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EXTENSION OF RIGHT-RADAU-TYPE INEQUALITIES TO FRACTAL SETS

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Abstract. In this study, we introduce a novel local fractional integral identity and utilize it to extend classical right-Radau-type inequalities to fractal sets. Addressing such inequalities poses a significant challenge due to their inherent asymmetry, which complicates their analysis and generalization. By leveraging the concept of generalized convexity within the framework of local fractional integrals, we successfully overcome this difficulty and derive refined results that generalize the 2-point right-Radau inequality. The theoretical advancements are complemented by a practical application, demonstrating the efficacy and versatility of our approach in fractal analysis.

1 Introduction

Numerical integration methods are essential in scientific and engineering applications, where functions may lack closed-form antiderivatives or are only known at discrete points. Newton-Cotes formulas, such as the midpoint, trapezoidal, and Simpson's rules, provide straightforward approximations based on polynomial interpolation over equidistant nodes, making them suitable for regular functions. In contrast, Gaussian quadrature offers higher accuracy by optimizing node placement, achieving superior precision with fewer evaluations. This efficiency makes Gauss methods particularly advantageous for complex or computationally expensive integrands. Together, these techniques provide versatile tools for accurate numerical integration tailored to various problem requirements.

Convexity is a fundamental notion in mathematical analysis with significant implications in various fields such as optimization, economics, and approximation theory. A function $\Lambda : I \to \mathbb{R}$, defined on an interval I, is said to be convex if for all $u_1, u_2 \in I$ and $t \in [0, 1]$, the following inequality holds:

$$\Lambda(tu_1 + (1-t)u_2) \leq t\Lambda(u_1) + (1-t)\Lambda(u_2).$$

The concept of convexity serves as a robust and versatile tool in deriving error bounds for quadrature formulas, playing a pivotal role in the analysis of numerical integration methods. Indeed, convexity has been extensively employed to establish various inequalities related to Newton-Cotes formulas, providing valuable insights into their accuracy [2, 9, 15, 16, 18]. However, when it comes to error estimation for Gaussian quadrature rules, the literature reveals a striking gap. Apart from a few notable works, the application of convexity to derive sharp error bounds for Gauss-type formulas remains underexplored. This lack of comprehensive research highlights the need for further investigation, as such results could significantly enhance our understanding and improve the reliability of Gaussian quadrature in practical applications.

In [19], R. Radau defined the following two Gaussian-type quadrature formulas, which extend the classical Gauss quadrature to handle specific cases involving endpoint singularities:

The 2-point left-Radau rule is given by:

$$\int_{\lambda}^{\omega} \Lambda(u)\, du \approx \frac{1}{4}\left(\Lambda(\lambda) + 3\Lambda\left(\frac{\lambda+2\omega}{3}\right)\right).$$

The 2-point right-Radau rule is given by:

$$\int_{a}^{b} f(x)\, w(x)\, dx \approx \frac{1}{4}\left(3\Lambda\left(\frac{2\lambda+\omega}{3}\right) + \Lambda(\omega)\right).$$

The 2-point left-Radau rule is particularly designed for functions that are not defined at the right endpoint of the integration interval, while the 2-point right-Radau rule applies to functions that are not defined at the left endpoint. These formulas adapt the node distributions to accommodate such cases, ensuring accurate approximations even when the integrand has singular behavior at one of the endpoints. However, studying the error bounds of these rules remains somewhat delicate due to their non-symmetric nature.

Meftah et al. [14] provided the following error bounds of the 2-point left-Radau rule for differentiable convex function as follows:

Theorem 1.1. *Let* $\Lambda : [\lambda, \omega] \to \mathbb{R}$ *be a differentiable function on* $[\lambda, \omega]$ *such that* $\Lambda' \in L^1([\lambda, \omega])$. *If* $|\Lambda'|$ *is convex on* $[\lambda, \omega]$, *then we have*

$$\left|\frac{1}{4}\left(\Lambda(\omega) + 3\Lambda\left(\frac{2\lambda+\omega}{3}\right)\right) - \frac{1}{\omega-\lambda}\int_{\lambda}^{\omega} \Lambda(u) du\right| \leq \frac{25(\omega-\lambda)}{144}\left(\frac{157|\Lambda(\lambda)|+379\left|\Lambda\left(\frac{\lambda+2\omega}{3}\right)\right|+64|\Lambda(\omega)|}{600}\right).$$

In [20], the authors established the following 2-point right-Radau-type inequality.

Theorem 1.2. *Let* $\Lambda : [\lambda, \omega] \to \mathbb{R}$ *be a differentiable function on* $[\lambda, \omega]$ *such that* $\Lambda' \in L^1([\lambda, \omega])$. *If* $|\Lambda'|$ *is convex on* $[\lambda, \omega]$, *then we have*

$$\left|\frac{1}{4}\left(3\Lambda\left(\frac{2\lambda+\omega}{3}\right) + \Lambda(\omega)\right) - \frac{1}{\omega-\lambda}\int_{\lambda}^{\omega} \Lambda(u) du\right| \leq \frac{\omega-\lambda}{9}\left(\frac{1}{6}|\Lambda(\lambda)| + \frac{379}{384}\left|\Lambda\left(\frac{2\lambda+\omega}{3}\right)\right| + \frac{157}{384}|\Lambda(\omega)|\right).$$

Among the few works that have explored the application of convexity for error estimation in Gaussian quadrature formulas, some notable contributions stand out. Liu et al. [13] extended the results from [20] to the context of Riemann-Liouville fractional integrals, demonstrating the potential of these approaches in more general settings. Additionally, Berkane and coauthors [3] studied multiplicative right-Radau inequalities, further enriching the theoretical framework surrounding such inequalities, while Zhou and Du studied the same inequality via multiplicative Hadamard k-fractional integrals in [30]. Despite these valuable efforts, the body of research in this area remains limited, emphasizing the need for further exploration of convexity in the context of Gauss quadrature formulas.

Fractal sets are mathematical constructs characterized by their complex, self-similar structures that exhibit fine-scale irregularities. These sets often defy traditional geometric descriptions and are commonly found in nature, such as in the shapes of coastlines, mountains, and trees. Due to their non-differentiable and non-integer dimensional properties, classical calculus is insufficient for analyzing fractal phenomena. To address this limitation, fractal calculus has emerged as a powerful tool for studying functions and equations on fractal sets [7, 24, 25]. By extending traditional calculus to handle local fractional derivatives and integrals, this framework enables the modeling of physical processes on fractal domains, providing new insights into problems in areas such as anomalous diffusion, fractal heat transfer, and signal processing on fractal media. Thus, fractal calculus serves as a bridge between fractal geometry and mathematical analysis, offering a robust approach to understanding and describing the behavior of systems with fractal characteristics.

Fractional calculus is a generalization of classical calculus that extends the concepts of differentiation and integration to non-integer orders. Unlike traditional calculus, which deals with integer-order derivatives and integrals, fractional calculus allows for operations of arbitrary real or complex order, providing a powerful tool for modeling systems with memory, hereditary properties, or anomalous behavior. Fractal calculus, on the other hand, focuses on mathematical analysis on fractal sets, which are characterized by their non-integer dimensions and self-similarity.

