcal{D}^{\theta_2}$.

The tangent bundle and the normal bundle of a pointwise bi-slant submanifold admits an orthogonal decomposition as

$$TM^m = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \qquad\qquad T^\perp M^m = F\mathcal{D}^{\theta_1} \oplus F\mathcal{D}^{\theta_2} + \mu \tag{3.1}$$

where μ is the orthogonal complementary distribution of $F\mathcal{D}^{\theta_1} \oplus F\mathcal{D}^{\theta_2}$ in $T^\perp M^m$ and is an invariant subbundle of $T^\perp M^m$ with respect to J. It is easy to observe that for $i = 1, 2$,

$$\begin{aligned} P\mathcal{D}^{\theta_i} &= \mathcal{D}^{\theta_i} \qquad t(F\mathcal{D}^{\theta_i}) = \mathcal{D}^{\theta_i} \qquad t(\mu) = \{0\} \\ f(F\mathcal{D}^{\theta_i}) &= F\mathcal{D}^{\theta_i} \qquad f(\mu) = \mu \end{aligned} \tag{3.2}$$

Let M^m be a pointwise bi-slant manifold isometrically immersed in an lcK manifold $(\widetilde{M}^{2n}, J, g)$ such that the distributions $\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2}$ are both involutive. Let $M^{2n_1}_{\theta_1}$ and $M^{2n_2}_{\theta_2}$ respectively denote the leaves of $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$, where $2n_1 = \dim_{\mathbb{R}} \mathcal{D}^{\theta_1}$ and $2n_2 = \dim_{\mathbb{R}} \mathcal{D}^{\theta_2}$. We say M^m is a

- *mixed totally geodesic pointwise bi-slant submanifold* if $h(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2}) = \{0\}$.
- *pointwise bi-slant product submanifold* if M^m can be expressed locally as $M_{\theta_1} \times M_{\theta_2}$.
- *pointwise bi-slant warped product submanifold* if M^m can be expressed locally as the warped product $M_{\theta_1} \times {}_\lambda M_{\theta_2}$ for some smooth function $\lambda : M_{\theta_1} \to (0, \infty)$.

Let X, Y be smooth vector fields in $\mathcal{D}^{\theta_1}$ and Z, W be smooth vector fields in $\mathcal{D}^{\theta_2}$. Then we have,

Theorem 3.1. *Let M^m be a pointwise bi-slant submanifold of an lcK manifold $\widetilde{M}^{2n}$.*

- *the slant distribution $\mathcal{D}^{\theta_1}$ is involutive if and only if*

$$\begin{aligned} g(\mathfrak{A}_{FPX}Z - \mathfrak{A}_{FX}PZ, Y) - g(\mathfrak{A}_{FPY}Z - \mathfrak{A}_{FY}PZ, X) \\ = g(\nabla^\perp_X FY, FZ) - g(\nabla^\perp_Y FX, FZ) \end{aligned} \tag{3.3}$$

- *the leaves of the slant distribution $\mathcal{D}^{\theta_1}$ are totally geodesic in M^m if and only if*

$$\omega(\mathcal{D}^{\theta_2}) = \{0\} \text{ and } g(\mathfrak{A}_{FPX}Z - \mathfrak{A}_{FX}PZ, Y) + g(\nabla^\perp_Y FX, FZ) = 0 \tag{3.4}$$

- *the leaves of the slant distribution $\mathcal{D}^{\theta_1}$ are totally umbilic in M^m if and only if*

$$\begin{aligned} g(\mathfrak{A}_{FPX}Z - \mathfrak{A}_{FX}PZ, Y) + g(\nabla^\perp_Y FX, FZ) \\ = \sin^2\theta_1 \left(\frac{1}{2}\omega(Z) + g(\mathcal{H}, Z)\right) g(X, Y) \end{aligned} \tag{3.5}$$

for some smooth vector field $\mathcal{H} \in \mathcal{D}^{\theta_2}$.

Proof. From (2.5) and (2.10), we have

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\overline{\nabla}_X PY + \overline{\nabla}_X FY - \frac{1}{2}g(X,Y)JB - \frac{1}{2}g(PX,Y)B, JZ) \\ &= -g(J\overline{\nabla}_X PY, Z) - g(\mathfrak{A}_{FY}X, PZ) + g(\nabla^\perp_X FY, FZ) \\ &\quad - \frac{1}{2}g(X,Y)g(B,Z) - \frac{1}{2}g(PX,Y)g(B,JZ) \\ &= -g(\overline{\nabla}_X JPY, Z) + \frac{1}{2}g(PX,PY)g(B,Z) - g(\mathfrak{A}_{FY}X, PZ) \\ &\quad - \frac{1}{2}g(X,Y)g(B,Z) + g(\nabla^\perp_X FY, FZ) \\ &= \cos^2\theta_1 g(\nabla_X Y, Z) + g(\mathfrak{A}_{FPY}X, Z) + \frac{1}{2}\sin^2\theta_1 g(X,Y)g(B,Z) \\ &\quad - g(\mathfrak{A}_{FY}X, PZ) + g(\nabla^\perp_X FY, FZ) \end{aligned}$$

i.e. we have

$$\sin^2\theta_1 \left(g(\nabla_X Y, Z) + \frac{1}{2}g(X,Y)g(B,Z)\right) = g(\mathfrak{A}_{FPY}Z - \mathfrak{A}_{FY}PZ, X) + g(\nabla^\perp_X FY, FZ)$$