The relationship between fractional calculus and fractal calculus has been a subject of significant interest, as both fields deal with phenomena that defy classical mathematical descriptions. Recent studies, such as the work by [insert citation here] [12], have explored this connection by examining the interaction between fractional operators and fractal functions. Specifically, the fractal dimension of a function plays a crucial role in understanding how fractional calculus affects its geometric properties. For instance, the Riemann–Liouville fractional integral tends to reduce the fractal dimension of a continuous function, while the fractional derivative can increase it, provided the derivative exists. These observations suggest that fractional calculus can be seen as a tool that modifies the geometric complexity of functions, offering insights into their fractal nature.

Moreover, the study [12] highlights that for Hölder continuous functions, the fractal dimension of the fractional integral decreases linearly, whereas the fractal dimension of the fractional derivative increases linearly. This behavior provides a deeper understanding of the mechanisms underlying fractional calculus and supports the rationality of its definitions. Additionally, investigations into other fractional operators, such as the Weyl-Marchaud fractional derivative and the Weyl fractional integral, further enrich our comprehension of these relationships.

In summary, fractional calculus and fractal calculus are interconnected through their shared focus on non-classical structures and behaviors. By bridging these two domains, researchers can reveal the internal relationships between fractional operators and fractal geometries, offering new perspectives for modeling complex systems in science and engineering.

Yang proposed the idea of generalized convexity tailored for fractal sets in [23]. The concept is formulated as follows:

Definition 1.3. Consider $\Lambda : I \subseteq \mathbb{R} \to \mathbb{R}^{\varrho}$. Λ is said generalized convex on I if for any $u_1, u_2 \in I$ and $t \in [0,1]$, the following inequality

$$\Lambda\left(tu_1 + (1-t)\,u_2\right) \leq t^{\varrho}\Lambda\left(u_1\right) + (1-t)^{\varrho}\,\Lambda\left(u_2\right)$$

holds.

Since the introduction of this concept, numerous research endeavors have been directed towards extending classical findings regarding integral inequalities to the realm of fractals. Notably, Lakhdari et al., as depicted in [8], used generalized convexity to establish corrected Simpson-type inequalities for local fractional integrals. Xu et al., documented in [28], formulated a one-parameter identity from which they derived some dual-simpson- and corrected dual-Simpson-type inequalities on fractal set. Li et al. made significant contributions by offering multiparameterized fractal inequalities via generalized (s,P)-convexity in [11], while in [29], the authors introduced Hermite-Hadamard as well as multiparameterized fractal inequalities via generalized α-convexity. Lakhdari et al. explored Hermite-Hadamard and Milne-type inequalities via fractal-fractional integrals in [10]. For further exploration of this topic, interested readers are directed to references [1, 6, 21, 26].

In [4], Bin-Mohsin et al. extended the result obtained by Meftah et al in Theorem 1.1 to the case of fractal sets in the following manner:

Theorem 1.4. *Let $\Lambda : [\lambda,\omega] \to \mathbb{R}^{\varrho}$ be a differentiable function on $[\lambda,\omega]$ such that $\Lambda \in D_{\varrho}[\lambda,\omega]$ and $\Lambda^{(\varrho)} \in C_{\varrho}[\lambda,\omega]$ with $\lambda < \omega$. If $\left|\Lambda^{(\varrho)}\right|$ is generalized convex on $[\lambda,\omega]$, then we have*

$$\begin{aligned}
&\left|\left(\tfrac{1}{4}\right)^{\varrho}\left(\Lambda(\lambda) + 3^{\varrho}\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)\right) - \tfrac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right| \\
&\leq \tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{\left(\left(\tfrac{31}{32}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)} - \left(\tfrac{7}{32}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)\left|\Lambda^{(\varrho)}(\lambda)\right|\right. \\
&\quad + \left(\left(\tfrac{43}{32}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)} - \left(\tfrac{27}{32}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|
\end{aligned}$$

$$+\left(2^{\varrho}\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}+\left(\frac{1}{4}\right)^{\varrho}\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|$$
$$+\left(\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}-\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}(\omega)\right|\Big\}.$$

Remark 1.5. When $\varrho \to 1$, and noting that $\left|\Lambda'\left(\frac{2\lambda+\omega}{3}\right)\right| \le \frac{1}{2}\left(|\Lambda'(\lambda)|+\left|\Lambda'\left(\frac{\lambda+2\omega}{3}\right)\right|\right)$ due to the convexity of $|\Lambda'|$, Theorem 1.4 reduces to Theorem 1.1.

In this article, we extend the 2-point right-Radau inequality to the framework of local fractional integrals. We commence by introducing a novel local fractional integral identity, which serves as the foundation for our subsequent results. The study concludes with several applications.

The paper is organized as follows: In Section 2, we recall some essential definitions and tools from fractal calculus that will be used throughout the article. The main results of the paper are presented in Section 3, where we introduce and analyze the core contributions. Section 4 discusses additional results obtained using the generalized Hölder inequality and the generalized power mean inequality, highlighting their implications in the context of the study. Section 5 is dedicated an application involving special means, illustrating the practical relevance of our findings. Finally, the paper concludes with a summary of the key results and potential future research directions in Section 6.

2 Preliminaries

In this section, we gather some fundamental definitions, notations, and key results from fractal calculus that will serve as the foundation for our subsequent analysis.

Definition 2.1 ([23]). A non-differentiable function $\Lambda : \mathbb{R} \to \mathbb{R}^{\varrho}$ is local fractional continuous at u_0 , if

$$\forall \epsilon > 0, \exists \delta > 0 : |\Lambda(u) - \Lambda(u_0)| < \epsilon^{\varrho}$$

holds for $|u-u_0| < \delta$, where $\epsilon, \delta \in \mathbb{R}$. We denote the set of all locally fractional continuous functions on (λ, ω) by $C_{\varrho}(\lambda, \omega)$.

Definition 2.2 ([23]). The local fractional derivative of $\Lambda(u)$ of order ϱ at $u = u_0$ is defined as:

$$\Lambda^{(\varrho)}(u_0) = \left.\frac{d^{\varrho}\Lambda(u)}{du^{\varrho}}\right|_{u=u_0} = \lim_{u\to u_0}\frac{\Delta^{\varrho}(\Lambda(u)-\Lambda(u_0))}{(u-u_0)^{\varrho}},$$

where $\Delta^{\varrho}(\Lambda(u)-\Lambda(u_0)) \cong \Gamma(\varrho+1)(\Lambda(u)-\Lambda(u_0))$.

If there exists $\Lambda^{k\varrho}(u) = \overbrace{D^{\varrho}D^{\varrho}...D^{\varrho}}^{k \text{ times}}\Lambda(u)$ for any $u \in I \subseteq R$, then we say that $\Lambda \in D_{k\varrho}(I)$, where $k = 1, 2, 3, ...$

Definition 2.3 ([23]). Let $\Lambda(u) \in C_{\varrho}[\lambda, \omega]$. Then the local fractional integral is defined by,

$${}_{\lambda}I_{\omega}^{\varrho}\Lambda(u) = \frac{1}{\Gamma(\varrho+1)}\int_{\lambda}^{\omega}\Lambda(t)(dt)^{\varrho} = \frac{1}{\Gamma(\varrho+1)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}\Lambda(u_j)(\Delta u_j)^{\varrho},$$

with $\Delta u_j = u_{j+1} - u_j$ and $\Delta t = \max\{\Delta u_1, \Delta u_2, ..., \Delta u_{N-1}\}$, where $[u_j, u_{j+1}], j = 0, 1, ..., N-1$ and $\lambda = u_0 < u_1 < ... < u_N = \omega$ is partition of interval $[\lambda, \omega]$.

Here, it follows that ${}_{\lambda}I_{\omega}^{\varrho}\Lambda(u) = 0$ if $\lambda = \omega$ and ${}_{\lambda}I_{\omega}^{\varrho}\Lambda(u) = -{}_{\omega}I_{\lambda}^{\varrho}\Lambda(u)$ if $\lambda < \omega$. If for any $u \in [\lambda, \omega]$, there exists ${}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)$, then we denoted by $\Lambda(u) \in I_u^{\varrho}[\lambda, \omega]$.

Lemma 2.4 ([23]).

1. Suppose that $\Lambda(u) = \Omega^{(\varrho)}(u) \in C_{\varrho}[\lambda, \omega]$ *, then we have*

$${}_{\lambda}I_{\omega}^{\varrho}\Lambda(u) = \Omega(\omega) - \Omega(\lambda).$$

2. Suppose that $\Lambda, \Omega \in D_{\varrho}[\lambda, \omega]$ *and* $\Lambda^{(\varrho)}(u), \Omega^{(\varrho)}(u) \in C_{\varrho}[\lambda, \omega]$*, then we have*

$${}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\Omega^{(\varrho)}(u) = \Lambda(u)\Omega(u)|_{\lambda}^{\omega} - {}_{\lambda}I_{\omega}^{\varrho}\Lambda^{(\varrho)}(u)\Omega(u).$$

Lemma 2.5 ([23]). *For $s \in \mathbb{R}$, the following identities hold*

$$\frac{d^{\varrho}u^{s\varrho}}{du^{\varrho}} = \frac{\Gamma(1+s\varrho)}{\Gamma(1+(s-1)\varrho)} u^{(s-1)\varrho},$$

$$\frac{1}{\Gamma(1+\varrho)} \int_{\lambda}^{\omega} u^{s\varrho} (du)^{\varrho} = \frac{\Gamma(1+s\varrho)}{\Gamma(1+(s+1)\varrho)} \left(\omega^{(s+1)\varrho} - \lambda^{(s+1)\varrho}\right).$$

Lemma 2.6 (Generalized Hölder's inequality [5]). *Let $\Lambda, \Omega \in C_{\varrho}[\lambda, \omega]$ and $|\Lambda(u)|^{p}, |\Omega(u)|^{q}$ where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, are both integrable under the frame of the fractal spaces, then we have*

$$\frac{1}{\Gamma(1+\varrho)} \int_{\lambda}^{\omega} |\Lambda(u)\,\Omega(u)| (du)^{\varrho} \le \left(\frac{1}{\Gamma(1+\varrho)} \int_{\lambda}^{\omega} |\Lambda(u)|^{p} (du)^{\varrho}\right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(1+\varrho)} \int_{\lambda}^{\omega} |\Omega(u)|^{q} (du)^{\varrho}\right)^{\frac{1}{q}}.$$

Lemma 2.7 (Generalized power mean inequality [27]). *Let $\Lambda, \Omega \in C_{\varrho}[\lambda, \omega]$ and $|\Lambda(u)|, |\Lambda(u)|\,|\Omega(u)|^{q}$ where $q > 1$ are both integrable under the frame of the fractal spaces, then we have*

$$\begin{aligned}
&\frac{1}{\Gamma(1+\varrho)} \int_{\lambda}^{\omega} |\Lambda(u)\,\Omega(u)| (du)^{\varrho} \\
&\le \left(\frac{1}{\omega-\lambda}\right)^{\varrho} \left(\left(\frac{1}{\Gamma(1+\varrho)} \int_{\lambda}^{\omega} (\omega-u)^{\varrho} |\Lambda(u)| (du)^{\varrho}\right)^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(1+\varrho)} \int_{\lambda}^{\omega} (\omega-u)^{\varrho} |\Lambda(u)|\,|\Omega(u)|^{q} (du)^{\varrho}\right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\frac{1}{\Gamma(1+\varrho)} \int_{\lambda}^{\omega} (u-\lambda)^{\varrho} |\Lambda(u)| (du)^{\varrho}\right)^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(1+\varrho)} \int_{\lambda}^{\omega} (u-\lambda)^{\varrho} |\Lambda(u)|\,|\Omega(u)|^{q} (du)^{\varrho}\right)^{\frac{1}{q}} \right).
\end{aligned}$$

3 Main results

This section is devoted to the main results of the article, where we begin by introducing a new fractal identity. Using this identity, we establish fractal left-Radau-type inequalities via the concept of generalized convexity.

Lemma 3.1. *Let $\Lambda : I \to \mathbb{R}^{\varrho}$ be a differentiable function on I°, $\lambda, \omega \in I^{\circ}$ with $\lambda < \omega$, and $\Lambda^{(\varrho)} \in C_{\varrho}[\lambda, \omega]$, then the following equality holds for $\varrho > 0$.*

$$\begin{aligned}
&\left(\tfrac{1}{4}\right)^{\varrho} \left(3^{\varrho}\Lambda\left(\tfrac{2\lambda+\omega}{3}\right) + \Lambda(\omega)\right) - \frac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\, {}_{\lambda}I_{\omega}^{\varrho}\Lambda(u) \\
&= \frac{(\omega-\lambda)^{\varrho}}{9^{\varrho}} \left(\frac{1}{\Gamma(\varrho+1)} \int_{0}^{1} t^{\varrho} \Lambda^{(\varrho)}\left((1-t)\lambda + t\tfrac{2\lambda+\omega}{3}\right) (dt)^{\varrho} \right. \\
&\quad - \frac{1}{\Gamma(\varrho+1)} \int_{0}^{1} \left(\tfrac{5}{4} - t\right)^{\varrho} \Lambda^{(\varrho)}\left((1-t)\tfrac{2\lambda+\omega}{3} + t\tfrac{\lambda+2\omega}{3}\right) (dt)^{\varrho} \\
&\quad \left. + \frac{1}{\Gamma(\varrho+1)} \int_{0}^{1} \left(t - \tfrac{1}{4}\right)^{\varrho} \Lambda^{(\varrho)}\left((1-t)\tfrac{\lambda+2\omega}{3} + t\omega\right) (dt)^{\varrho} \right),
\end{aligned}$$

where

$${}_{\lambda}I_{\omega}^{\varrho}\Lambda(u) = \frac{1}{\Gamma(\varrho+1)} \int_{\lambda}^{\frac{2\lambda+\omega}{3}} \Lambda(u)(du)^{\varrho} + \frac{1}{\Gamma(\varrho+1)} \int_{\frac{2\lambda+\omega}{3}}^{\frac{\lambda+2\omega}{3}} \Lambda(u)(du)^{\varrho} + \frac{1}{\Gamma(\varrho+1)} \int_{\frac{\lambda+2\omega}{3}}^{\omega} \Lambda(u)(du)^{\varrho}. \quad (3.1)$$