Hence, the result follows. □

Similarly, we have

Theorem 3.2. *Let M^m be a pointwise bi-slant submanifold of an lcK manifold $\widetilde{M}^{2n}$. Then*

- *the slant distribution $\mathcal{D}^{\theta_2}$ is involutive if and only if*

$$\begin{aligned} g(\mathfrak{A}_{FPZ}X - \mathfrak{A}_{FZ}PX, W) - g(\mathfrak{A}_{FPW}X - \mathfrak{A}_{FW}PX, Z) \\ = g(\nabla^{\perp}_{Z}FW, FX) - g(\nabla^{\perp}_{W}FZ, FX) \end{aligned} \tag{3.6}$$

- *the leaves of the slant distribution $\mathcal{D}^{\theta_2}$ are totally geodesic in M^m if and only if*

$$\omega(\mathcal{D}^{\theta_1}) = \{0\} \text{ and } g(\mathfrak{A}_{FPZ}X - \mathfrak{A}_{FZ}PX, W) + g(\nabla^{\perp}_{W}FZ, FX) = 0 \tag{3.7}$$

- *the leaves of the slant distribution $\mathcal{D}^{\theta_2}$ are totally umbilic in M^m if and only if*

$$\begin{aligned} g(\mathfrak{A}_{FPZ}X - \mathfrak{A}_{FZ}PX, W) + g(\nabla^{\perp}_{W}FZ, FX) \\ = \sin^2\theta_2 \left(\frac{1}{2}\omega(X) + g(\mathcal{H}, X)\right) g(Z, W) \end{aligned} \tag{3.8}$$

  for some smooth vector field $\mathcal{H} \in \mathcal{D}^{\theta_1}$.

Notations: Let $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ be the slant distributions on a pointwise bi-slant submanifold M^m of an lcK manifold $\widetilde{M}^{2n}$ such that both distributions are involutive and let M_{θ_1} and M_{θ_2} respectively denote the leaves of the distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ respectively. Then $\mathcal{D}^{\theta_1}(p,q) = T_{(p,q)}(M_{\theta_1} \times \{q\})$ and $\mathcal{D}^{\theta_2}(p,q) = T_{(p,q)}(\{p\} \times M_{\theta_2})$. Let $\mathcal{L}(M_{\theta_1})$ and $\mathcal{L}(M_{\theta_2})$ respectively denote the set of lifts of vector fields from M_{θ_1} and M_{θ_2} to M. Then $X \in \mathcal{L}(M_{\theta_1})$ if and only if $X|_{\{p\}\times M_{\theta_2}}$ is constant for every $p \in M_{\theta_1}$. Similarly, $Z \in \mathcal{L}(M_{\theta_2})$ if and only if $Z|_{M_{\theta_1}\times\{q\}}$ is constant for every $q \in M_{\theta_2}$. Also, if $\pi_{\theta_1} : M_{\theta_1} \times M_{\theta_2} \to M_{\theta_1}$ and $\pi_{\theta_2} : M_{\theta_1} \times M_{\theta_2} \to M_{\theta_2}$ are the canonical projections, we have $d\pi_{\theta_1}(\mathcal{L}(M_{\theta_1})) = TM_{\theta_1}$ and $d\pi_{\theta_2}(\mathcal{L}(M_{\theta_2})) = TM_{\theta_2}$. It is clear that a general vector field in $\mathcal{D}^{\theta_1}$ (respectively $\mathcal{D}^{\theta_2}$) need not be in $\mathcal{L}(M_{\theta_1})$ (respectively $\mathcal{L}(M_{\theta_2})$).

From here on we use X, Y to denote smooth vector fields in $\mathcal{L}(M_{\theta_1})$ and Z, W to denote smooth vector fields in $\mathcal{L}(M_{\theta_2})$

4 Some Lemmas

We give the following lemmas which will be used to prove our main results.

Lemma 4.1. *Given a pointwise bi-slant warped product submanifold $M = M_{\theta_1} \times {}_{\lambda}M_{\theta_2}$ in an lcK manifold $(\widetilde{M}^{2n}, J, g)$, we have for all $X, Y \in \mathcal{L}(M_{\theta_1})$ and $Z, W \in \mathcal{L}(M_{\theta_2})$,*

$$g(h(X,Z), FY) = g(h(Y,Z), FX) \tag{4.1}$$

$$g(h(X,Z), FW) = g(h(X,W), FZ) \tag{4.2}$$

$$g(h(X,Y), FZ) = g(h(X,Z), FY) - \frac{1}{2}g(X,Y)g(B, FZ) \tag{4.3}$$

$$g(h(Z,W), FX) = g(h(X,Z), FW) - \frac{1}{2}g(Z,W)g(B, FX) \tag{4.4}$$

$$X(\ln\lambda) = \frac{1}{2}g(B, X) \tag{4.5}$$

$$g(B, Z) = 0 \tag{4.6}$$

Proof. For all $X, Y \in \mathcal{L}(M_{\theta_1})$ and $Z, W \in \mathcal{L}(M_{\theta_2})$, we have using (2.5) and (2.13),

$$\begin{aligned} g(h(X,Z), FW) &= g(\overline{\nabla}_X Z, JW - PW) \\ &= -g(J\overline{\nabla}_X Z, W) - g(\nabla_X Z, PW) \\ &= -g(\overline{\nabla}_X JZ, W) - X(\ln\lambda)g(Z, PW) \end{aligned}$$