Proof. Let

$$\mathcal{J} = \mathcal{J}_1 - \mathcal{J}_2 + \mathcal{J}_3, \tag{3.2}$$

where

$$\mathcal{J}_1 = \frac{1}{\Gamma(\varrho+1)} \int_0^1 t^{\varrho} \Lambda^{(\varrho)} \left((1-t)\lambda + t\frac{2\lambda+\omega}{3} \right) (dt)^{\varrho},$$

$$\mathcal{J}_2 = \frac{1}{\Gamma(\varrho+1)} \int_0^1 \left(\frac{5}{4} - t\right)^{\varrho} \Lambda^{(\varrho)} \left((1-t)\frac{2\lambda+\omega}{3} + t\frac{\lambda+2\omega}{3} \right) (dt)^{\varrho}$$

and

$$\mathcal{J}_3 = \frac{1}{\Gamma(\varrho+1)} \int_0^1 \left(t - \frac{1}{4}\right)^{\varrho} \Lambda^{(\varrho)} \left((1-t)\frac{\lambda+2\omega}{3} + t\omega \right) (dt)^{\varrho}.$$

Using Lemma 2.4, $\mathcal{J}_1$ gives

$$\begin{aligned}
\mathcal{J}_1 = & \frac{3^{\varrho}}{(\omega-\lambda)^{\varrho}} t^{\varrho} \Lambda \left((1-t)\lambda + t\frac{2\lambda+\omega}{3} \right) \Big|_0^1 \\
& - \frac{3^{\varrho}\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}} \int_0^1 \Lambda \left((1-t)\lambda + t\frac{2\lambda+\omega}{3} \right) (dt)^{\varrho} \\
= & \frac{3^{\varrho}}{(\omega-\lambda)^{\varrho}} \Lambda \left(\frac{2\lambda+\omega}{3} \right) - \frac{9^{\varrho}\Gamma(\varrho+1)}{(\omega-\lambda)^{2\varrho}\Gamma(\varrho+1)} \int_{\lambda}^{\frac{2\lambda+\omega}{3}} \Lambda(u) (du)^{\varrho}.
\end{aligned} \tag{3.3}$$

Similarly, we obtain

$$\begin{aligned}
\mathcal{J}_2 = & \frac{3^{\varrho}}{(\omega-\lambda)^{\varrho}} \left(\frac{5}{4} - t\right)^{\varrho} \Lambda \left((1-t)\frac{2\lambda+\omega}{3} + t\frac{\lambda+2\omega}{3} \right) \Big|_0^1 \\
& + \frac{3^{\varrho}\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}} \int_0^1 \Lambda \left((1-t)\frac{2\lambda+\omega}{3} + t\frac{\lambda+2\omega}{3} \right) (dt)^{\varrho} \\
= & \frac{3^{\varrho}}{4^{\varrho}(\omega-\lambda)^{\varrho}} \Lambda \left(\frac{\lambda+2\omega}{3} \right) - \frac{15^{\varrho}}{4^{\varrho}(\omega-\lambda)^{\varrho}} \Lambda \left(\frac{2\lambda+\omega}{3} \right) \\
& + \frac{9^{\varrho}\Gamma(\varrho+1)}{(\omega-\lambda)^{2\varrho}\Gamma(\varrho+1)} \int_{\frac{2\lambda+\omega}{3}}^{\frac{\lambda+2\omega}{3}} \Lambda(u) (du)^{\varrho}
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\mathcal{J}_3 = & \frac{3^{\varrho}}{(\omega-\lambda)^{\varrho}} \left(t - \frac{1}{4}\right)^{\varrho} \Lambda \left((1-t)\frac{\lambda+2\omega}{3} + t\omega \right) \Big|_0^1 \\
& - \frac{3^{\varrho}\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}} \int_0^1 \Lambda \left((1-t)\frac{\lambda+2\omega}{3} + t\omega \right) (dt)^{\varrho} \\
= & \frac{9^{\varrho}}{4^{\varrho}(\omega-\lambda)^{\varrho}} \Lambda(\omega) + \frac{3^{\varrho}}{4^{\varrho}(\omega-\lambda)^{\varrho}} \Lambda \left(\frac{\lambda+2\omega}{3} \right) \\
& - \frac{9^{\varrho}\Gamma(\varrho+1)}{(\omega-\lambda)^{2\varrho}\Gamma(\varrho+1)} \int_{\frac{\lambda+2\omega}{3}}^{\omega} \Lambda(u) (du)^{\varrho}.
\end{aligned} \tag{3.5}$$

Using (3.3)-(3.5) in (3.2), then multiplying the resulting equality by $\frac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}$, we get the desired result. □

Theorem 3.2. *Let* $\Lambda : [\lambda, \omega] \to \mathbb{R}^{\varrho}$ *be a differentiable function on* $[\lambda, \omega]$ *such that* $\Lambda \in D_{\varrho}[\lambda, \omega]$ *and* $\Lambda^{(\varrho)} \in C_{\varrho}[\lambda, \omega]$ *with* $\lambda < \omega$. *If* $\left|\Lambda^{(\varrho)}\right|$ *is generalized convex on* $[\lambda, \omega]$, *then we have*

$$\begin{aligned}
&\left|\left(\tfrac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\tfrac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right| \\
\leq& \tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{\left(\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}-\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}(\lambda)\right|\right. \\
&+\left(2^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}+\left(\tfrac{1}{4}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right| \\
&+\left(\left(\tfrac{43}{32}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}-\left(\tfrac{27}{32}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right| \\
&\left.+\left(\left(\tfrac{31}{32}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}-\left(\tfrac{7}{32}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)\left|\Lambda^{(\varrho)}(\omega)\right|\right\}.
\end{aligned}$$

Proof. From Lemma 3.1 and properties of modulus, we have

$$\begin{aligned}
&\left|\left(\tfrac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\tfrac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right| \\
\leq& \tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{\tfrac{1}{\Gamma(\varrho+1)}\int_0^1 |t|^{\varrho}\left|\Lambda^{(\varrho)}\left((1-t)\lambda+t\tfrac{2\lambda+\omega}{3}\right)\right|(dt)^{\varrho}\right. \\
&+\tfrac{1}{\Gamma(\varrho+1)}\int_0^1 \left|\tfrac{5}{4}-t\right|^{\varrho}\left|\Lambda^{(\varrho)}\left((1-t)\tfrac{2\lambda+\omega}{3}+t\tfrac{\lambda+2\omega}{3}\right)\right|(dt)^{\varrho} \\
&\left.+\tfrac{1}{\Gamma(\varrho+1)}\int_0^1 \left|t-\tfrac{1}{4}\right|^{\varrho}\left|\Lambda^{(\varrho)}\left((1-t)\tfrac{\lambda+2\omega}{3}+t\omega\right)\right|(dt)^{\varrho}\right\}.
\end{aligned}$$