$$
\begin{aligned}
&= -g(\overline{\nabla}_X PZ, W) - g(\overline{\nabla}_X FZ, W) - X(\ln\lambda)g(Z, PW)\\
&= -X(\ln\lambda)g(PZ, W) + g(\mathfrak{A}_{FZ}X, W) - X(\ln\lambda)g(Z, PW)\\
&= g(h(X, W), FZ)
\end{aligned}
$$

which gives (4.2). Repeating the above calculation, we have

$$
\begin{aligned}
g(h(X,Z), FW) &= g(\overline{\nabla}_Z X, JW - PW)\\
&= -g(J\overline{\nabla}_Z X, W) - g(\nabla_Z X, PW)\\
&= -g(\overline{\nabla}_Z JX, W) - \frac{1}{2}g(JB, X)g(Z, W) - \frac{1}{2}g(B, X)g(JZ, W)\\
&\quad - X(\ln\lambda)g(Z, PW)\\
&= -g(\overline{\nabla}_Z PX, W) - g(\overline{\nabla}_Z FX, W) - \frac{1}{2}g(JB, X)g(Z, W)\\
&\quad - \frac{1}{2}g(B, X)g(JZ, W) - X(\ln\lambda)g(Z, PW)\\
&= -PX(\ln\lambda)g(Z, W) + g(\mathfrak{A}_{FX}Z, W) + \frac{1}{2}g(B, JX)g(Z, W)\\
&\quad - \frac{1}{2}g(B, X)g(PZ, W) - X(\ln\lambda)g(Z, PW)
\end{aligned}
$$

Using (4.2) and comparing symmetric and skew symmetric terms in Z and W we have,

$$
X(\ln\lambda) = \frac{1}{2}g(B, X)
$$

which gives (4.5) and

$$
g(h(X,Z), FW) = g(h(Z,W), FX) - PX(\ln\lambda)g(Z,W) + \frac{1}{2}g(B, PX + FX)g(Z, W)
$$

which on substituting from (4.5) gives (4.4). Similarly,

$$
\begin{aligned}
g(h(X,Z), FY) &= g(\overline{\nabla}_Z X, JY - PY)\\
&= -g(J\overline{\nabla}_Z X, Y) - g(\nabla_Z X, PY)\\
&= -g(\overline{\nabla}_Z JX, Y)\\
&= -g(\overline{\nabla}_Z PX, Y) - g(\overline{\nabla}_Z FX, Y)\\
&= g(\mathfrak{A}_{FX}Z, Y)\\
&= g(h(Y, Z), FX)
\end{aligned}
$$

which gives (4.1). Repeating the above calculation, we have

$$
\begin{aligned}
g(h(X,Z), FY) &= g(\overline{\nabla}_X Z, JY - PY)\\
&= -g(J\overline{\nabla}_X Z, Y) - g(\nabla_X Z, PY)\\
&= -g(\overline{\nabla}_X JZ, Y) - \frac{1}{2}g(JB, Z)g(X, Y) - \frac{1}{2}g(B, Z)g(JX, Y)\\
&= -g(\overline{\nabla}_X PZ, Y) - g(\overline{\nabla}_X FZ, Y) - \frac{1}{2}g(JB, Z)g(X, Y)\\
&\quad - \frac{1}{2}g(B, Z)g(JX, Y)
\end{aligned}
$$

$$= g(\mathfrak{A}_{FZ}X, Y) + \frac{1}{2}g(B, JZ)g(X, Y) - \frac{1}{2}g(B, Z)g(PX, Y)$$

Using (4.1) and comparing symmetric and skew symmetric terms in X and Y we have,

$$\frac{1}{2}g(B, Z) = 0$$

which gives (4.6) and

$$g(h(X, Z), FY) = g(h(X, Y), FZ) + \frac{1}{2}g(B, PZ + FZ)g(X, Y)$$

which on substituting from (4.6) gives (4.3). □

From (4.5) and (4.6) we have

Corollary 4.2. *Given a pointwise bi-slant warped product submanifold $M = M_{\theta_1} \times {}_\lambda M_{\theta_2}$ in an lcK manifold $(\widetilde{M}^{2n}, J, g)$, we have the Lee vector field B is orthogonal to the second factor and the warping function λ satisfies grad$(\ln \lambda) = \frac{1}{2}B^T$, where B^T denotes the tangential part of the Lee vector field along M.*

The above result extends the corresponding result for pointwise semi-slant warped product submanifolds of lcK manifolds proved in [1].