Using the generalized convexity of $\left|\Lambda^{(\varrho)}\right|$, we get

$$\begin{aligned}
&\left|\left(\tfrac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\tfrac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right| \\
\leq& \tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{\tfrac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}(\lambda)\right|+t^{\varrho}\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|\right)(dt)^{\varrho}\right. \\
&+\tfrac{1}{\Gamma(\varrho+1)}\int_0^1 \left|\tfrac{5}{4}-t\right|^{\varrho}\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|+t^{\varrho}\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right|\right)(dt)^{\varrho} \\
&\left.+\tfrac{1}{\Gamma(\varrho+1)}\int_0^1 \left|t-\tfrac{1}{4}\right|^{\varrho}\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right|+t^{\varrho}\left|\Lambda^{(\varrho)}(\omega)\right|\right)(dt)^{\varrho}\right\} \\
=& \tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{\left(\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}-\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}(\lambda)\right|\right. \\
&+\left(2^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}+\left(\tfrac{1}{4}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right| \\
&+\left(\left(\tfrac{43}{32}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}-\left(\tfrac{27}{32}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right| \\
&\left.+\left(\left(\tfrac{31}{32}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}-\left(\tfrac{7}{32}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)\left|\Lambda^{(\varrho)}(\omega)\right|\right\},
\end{aligned}$$

where we have used

$$\tfrac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}(1-t)^{\varrho}(dt)^{\varrho}=\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}-\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}, \tag{3.6}$$

$$\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{2\varrho}\,(dt)^\varrho=\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}, \tag{3.7}$$

$$\frac{1}{\Gamma(\varrho+1)}\int_0^1 \left|\tfrac{5}{4}-t\right|^\varrho (1-t)^\varrho\,(dt)^\varrho=\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}+\left(\tfrac{1}{4}\right)^\varrho\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}, \tag{3.8}$$

$$\frac{1}{\Gamma(\varrho+1)}\int_0^1 \left|\tfrac{5}{4}-t\right|^\varrho t^\varrho\,(dt)^\varrho=\left(\tfrac{5}{4}\right)^\varrho\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}-\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}, \tag{3.9}$$

$$\frac{1}{\Gamma(\varrho+1)}\int_0^1 \left|t-\tfrac{1}{4}\right|^\varrho (1-t)^\varrho\,(dt)^\varrho=\left(\tfrac{5}{32}\right)^\varrho\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}+\left(\tfrac{3}{32}\right)^\varrho\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)} \tag{3.10}$$

and

$$\frac{1}{\Gamma(\varrho+1)}\int_0^1 \left|t-\tfrac{1}{4}\right|^\varrho t^\varrho\,(dt)^\varrho=\left(\tfrac{31}{32}\right)^\varrho\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}-\left(\tfrac{7}{32}\right)^\varrho\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}. \tag{3.11}$$

Thus, the proof is completed. □

Corollary 3.3. *In Theorem 3.2, using the generalized convexity of* $\left|\Lambda^{(\varrho)}\right|$, *we get*

$$\begin{aligned}&\left|\left(\tfrac{1}{4}\right)^\varrho\left(3^\varrho\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\frac{\Gamma(\varrho+1)}{(\omega-\lambda)^\varrho}\,{}_\lambda I_\omega^\varrho\Lambda(u)\right|\\&\le\frac{(\omega-\lambda)^\varrho}{27^\varrho}\left\{\left(\left(\tfrac{155}{32}\right)^\varrho\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}+\left(\tfrac{5}{32}\right)^\varrho\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}(\lambda)\right|\right.\\&\quad\left.+\left(\left(\tfrac{73}{32}\right)^\varrho\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}+\left(\tfrac{103}{32}\right)^\varrho\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}(\omega)\right|\right\}.\end{aligned}$$

Corollary 3.4. *Letting* $\varrho\to1$, *Theorem 3.2 yields*

$$\begin{aligned}&\left|\tfrac{1}{4}\left(3\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\frac{1}{\omega-\lambda}\int_\lambda^\omega\Lambda(u)du\right|\\&\le\frac{\omega-\lambda}{9}\left(\tfrac{1}{6}\left|\Lambda(\lambda)\right|+\tfrac{19}{24}\left|\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)\right|+\tfrac{75}{192}\left|\Lambda\left(\tfrac{\lambda+2\omega}{3}\right)\right|+\tfrac{1}{4}\left|\Lambda(\omega)\right|\right).\end{aligned}$$

Remark 3.5. Corollary 3.4 can be seen as a refinement of the result presented in Theorem 1.2. Indeed, the latter can be recovered by observing that $\left|\Lambda'\left(\frac{\lambda+2\omega}{3}\right)\right|\le\frac{1}{2}\left(\left|\Lambda'\left(\frac{2\lambda+\omega}{3}\right)\right|+\left|\Lambda'(\omega)\right|\right)$. This inequality highlights the connection between the two results and underscores the improved precision offered by Theorem 3.2.

4 Additional results

In this section, we present some additional results obtained by using the generalized Hölder inequality and the generalized power mean inequality through the concepts of convexity and concavity.

Theorem 4.1. *Assume that all the assumptions of Theorem 3.2 are satisfied. If* $\left|\Lambda^{(\varrho)}\right|^q$ *is generalized convex. Then we have*

$$\left|\left(\tfrac{1}{4}\right)^\varrho\left(3^\varrho\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\frac{\Gamma(\varrho+1)}{(\omega-\lambda)^\varrho}\,{}_\lambda I_\omega^\varrho\Lambda(u)\right|$$

$$\leq \tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left(\tfrac{\Gamma(1+p\varrho)}{\Gamma(1+(p+1)\varrho)}\right)^{\frac{1}{p}}\left(\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)^{\frac{1}{q}}\left\{\left(\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$+\left(\tfrac{5^{p+1}-1}{4^{p+1}}\right)^{\frac{\varrho}{p}}\left(\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^{q}+\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$\left.+\left(\tfrac{1+3^{p+1}}{4^{p+1}}\right)^{\frac{\varrho}{p}}\left(\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^{q}+\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)^{\frac{1}{q}}\right\},$$

where $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.1, properties of modulus among with generalized Hölder inequality, we have

$$\left|\left(\tfrac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\tfrac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right|$$
$$\leq \tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1|t|^{p\varrho}(dt)^{\varrho}\right)^{\frac{1}{p}}\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left|\Lambda^{(\varrho)}\left((1-t)\lambda+t\tfrac{2\lambda+\omega}{3}\right)\right|^{q}(dt)^{\varrho}\right)^{\frac{1}{q}}\right.$$
$$+\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left|\tfrac{5}{4}-t\right|^{p\varrho}(dt)^{\varrho}\right)^{\frac{1}{p}}\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left|\Lambda^{(\varrho)}\left((1-t)\tfrac{2\lambda+\omega}{3}+t\tfrac{\lambda+2\omega}{3}\right)\right|^{q}(dt)^{\varrho}\right)^{\frac{1}{q}}$$
$$\left.+\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left|t-\tfrac{1}{4}\right|^{p\varrho}(dt)^{\varrho}\right)^{\frac{1}{p}}\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left|\Lambda^{(\varrho)}\left((1-t)\tfrac{\lambda+2\omega}{3}+t\omega\right)\right|^{q}(dt)^{\varrho}\right)^{\frac{1}{q}}\right\}.$$

Using the generalized convexity of $\left|\Lambda^{(\varrho)}\right|^{q}$, we get

$$\left|\left(\tfrac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\tfrac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right|$$
$$\leq \tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1|t|^{p\varrho}(dt)^{\varrho}\right)^{\frac{1}{p}}\right.$$
$$\times\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+t^{\varrho}\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^{q}\right)(dt)^{\varrho}\right)^{\frac{1}{q}}$$
$$+\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left|\tfrac{5}{4}-t\right|^{p\varrho}(dt)^{\varrho}\right)^{\frac{1}{p}}$$
$$\times\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^{q}+t^{\varrho}\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right|^{q}\right)(dt)^{\varrho}\right)^{\frac{1}{q}}$$
$$+\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left|t-\tfrac{1}{4}\right|^{p\varrho}(dt)^{\varrho}\right)^{\frac{1}{p}}$$
$$\left.\times\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right|^{q}+t^{\varrho}\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)(dt)^{\varrho}\right)^{\frac{1}{q}}\right\}$$
$$=\tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left(\tfrac{\Gamma(1+p\varrho)}{\Gamma(1+(p+1)\varrho)}\right)^{\frac{1}{p}}\left(\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)^{\frac{1}{q}}\left\{\left(\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{5^{p+1}-1}{4^{p+1}}\right)^{\frac{\varrho}{p}}\left(\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}+\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1+3^{p+1}}{4^{p+1}}\right)^{\frac{\varrho}{p}}\left(\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}+\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)^{\frac{1}{q}}\Bigg\},$$

where we have used

$$\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{p\varrho}\,(dt)^{\varrho}=\frac{\Gamma(1+p\varrho)}{\Gamma(1+(p+1)\varrho)}, \tag{4.1}$$

$$\frac{1}{\Gamma(\varrho+1)}\int_0^1 \left|\frac{5}{4}-t\right|^{p\varrho}(dt)^{\varrho}=\frac{\Gamma(1+p\varrho)}{\Gamma(1+(p+1)\varrho)}\left(\left(\frac{5}{4}\right)^{p+1}-\left(\frac{1}{4}\right)^{p+1}\right)^{\varrho}, \tag{4.2}$$

$$\frac{1}{\Gamma(\varrho+1)}\int_0^1 \left|t-\frac{1}{4}\right|^{p\varrho}(dt)^{\varrho}=\frac{\Gamma(1+p\varrho)}{\Gamma(1+(p+1)\varrho)}\left(\left(\frac{1}{4}\right)^{p+1}+\left(\frac{3}{4}\right)^{p+1}\right)^{\varrho} \tag{4.3}$$

and

$$\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}\,(dt)^{\varrho}=\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\,(dt)^{\varrho}=\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}.$$

The proof is completed. □

Corollary 4.2. *In Theorem 4.1, using the generalized convexity of* $\left|\Lambda^{(\varrho)}\right|^q$, *we get*

$$\left|\left(\frac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\frac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\frac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right|$$
$$\leq\frac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left(\frac{\Gamma(1+p\varrho)}{\Gamma(1+(p+1)\varrho)}\right)^{\frac{1}{p}}\left(\frac{2^{\varrho}\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)^{\frac{1}{q}}\left\{\left(\frac{5^{\varrho}\left|\Lambda^{(\varrho)}(\lambda)\right|+\left|\Lambda^{(\varrho)}(\omega)\right|}{6^{\varrho}}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{5^{p+1}+1}{4^{p+1}}\right)^{\frac{\varrho}{p}}\left(\frac{\left|\Lambda^{(\varrho)}(\lambda)\right|+\left|\Lambda^{(\varrho)}(\omega)\right|}{2^{\varrho}}\right)^{\frac{1}{q}}+\left(\frac{3^{p+1}+1}{4^{p+1}}\right)^{\frac{\varrho}{p}}\left(\frac{\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+2^{\varrho}\left|\Lambda^{(\varrho)}(\omega)\right|^{q}}{3^{\varrho}}\right)^{\frac{1}{q}}\right\}.$$

Theorem 4.3. *Assume that all the assumptions of Theorem 3.2 are satisfied. If* $\left|\Lambda^{(\varrho)}\right|^q$ *is generalized convex for* $q>1$. *Then we have*

$$\left|\left(\frac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\frac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\frac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right|$$
$$\leq\frac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left(\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)^{1-\frac{1}{q}}\left\{\left(\left(\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}-\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{3}{2}\right)^{\varrho\left(1-\frac{1}{q}\right)}\left(\left(\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}+\frac{\Gamma(1+\varrho)}{4^{\varrho}\Gamma(1+2\varrho)}\right)\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}+\left(\frac{5^{\varrho}\Gamma(1+\varrho)}{4^{\varrho}\Gamma(1+2\varrho)}-\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\left(\frac{5}{8}\right)^{\varrho\left(1-\frac{1}{q}\right)}\left(\left(\frac{3^{\varrho}\Gamma(1+\varrho)}{32^{\varrho}\Gamma(1+2\varrho)}+\frac{5^{\varrho}\Gamma(1+2\varrho)}{32^{\varrho}\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}\right.$$
$$\left.\left.++\left(\frac{31^{\varrho}\Gamma(1+2\varrho)}{32^{\varrho}\Gamma(1+3\varrho)}-\frac{7^{\varrho}\Gamma(1+\varrho)}{32^{\varrho}\Gamma(1+2\varrho)}\right)\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)^{\frac{1}{q}}\right\}.$$

Proof. From Lemma 3.1, properties of modulus among with generalized power mean inequality, we have

$$\left|\left(\frac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\frac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\frac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right|$$

$$\leq \frac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{\left(\frac{1}{\Gamma(\varrho+1)}\int_{0}^{1}|t|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\varrho+1)}\int_{0}^{1}|t|^{\varrho}\left|\Lambda^{(\varrho)}\left((1-t)\lambda+t\frac{2\lambda+\omega}{3}\right)\right|^{q}(dt)^{\varrho}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_{0}^{1}\left|\frac{5}{4}-t\right|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\varrho+1)}\int_{0}^{1}\left|\frac{5}{4}-t\right|^{\varrho}\left|\Lambda^{(\varrho)}\left((1-t)\frac{2\lambda+\omega}{3}+t\frac{\lambda+2\omega}{3}\right)\right|^{q}(dt)^{\varrho}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1}{\Gamma(\varrho+1)}\int_{0}^{1}\left|t-\frac{1}{4}\right|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\varrho+1)}\int_{0}^{1}\left|t-\frac{1}{4}\right|^{\varrho}\left|\Lambda^{(\varrho)}\left((1-t)\frac{\lambda+2\omega}{3}+t\omega\right)\right|^{q}(dt)^{\varrho}\right)^{\frac{1}{q}}\right\}.$$

Using the generalized convexity of $\left|\Lambda^{(\varrho)}\right|^{q}$, we get