Corollary 4.3 ([1]). *Given $M = M_T \times {}_\lambda M_\theta$ is a pointwise semi-slant warped product submanifold of an lcK manifold $(\widetilde{M}^{2n}, J, g)$, we have the Lee vector field B is orthogonal to the pointwise slant distribution.*

Remark 4.4. Given a pointwise bi-slant warped product submanifold $M_{\theta_1} \times {}_\lambda M_{\theta_2}$ of an l.c.K manifold $\widetilde{M}^{2n}$, let $\{X_i, \beta_1 X_i\}_{i=1}^{p}$ and $\{Z_j, \beta_2 PZ_j\}_{j=1}^{q}$ respectively be local orthonormal frames of TM_{θ_1} and TM_{θ_2}. Then a local orthonormal frame of $\widetilde{M}^{2n}$ is

$$\left\{\widehat{X_i} = X_i, \widehat{PX_i} = \beta_1 PX_i\right\} \cup \left\{\widehat{Z_j} = \frac{Z_j}{\lambda}, \widehat{PZ_j} = \frac{\beta_2 PZ_j}{\lambda}\right\} \cup \left\{\widehat{\xi_k}, \widehat{J\xi_k}\right\}$$
$$\cup \left\{\widehat{FX_i} = \alpha_1 FX_i, \widehat{FPX_i} = \alpha_1\beta_1 FPX_i\right\} \cup \left\{\widehat{FZ_j} = \frac{\alpha_2 FZ_j}{\lambda}, \widehat{FPZ_j} = \frac{\alpha_2\beta_2 FPZ_j}{\lambda}\right\}$$

where $\alpha_i = \csc\theta_i$, $\beta_i = \sec\theta_i$ for $i = 1, 2$ and

$$\left\{\widehat{X_i}\widehat{PX_i} : 1 \le i \le n_1\right\} \text{ is an orthonormal basis of } \mathcal{D}^{\theta_1}$$
$$\left\{\widehat{Z_j}, \widehat{PZ_j} : 1 \le j \le n_2\right\} \text{ is an orthonormal basis of } \mathcal{D}^{\theta_2}$$
$$\left\{\widehat{FX_i}, \widehat{FPX_i} : 1 \le j \le n_1\right\} \text{ is an orthonormal basis of } F\mathcal{D}^{\theta_1}$$
$$\left\{\widehat{FZ_j}, \widehat{FPZ_j} : 1 \le j \le n_2\right\} \text{ is an orthonormal basis of } F\mathcal{D}^{\theta_2}$$
$$\left\{\widehat{\xi_k}, \widehat{J\xi_k} : 1 \le s \le \frac{n - 2n_1 - 2n_2}{2}\right\} \text{ is an orthonormal basis of } \mu$$

However, while $Z_j, \beta PZ_j \in \mathcal{L}(M_{\theta_2})$ we have $\widehat{Z_j}, \widehat{PZ_j} \notin \mathcal{L}(M_{\theta_2})$ in general, as λ is a function on M_{θ_1}. Also, note that

$$J\left(\widehat{Z_j}\right) = J\left(\frac{Z_j}{\lambda}\right) = \frac{PZ_j}{\lambda} + \frac{FZ_j}{\lambda} = \cos\theta_2\widehat{PZ_j} + \sin\theta_2\widehat{FZ_j}$$
$$J\left(\widehat{PZ_j}\right) = J\left(\sec\theta_2\frac{PZ_j}{\lambda}\right) = \frac{\sec\theta_2 P^2 Z_j}{\lambda} + \frac{\sec\theta_2 FPZ_j}{\lambda} = -\cos\theta_2\widehat{Z_j} + \sin\theta_2\widehat{FPZ_j}$$

5 Main Results

We first give a characterisation for pointwise bi-slant warped product submanifolds of lcK manifolds.

Theorem 5.1. *Let M^m be a pointwise bi-slant submanifold of an lcK manifold $\widetilde{M}^{2n}$. Then the following are equivalent*

(i) M^m is a pointwise bi-slant warped product submanifold $M_{\theta_1} \times {}_\lambda M_{\theta_2}$ of $\widetilde{M}^{2n}$

(ii) For every $X, Y \in \mathfrak{L}(M_{\theta_1})$ and $Z, W \in \mathfrak{L}(M_{\theta_2})$ we have

$$\begin{aligned} \omega(\mathcal{D}^{\theta_2}) = \{0\} \text{ and } g\left(\mathfrak{A}_{FPX}Z - \mathfrak{A}_{FX}PZ, Y\right) + g(\nabla^{\perp}_{Y}FX, FZ) = 0 \\ g(\mathfrak{A}_{FPZ}X - \mathfrak{A}_{FZ}PX, W) + g(\nabla^{\perp}_{W}FZ, FX) \\ = \sin^2\theta_2 \left(\frac{1}{2}\omega(X) - X(\ln\lambda)\right) g(Z, W) \end{aligned} \tag{5.1}$$

for some smooth function $\lambda : M_{\theta_1} \to (0, \infty)$.

(iii) For every $X \in \mathfrak{L}(M_{\theta_1})$ and $Z \in \mathfrak{L}(M_{\theta_2})$ we have

$$\omega(\mathcal{D}^{\theta_2}) = \{0\} \text{ and } \nabla_X Z = \nabla_Z X = \frac{1}{2}\omega(X)Z \tag{5.2}$$

Also, in this case we have the mean curvature vector $\mathcal{H}$ of M_{θ_2} in M^m is

$$\mathcal{H} = -grad(\ln\lambda) = -\frac{1}{2}B^T \tag{5.3}$$

where B^T is the tangential component of B along M.