$$\left|\left(\frac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\frac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\frac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right|$$
$$\leq \frac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{\left(\frac{1}{\Gamma(\varrho+1)}\int_{0}^{1}|t|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\frac{1}{\Gamma(\varrho+1)}\int_{0}^{1}|t|^{\varrho}\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+t^{\varrho}\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}\right)(dt)^{\varrho}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_{0}^{1}\left|\frac{5}{4}-t\right|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\Gamma(\varrho+1)}\int_{0}^{1}\left|\frac{5}{4}-t\right|^{\varrho}\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}+t^{\varrho}\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}\right)(dt)^{\varrho}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_{0}^{1}\left|t-\frac{1}{4}\right|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}$$
$$\left.\times\left(\frac{1}{\Gamma(\varrho+1)}\int_{0}^{1}\left|t-\frac{1}{4}\right|^{\varrho}\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}+t^{\varrho}\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)(dt)^{\varrho}\right)^{\frac{1}{q}}\right\}$$
$$=\frac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{\left(\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)^{1-\frac{1}{q}}\left(\frac{\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}}{\Gamma(\varrho+1)}\int_{0}^{1}t^{\varrho}(1-t)^{\varrho}(dt)^{\varrho}+\frac{\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}}{\Gamma(\varrho+1)}\int_{0}^{1}t^{2\varrho}(dt)^{\varrho}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\left(\frac{3}{2}\right)^{\varrho}\right)^{\frac{q-1}{q}}\left(\frac{\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}}{\Gamma(\varrho+1)}\int_{0}^{1}\left|\frac{5}{4}-t\right|^{\varrho}(1-t)^{\varrho}(dt)^{\varrho}+\frac{\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}}{\Gamma(\varrho+1)}\int_{0}^{1}\left|\frac{5}{4}-t\right|^{\varrho}t^{\varrho}(dt)^{\varrho}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\left(\frac{5}{8}\right)^{\varrho}\right)^{\frac{q-1}{q}}\left(\frac{\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}}{\Gamma(\varrho+1)}\int_{0}^{1}\left|t-\frac{1}{4}\right|^{\varrho}(1-t)^{\varrho}(dt)^{\varrho}+\frac{\left|\Lambda^{(\varrho)}(\omega)\right|^{q}}{\Gamma(\varrho+1)}\int_{0}^{1}\left|t-\frac{1}{4}\right|^{\varrho}t^{\varrho}(dt)^{\varrho}\right)^{\frac{1}{q}}\right\}$$
$$=\frac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left(\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)^{1-\frac{1}{q}}\left\{\left(\left(\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}-\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{3}{2}\right)^{\varrho\left(1-\frac{1}{q}\right)}\left(\left(\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}+\frac{\Gamma(1+\varrho)}{4^{\varrho}\Gamma(1+2\varrho)}\right)\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}\left(\frac{5^{\varrho}\Gamma(1+\varrho)}{4^{\varrho}\Gamma(1+2\varrho)}-\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{5}{8}\right)^{\varrho\left(1-\frac{1}{q}\right)}\left(\left(\tfrac{3^{\varrho}\Gamma(1+\varrho)}{32^{\varrho}\Gamma(1+2\varrho)}+\tfrac{5^{\varrho}\Gamma(1+2\varrho)}{32^{\varrho}\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right|^{q}\right.$$

$$\left.+\,+\left(\tfrac{31^{\varrho}\Gamma(1+2\varrho)}{32^{\varrho}\Gamma(1+3\varrho)}-\tfrac{7^{\varrho}\Gamma(1+\varrho)}{32^{\varrho}\Gamma(1+2\varrho)}\right)\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)^{\frac{1}{q}}\Bigg\},$$

where we have used (3.6)-(3.11) and

$$\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\,(dt)^{\varrho}=\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)},\tag{4.4}$$

$$\frac{1}{\Gamma(\varrho+1)}\int_0^1 \left|\tfrac{5}{4}-t\right|^{\varrho}(dt)^{\varrho}=\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\left(\tfrac{3}{2}\right)^{\varrho}\tag{4.5}$$

and

$$\frac{1}{\Gamma(\varrho+1)}\int_0^1 \left|t-\tfrac{1}{4}\right|^{\varrho}(dt)^{\varrho}=\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\left(\tfrac{5}{8}\right)^{\varrho}.\tag{4.6}$$

This completes the proof. □

Corollary 4.4. *In Theorem 4.3, using the generalized convexity of* $\left|\Lambda^{(\varrho)}\right|^{q}$*, we get*

$$\left|\left(\tfrac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\tfrac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right|$$

$$\leq\tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left(\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}\right)^{1-\frac{1}{q}}\left\{\left(\left(\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}-\tfrac{\Gamma(1+2\varrho)}{3^{\varrho}\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+\tfrac{\Gamma(1+2\varrho)}{3^{\varrho}\Gamma(1+3\varrho)}\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$+\left(\tfrac{3^{\varrho}}{2^{\varrho}}\right)^{1-\frac{1}{q}}\left(\left(\tfrac{\Gamma(1+2\varrho)}{3^{\varrho}\Gamma(1+3\varrho)}+\tfrac{7^{\varrho}\Gamma(1+\varrho)}{12^{\varrho}\Gamma(1+2\varrho)}\right)\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+\left(\tfrac{11^{\varrho}\Gamma(1+\varrho)}{12^{\varrho}\Gamma(1+2\varrho)}-\tfrac{\Gamma(1+2\varrho)}{3^{\varrho}\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)^{\frac{1}{q}}$$

$$\left.+\left(\tfrac{5^{\varrho}}{8^{\varrho}}\right)^{1-\frac{1}{q}}\left(\left(\tfrac{\Gamma(1+\varrho)}{32^{\varrho}\Gamma(1+2\varrho)}+\tfrac{5^{\varrho}\Gamma(1+2\varrho)}{96^{\varrho}\Gamma(1+3\varrho)}\right)\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+\left(\tfrac{103^{\varrho}\Gamma(1+2\varrho)}{96^{\varrho}\Gamma(1+3\varrho)}-\tfrac{5^{\varrho}\Gamma(1+\varrho)}{32^{\varrho}\Gamma(1+2\varrho)}\right)\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)^{\frac{1}{q}}\right\}.$$

Theorem 4.5. *Assume that all the assumptions of Theorem 3.2 are satisfied. If* $\left|\Lambda^{(\varrho)}\right|^{q}$ *is generalized concave. Then we have*

$$\left|\left(\tfrac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\tfrac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right|$$

$$\leq\tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left(\tfrac{(\omega-\lambda)^{\varrho}}{3^{\varrho}\Gamma(\varrho+1)}\right)^{\frac{1}{q}}\left(\tfrac{\Gamma(1+p\varrho)}{\Gamma(1+(p+1)\varrho)}\right)^{\frac{1}{p}}\left\{\left(\tfrac{5^{p+1}-1}{4^{p+1}}\right)^{\frac{\varrho}{p}}\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+\omega}{2}\right)\right|\right.$$

$$\left.+\left|\Lambda^{(\varrho)}\left(\tfrac{5\lambda+\omega}{6}\right)\right|+\left(\tfrac{1+3^{p+1}}{4^{p+1}}\right)^{\frac{\varrho}{p}}\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+5\omega}{6}\right)\right|\right\},$$

where $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.1, properties of modulus, generalized Hölder inequality and generalized concavity of $\left|\Lambda^{(\varrho)}\right|^{q}$, we have

$$\left|\left(\tfrac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\tfrac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right|$$

$$\leq\tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1|t|^{p\varrho}(dt)^{\varrho}\right)^{\frac{1}{p}}\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left|\Lambda^{(\varrho)}\left((1-t)\lambda+t\tfrac{2\lambda+\omega}{3}\right)\right|^{q}(dt)^{\varrho}\right)^{\frac{1}{q}}\right.$$

$$+\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left|\tfrac{5}{4}-t\right|^{p\varrho}(dt)^{\varrho}\right)^{\frac{1}{p}}\left(\tfrac{1}{\Gamma(\varrho+1)}\int_0^1\left|\Lambda^{(\varrho)}\left((1-t)\tfrac{2\lambda+\omega}{3}+t\tfrac{\lambda+2\omega}{3}\right)\right|^{q}(dt)^{\varrho}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 \left|t-\tfrac{1}{4}\right|^{p\varrho}(dt)^\varrho\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 \left|\Lambda^{(\varrho)}\left((1-t)\tfrac{\lambda+2\omega}{3}+t\omega\right)\right|^q (dt)^\varrho\right)^{\frac{1}{q}}\Bigg\}$$