Proof. (i)⇔(ii) This follows from Theorem 3.1, Theorem 3.2 and grad$(\ln\lambda) \in \mathfrak{L}(M_{\theta_1})$ which implies

$$\begin{aligned} g(\nabla_Z(\text{grad}(\ln\lambda)), X) &= ZX(\ln\lambda) - g(\text{grad}(\ln\lambda), \nabla_Z X) \\ &= [Z, X](\ln\lambda) - \nabla_Z X(\ln\lambda) \\ &= -\nabla_X Z(\ln\lambda) \\ &= g(Z, \nabla_X(\text{grad}(\ln\lambda))) \\ &= 0 \end{aligned}$$

as $Z(\ln\lambda) = 0$ and M_{θ_1} is totally geodesic in M. Also, (5.3) follows from Lemma 4.1 (4.5).
(i)⇔(iii) Assume (i), i.e. let $M = M_{\theta_1} \times {}_\lambda M_{\theta_2}$ be a pointwise bi-slant warped product submanifold. Then (5.2) and (5.3) follow from (2.13) and (4.5).

For the converse part, assume (iii), i.e. let M^m be a pointwise bi-slant submanifold of an lcK manifold $\widetilde{M}^{2n}$ such that (5.2) holds. Then for all $X, Y \in \mathfrak{L}(M_{\theta_1})$ and $Z, W \in \mathfrak{L}(M_{\theta_2})$ we have

$$\begin{aligned} g([X, Y], Z) &= g(\nabla_X Y - \nabla_Y X, Z) \\ &= -g(\nabla_X Z, Y) + g(\nabla_Y Z, X) \\ &= 0 \end{aligned}$$

which implies $\mathcal{D}^{\theta_1}$ is involutive.

$$g(\nabla_X Y, Z) = -g(\nabla_X Z, Y) = 0$$

which implies leaves of $\mathcal{D}^{\theta_1}$ are totally geodesic in M.

$$g([Z, W], X) = g(\nabla_Z W - \nabla_W Z, X)$$

$$= -g(\nabla_Z X, W) + g(\nabla_W X, Z)$$
$$= -\frac{1}{2}\omega(X)g(Z,W) + \frac{1}{2}\omega(X)g(W,Z) = 0$$

which implies $\mathcal{D}^{\theta_2}$ is involutive.

$$\begin{aligned} g(\nabla_Z W, X) &= -g(\nabla_Z X, W) \\ &= -\frac{1}{2}\omega(X)g(Z,W) \\ &= -\frac{1}{2}g(Z,W)g(B^T, X) \end{aligned}$$

which implies leaves of $\mathcal{D}^{\theta_2}$ are totally umbilical in M with mean curvature vector $-\frac{1}{2}B^T$. Finally, we have

$$g\left(\nabla_Z B^T, X\right) = \frac{1}{2}\omega\left(B^T\right)g(Z,X) = 0$$

which implies B^T is parallel in the normal bundle of M_{θ_2} in M.

Hence by Theorem 2.3 we have $M = M_{\theta_1} \times {}_\lambda M_{\theta_2}$ is a pointwise bi-slant warped product submanifold. □

Alghamdi, Chen, Uddin [1] gave a characterisation for a pointwise semi-slant submanifold of an lcK manifold to be locally a warped product submanifold $M_T \times {}_\lambda M_\theta$. Their characterisation follows as a corollary of our result.

Corollary 5.2 ([1]). *Let M be a pointwise semi-slant submanifold with invariant distribution $\mathcal{D}$ and a pointwise slant distribution $\mathcal{D}^\theta$ of an lcK manifold $(\widetilde{M}^{2n}, J, g)$. Then M is locally a warped product submanifold of the form $M_T \times {}_\lambda M_\theta$ if and only if for all $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\theta)$*

$$\mathfrak{A}_{FZ}JX - \mathfrak{A}_{FPZ}X = \sin^2\theta\left(X(\ln\lambda) - \frac{1}{2}\omega(X)\right)Z \tag{5.4}$$

We conclude our study of pointwise bi-slant warped product submanifolds of lcK manifolds by giving a Chen's type inequality for the squared norm of the second fundamental form.

Theorem 5.3. *Let $M = M_{\theta_1} \times {}_\lambda M_{\theta_2}$ be a pointwise bi-slant warped product submanifold in an lcK manifold $(\widetilde{M}^{2n}, J, g)$. Then the norm of the second fundamental form satisfies the inequality*

$$\begin{aligned} ||h||^2 \geq{}& \frac{n_1}{2}\sin^2\theta_2 \left\|B|_{F\mathcal{D}^{\theta_2}}\right\|^2 + \frac{n_2}{2}\sin^2\theta_1\left\|B|_{F\mathcal{D}^{\theta_1}}\right\|^2 + \sin\theta_2\, g\Big(H_{\mathcal{D}^{\theta_1}}|_{F\mathcal{D}^{\theta_2}}, B|_{F\mathcal{D}^{\theta_2}}\Big) \\ &+ \sin\theta_1\, g\Big(H_{\mathcal{D}^{\theta_2}}|_{F\mathcal{D}^{\theta_1}}, B|_{F\mathcal{D}^{\theta_1}}\Big) \end{aligned} \tag{5.5}$$

where $2n_1 = \dim_{\mathbb{R}}\mathcal{D}^{\theta_1}$, $2n_2 = \dim_{\mathbb{R}}\mathcal{D}^{\theta_2}$ and $H_{\mathcal{D}^{\theta_1}}$ and $H_{\mathcal{D}^{\theta_2}}$ are respectively the components of the mean curvature vector H of M in $\widetilde{M}^{2n}$ along $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$.
If equality holds then we have

- *Image$(h) \subseteq (F\mathcal{D}^{\theta_1} \oplus F\mathcal{D}^{\theta_2})$, and*
- *M is minimal in $\widetilde{M}^{2n}$, if and only if, M is mixed-totally geodesic in $\widetilde{M}^{2n}$.*

Proof.

$$\begin{aligned} ||h||^2 ={}& \left\|h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})\big|_{F\mathcal{D}^{\theta_1}}\right\|^2 + \left\|h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_2})\big|_{F\mathcal{D}^{\theta_1}}\right\|^2 + \left\|h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2})\big|_{F\mathcal{D}^{\theta_1}}\right\|^2 \\ &+ \left\|h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})\big|_{F\mathcal{D}^{\theta_2}}\right\|^2 + \left\|h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_2})\big|_{F\mathcal{D}^{\theta_2}}\right\|^2 + \left\|h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2})\big|_{F\mathcal{D}^{\theta_2}}\right\|^2 \end{aligned}$$

$$+ \left\| h(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_1})|_{\mu} \right\|^2 + \left\| h(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2})|_{\mu} \right\|^2 + \left\| h(\mathcal{D}^{\theta_2}, \mathcal{D}^{\theta_2})|_{\mu} \right\|^2$$

From (4.3) and Remark 4.4 we have

$$g\left(h\left(\widehat{X_i}, \widehat{Z_p}\right), \widehat{FX_j}\right) = \frac{\csc\theta_1}{\lambda}\left\{g\left(h\left(X_i, X_j\right), FZ_p\right) + \frac{1}{2}\delta_{ij} g\left(B, FZ_p\right)\right\}$$

$$\Longrightarrow \sin\theta_1 g\left(h\left(\widehat{X_i}, \widehat{Z_p}\right), \widehat{FX_j}\right) = \sin\theta_2 g\left(h\left(\widehat{X_i}, \widehat{X_j}\right), \widehat{FZ_p}\right) + \frac{1}{2}\sin\theta_2 \delta_{ij} g\left(B, \widehat{FZ_p}\right)$$

Similarly,

$$\sin\theta_1 g\left(h\left(\widehat{X_i}, \widehat{PZ_p}\right), \widehat{FX_j}\right) = \sin\theta_2 g\left(h\left(\widehat{X_i}, \widehat{X_j}\right), \widehat{FPZ_p}\right) + \frac{1}{2}\sin\theta_2 \delta_{ij} g\left(B, \widehat{FPZ_p}\right)$$

$$\sin\theta_1 g\left(h\left(\widehat{PX_i}, \widehat{Z_p}\right), \widehat{FPX_j}\right) = \sin\theta_2 g\left(h\left(\widehat{PX_i}, \widehat{PX_j}\right), \widehat{FZ_p}\right) + \frac{1}{2}\sin\theta_2 \delta_{ij} g\left(B, \widehat{FZ_p}\right)$$

$$\sin\theta_1 g\left(h\left(\widehat{PX_i}, \widehat{PZ_p}\right), \widehat{FPX_j}\right) = \sin\theta_2 g\left(h\left(\widehat{PX_i}, \widehat{PX_j}\right), \widehat{FPZ_p}\right) + \frac{1}{2}\sin\theta_2 \delta_{ij} g\left(B, \widehat{FPZ_p}\right)$$

$$\sin\theta_1 g\left(h\left(\widehat{PX_i}, \widehat{Z_p}\right), \widehat{FX_j}\right) = \sin\theta_2 g\left(h\left(\widehat{PX_i}, \widehat{X_j}\right), \widehat{FZ_p}\right)$$

$$\sin\theta_1 g\left(h\left(\widehat{PX_i}, \widehat{PZ_p}\right), \widehat{FX_j}\right) = \sin\theta_2 g\left(h\left(\widehat{PX_i}, \widehat{X_j}\right), \widehat{FPZ_p}\right)$$

$$\sin\theta_1 g\left(h\left(\widehat{X_i}, \widehat{Z_p}\right), \widehat{FPX_j}\right) = \sin\theta_2 g\left(h\left(\widehat{X_i}, \widehat{PX_j}\right), \widehat{FZ_p}\right)$$

$$\sin\theta_1 g\left(h\left(\widehat{X_i}, \widehat{PZ_p}\right), \widehat{FPX_j}\right) = \sin\theta_2 g\left(h\left(\widehat{X_i}, \widehat{PX_j}\right), \widehat{FPZ_p}\right)$$

which implies

$$\begin{aligned}
\left\| h(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_1}} \right\|^2 &= \cos^2\theta_1 \left\| h(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_1}} \right\|^2 + \sin^2\theta_1 \left\| h(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_1}} \right\|^2 \\
&= \cos^2\theta_1 \left\| h(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_1}} \right\|^2 + \sin^2\theta_2 \left\| h(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_1})|_{F\mathcal{D}^{\theta_2}} \right\|^2 \\
&\quad + \frac{1}{2}\sin^2\theta_2 \sum_{i,p}\left\{ g\left(B, \widehat{FZ_p}\right)^2 + g\left(B, \widehat{FPZ_p}\right)^2 \right\} \\
&\quad + \sin\theta_2 \sum_{i,p}\left\{ g\left(h\left(\widehat{X_i}, \widehat{X_i}\right), \widehat{FZ_p}\right) g\left(B, \widehat{FZ_p}\right) \right. \\
&\qquad + g\left(h\left(\widehat{X_i}, \widehat{X_i}\right), \widehat{FPZ_p}\right) g\left(B, \widehat{FPZ_p}\right) \\
&\qquad + g\left(h\left(\widehat{PX_i}, \widehat{PX_i}\right), \widehat{FZ_p}\right) g\left(B, \widehat{FZ_p}\right) \\
&\qquad \left. + g\left(h\left(\widehat{PX_i}, \widehat{PX_i}\right), \widehat{FPZ_p}\right) g\left(B, \widehat{FPZ_p}\right) \right\}
\end{aligned}$$

i.e.

$$\begin{aligned}
\sin^2\theta_1 \left\| h(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_1}} \right\|^2 &= \sin^2\theta_2 \left\| h(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_1})|_{F\mathcal{D}^{\theta_2}} \right\|^2 + \frac{2n_1}{4}\sin^2\theta_2 \left\| B|_{F\mathcal{D}^{\theta_2}} \right\|^2 \\
&\quad + \sin\theta_2 g\left(\sum_i \left\{ h\left(\widehat{X_i}, \widehat{X_i}\right) + h\left(\widehat{PX_i}, \widehat{PX_i}\right) \right\}\Big|_{F\mathcal{D}^{\theta_2}}, B|_{F\mathcal{D}^{\theta_2}} \right) \\
&\geq \frac{n_1}{2}\sin^2\theta_2 \| B|_{F\mathcal{D}^{\theta_2}} \|^2 + \sin\theta_2 g\left(H_{\mathcal{D}^{\theta_1}}|_{F\mathcal{D}^{\theta_2}}, B|_{F\mathcal{D}^{\theta_2}}\right)
\end{aligned}$$

As done above we have

$$\sin^2\theta_2 \left\| h(\mathcal{D}^{\theta_1}, \mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_2}} \right\|^2 = \sin^2\theta_1 \left\| h(\mathcal{D}^{\theta_2}, \mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_1}} \right\|^2 + \frac{2n_2}{4}\sin^2\theta_1 \left\| B|_{F\mathcal{D}^{\theta_1}} \right\|^2$$

$$+\sin\theta_1 g\left(\sum_p \left\{h\left(\widehat{Z_p},\widehat{Z_p}\right)+h\left(\widehat{PZ_p},\widehat{PZ_p}\right)\right\}\Big|_{F\mathcal{D}^{\theta_1}}, B|_{F\mathcal{D}^{\theta_1}}\right)$$
$$\geq \frac{n_2}{2}\sin^2\theta_1\|B|_{F\mathcal{D}^{\theta_1}}\|^2+\sin\theta_1 g\left(H_{\mathcal{D}^{\theta_2}}|_{F\mathcal{D}^{\theta_1}}, B|_{F\mathcal{D}^{\theta_1}}\right)$$

Combining we have (5.5).

If equality holds in (5.5), then the only non-zero components of $||h||^2$ are

$$\left\|h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})|_{F\mathcal{D}^{\theta_2}}\right\|^2 \quad , \quad \left\|h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_1}}\right\|^2,$$
$$\left\|h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_1}}\right\|^2 \text{ and } \left\|h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_2}}\right\|^2.$$

which implies $\text{Image}(h)\subseteq(F\mathcal{D}^{\theta_1}\oplus F\mathcal{D}^{\theta_2})$. Also, from the calculations above we have,

$$\left\|h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_1})|_{F\mathcal{D}^{\theta_2}}\right\|^2=0 \text{ if and only if } \left\|h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_1}}\right\|^2=0$$
$$\text{and}$$
$$\left\|h(\mathcal{D}^{\theta_2},\mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_1}}\right\|^2=0 \text{ if and only if } \left\|h(\mathcal{D}^{\theta_1},\mathcal{D}^{\theta_2})|_{F\mathcal{D}^{\theta_2}}\right\|^2=0$$

Hence, M is minimal in $\widetilde{M}^{2n}$, if and only if, M is mixed-totally geodesic in $\widetilde{M}^{2n}$. □

6 Example

Consider $\mathbb{C}^n=\mathbb{E}^{2n},J,g_0)$ where $\mathbb{E}^{2n}$ is the Euclidean space of dimension $2n$ with coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n)$ equipped with the standard Euclidean metric g_0 and the canonical almost complex structure

$$J(x_1,\ldots,x_n,y_1,\ldots,y_n)=(-y_1,\ldots,-y_n,x_1,\ldots,x_n) \tag{6.1}$$

Then $\mathbb{C}^n=(\mathbb{E}^{2n},J,g_0)$ is a flat Kähler manifold.

We use the following result about pointwise slant immersions.

Theorem 6.1 ([20] (Proposition 2.2)). *Given a Kähler manifold $(\widetilde{M}^{2n},J,g_0)$, let M_θ be a pointwise slant submanifold. Then for any smooth function $f:\widetilde{M}\to(0,\infty)$, we have M_θ is again a pointwise slant submanifold of the globally conformal Kähler (gcK) manifold $(\widetilde{M}^{2n},J,e^{-f}g_0)$ with the same slant angle.*

Example 6.2. Let $\mathbb{C}^4=(\mathbb{E}^8,J,g_0)$ be as defined above. Consider an open subset of $\mathbb{E}^4$ with $u_1u_2\neq1$, $u_3u_4\neq1$, $(u_1-u_2)\in\left(0,\frac{\pi}{4}\right)$ and $(u_3-u_4)\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$. Define the 4-dimensional submanifold M of $\mathbb{C}^4$ given by

$$\begin{aligned} x_1&=u_1\cos u_2 \quad , \quad y_1=u_1\sin u_2\\ x_2&=u_2\cos u_1 \quad , \quad y_2=u_2\sin u_1\\ x_3&=u_3\cos u_4 \quad , \quad y_3=u_3\sin u_4\\ x_4&=u_4\cos u_3 \quad , \quad y_4=u_4\sin u_3 \end{aligned} \tag{6.2}$$

An orthonormal frame of the tangent bundle TM of M is

$$X_1=\frac{1}{\sqrt{1+u_2^2}}\left(\cos u_2\frac{\partial}{\partial x_1}-u_2\sin u_1\frac{\partial}{\partial x_2}+\sin u_2\frac{\partial}{\partial y_1}+u_2\cos u_1\frac{\partial}{\partial y_2}\right)$$
$$X_2=\frac{1}{\sqrt{1+u_1^2}}\left(-u_1\sin u_2\frac{\partial}{\partial x_1}+\cos u_1\frac{\partial}{\partial x_2}+u_1\cos u_2\frac{\partial}{\partial y_1}+\sin u_1\frac{\partial}{\partial y_2}\right)$$
$$X_3=\frac{1}{\sqrt{1+u_4^2}}\left(\cos u_4\frac{\partial}{\partial x_3}-u_4\sin u_3\frac{\partial}{\partial x_4}+\sin u_4\frac{\partial}{\partial y_3}+u_4\cos u_3\frac{\partial}{\partial y_4}\right)$$

$$X_4 = \frac{1}{\sqrt{1+u_3^2}} \left(-u_3 \sin u_4 \frac{\partial}{\partial x_3} + \cos u_3 \frac{\partial}{\partial x_4} + u_3 \cos u_4 \frac{\partial}{\partial y_3} + \sin u_3 \frac{\partial}{\partial y_4} \right)$$

Then, M is a proper pointwise bi-slant submanifold with slant distributions given by $\mathcal{D}^{\theta_1} = \text{Span}\{X_1, X_2\}$ and $\mathcal{D}^{\theta_2} = \text{Span}\{X_3, X_4\}$. Also, the slant functions are given by

$$\cos^2\theta_1 = \frac{(u_1u_2-1)^2\cos^2(u_1-u_2)}{(1+u_1^2)(1+u_2^2)} \text{ , and , } \cos^2\theta_2 = \frac{(u_3u_4-1)^2\cos^2(u_3-u_4)}{(1+u_3^2)(1+u_4^2)}$$

where $\theta_1 \neq \theta_2$. Indeed we have $u_3u_4 \neq 1$ and $(u_3-u_4) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ implies $\theta_2(u_3,u_4) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, while for $u_1 = \frac{1}{4}, u_2 = -\frac{1}{4}$ we have $\theta_1\left(\frac{1}{4}, -\frac{1}{4}\right) < \frac{\pi}{4}$. Thus, for any point $\left(\frac{1}{4}, -\frac{1}{4}, u_3, u_4\right)$ of M we have $\theta_1\left(\frac{1}{4}, -\frac{1}{4}\right) \neq \theta_2(u_3,u_4)$.

It is straightforward to check that $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ are both involutive and totally geodesic in M. Let M_{θ_1} and M_{θ_2} be the leaves of $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ respectively. Then we have M is the Riemannian product $M = M_{\theta_1} \times M_{\theta_2}$ and the metric g_M induced on M from $\mathbb{C}^4$ is given by

$$g_M = g_1 + g_2 \tag{6.3}$$

where

$$g_1 = (1+u_2^2)du_1^2 + (1+u_1^2)du_2^2 \text{ , and , } g_2 = (1+u_4^2)du_3^2 + (1+u_3^2)du_4^2 \tag{6.4}$$

Now, for any non-constant positive smooth function $f = f(x_1, x_2, y_1, y_2)$ on $\mathbb{C}^4$, depending only on coordinates x_1, x_2, y_1, y_2, consider the Riemannian metric $\tilde{g} = e^{-f}g_0$, conformal to the standard metric g_0. Then, $\widetilde{M} = (\mathbb{E}^8, J, \tilde{g})$ is a globally conformal Kähler manifold and the metric on M induced from $\widetilde{M}$ is the warped product metric

$$\tilde{g}_M = \tilde{g}_1 + e^{-f}g_2 \tag{6.5}$$

where

$$\tilde{g}_1 = e^{-f}g_1 \tag{6.6}$$

is conformal to g_1 by the choice of f.

Hence from Theorem 6.1 we have $(M, \tilde{g}_M)$ is a proper pointwise bi-slant warped product submanifold of $\widetilde{M} = (\mathbb{E}^8, J, \tilde{g})$.

Also, as $f = f(x_1, x_2, y_1, y_2)$ is a non-constant positive smooth function on $\mathbb{C}^4$, depending only on coordinates x_1, x_2, y_1, y_2, from (6.2) we have that restricted to the submanifold M, the Lee form ω of $\widetilde{M}$ is given by

$$\omega = df = \frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2 \tag{6.7}$$

Hence, it follows that the Lee-vector field B is orthogonal to $\mathcal{D}^{\theta_2}$ and the warping function $\lambda = -e^{-\frac{f}{2}}|_M$ satisfies $\text{grad}(\ln\lambda) = \frac{1}{2}\text{grad}(f|_M) = \frac{1}{2}B^T$.

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