$$\leq \tfrac{(\omega-\lambda)^\varrho}{9^\varrho}\left\{\left(\tfrac{\Gamma(1+p\varrho)}{\Gamma(1+(p+1)\varrho)}\right)^{\frac{1}{p}}\left(\tfrac{(\omega-\lambda)^\varrho}{3^\varrho\Gamma(\varrho+1)}\right)^{\frac{1}{q}}\left|\Lambda^{(\varrho)}\left(\tfrac{5\lambda+\omega}{6}\right)\right|\right.$$

$$+\left(\tfrac{\Gamma(1+p\varrho)}{\Gamma(1+(p+1)\varrho)}\left(\tfrac{5^{p+1}-1}{4^{p+1}}\right)^\varrho\right)^{\frac{1}{p}}\left(\tfrac{(\omega-\lambda)^\varrho}{3^\varrho\Gamma(\varrho+1)}\right)^{\frac{1}{q}}\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+\omega}{2}\right)\right|$$

$$\left.+\left(\tfrac{\Gamma(1+p\varrho)}{\Gamma(1+(p+1)\varrho)}\left(\left(\tfrac{1}{4}\right)^{p+1}+\left(\tfrac{3}{4}\right)^{p+1}\right)^\varrho\right)^{\frac{1}{p}}\left(\tfrac{(\omega-\lambda)^\varrho}{3^\varrho\Gamma(\varrho+1)}\right)^{\frac{1}{q}}\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+5\omega}{6}\right)\right|\right\}$$

$$=\tfrac{(\omega-\lambda)^\varrho}{9^\varrho}\left(\tfrac{(\omega-\lambda)^\varrho}{3^\varrho\Gamma(\varrho+1)}\right)^{\frac{1}{q}}\left(\tfrac{\Gamma(1+p\varrho)}{\Gamma(1+(p+1)\varrho)}\right)^{\frac{1}{p}}\left\{\left(\tfrac{5^{p+1}-1}{4^{p+1}}\right)^{\frac{\varrho}{p}}\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+\omega}{2}\right)\right|\right.$$

$$\left.+\left|\Lambda^{(\varrho)}\left(\tfrac{5\lambda+\omega}{6}\right)\right|+\left(\tfrac{1+3^{p+1}}{4^{p+1}}\right)^{\frac{\varrho}{p}}\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+5\omega}{6}\right)\right|\right\},$$

where we have used (4.1)-(4.3). The proof is completed. □

Theorem 4.6. *Assume that all the assumptions of Theorem 3.2 are satisfied. If $\left|\Lambda^{(\varrho)}\right|^q$ is generalized convex for $q>1$. Then we have*

$$\left|\left(\tfrac{1}{4}\right)^\varrho\left(3^\varrho\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\tfrac{\Gamma(\varrho+1)}{(\omega-\lambda)^\varrho}\,{}_\lambda I_\omega^\varrho\Lambda(u)\right|$$

$$\leq\tfrac{(\omega-\lambda)^\varrho}{9^\varrho}\left\{(\mathcal{M}_1)^{1-\frac{1}{q}}\left(\mathcal{M}_2\left|\Lambda^{(\varrho)}(\lambda)\right|^q+\mathcal{M}_3\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+(\mathcal{M}_4)^{1-\frac{1}{q}}\left(\mathcal{M}_3\left|\Lambda^{(\varrho)}(\lambda)\right|^q+\mathcal{M}_5\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+(\mathcal{N}_1)^{1-\frac{1}{q}}\left(\mathcal{N}_2\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^q+\mathcal{N}_3\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+(\mathcal{N}_4)^{1-\frac{1}{q}}\left(\mathcal{N}_3\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^q+\mathcal{N}_5\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+(\mathcal{K}_1)^{1-\frac{1}{q}}\left(\mathcal{K}_2\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right|^q+\mathcal{K}_3\left|\Lambda^{(\varrho)}(\omega)\right|^q\right)^{\frac{1}{q}}$$

$$\left.+(\mathcal{K}_4)^{1-\frac{1}{q}}\left(\mathcal{K}_3\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right|^q+\mathcal{K}_5\left|\Lambda^{(\varrho)}(\omega)\right|^q\right)^{\frac{1}{q}}\right\},$$

where $\mathcal{M}_i,\mathcal{N}_i$ and $\mathcal{K}_i$ are defined for $i=1,2,...,5$ by (4.7)-(4.21), respectively.

Proof. From Lemma 3.1, properties of modulus, generalized power mean inequality and generalized convexity of $\left|\Lambda^{(\varrho)}\right|^q$, we have

$$\left|\left(\tfrac{1}{4}\right)^\varrho\left(3^\varrho\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right)-\tfrac{\Gamma(\varrho+1)}{(\omega-\lambda)^\varrho}\,{}_\lambda I_\omega^\varrho\Lambda(u)\right|$$

$$\leq\tfrac{(\omega-\lambda)^\varrho}{9^\varrho}\left\{\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1(1-t)^\varrho t^\varrho(dt)^\varrho\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1(1-t)^\varrho t^\varrho\left|\Lambda^{(\varrho)}\left((1-t)\lambda+t\tfrac{2\lambda+\omega}{3}\right)\right|^q(dt)^\varrho\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{2\varrho}(dt)^\varrho\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{2\varrho}\left|\Lambda^{(\varrho)}\left((1-t)\lambda+t\tfrac{2\lambda+\omega}{3}\right)\right|^q(dt)^\varrho\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}\left|\tfrac{5}{4}-t\right|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}\left|\tfrac{5}{4}-t\right|^{\varrho}\left|\Lambda^{(\varrho)}\left((1-t)\tfrac{2\lambda+\omega}{3}+t\tfrac{\lambda+2\omega}{3}\right)\right|^{q}(dt)^{\varrho}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\left|\tfrac{5}{4}-t\right|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\left|\tfrac{5}{4}-t\right|^{\varrho}\left|\Lambda^{(\varrho)}\left((1-t)\tfrac{2\lambda+\omega}{3}+t\tfrac{\lambda+2\omega}{3}\right)\right|^{q}(dt)^{\varrho}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}\left|t-\tfrac{1}{4}\right|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}\left|t-\tfrac{1}{4}\right|^{\varrho}\left|\Lambda^{(\varrho)}\left((1-t)\tfrac{\lambda+2\omega}{3}+t\omega\right)\right|^{q}(dt)^{\varrho}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\left|t-\tfrac{1}{4}\right|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\left|t-\tfrac{1}{4}\right|^{\varrho}\left|\Lambda^{(\varrho)}\left((1-t)\tfrac{\lambda+2\omega}{3}+t\omega\right)\right|^{q}(dt)^{\varrho}\right)^{\frac{1}{q}}\right\}$$

$$\leq\frac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}t^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}t^{\varrho}\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+t^{\varrho}\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^{q}\right)(dt)^{\varrho}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{2\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{2\varrho}\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+t^{\varrho}\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^{q}\right)(dt)^{\varrho}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}\left|\tfrac{5}{4}-t\right|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}\left|\tfrac{5}{4}-t\right|^{\varrho}\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}\left(\tfrac{2\lambda+\omega}{3}\right)\right|^{q}+t^{\varrho}\left|\Lambda^{(\varrho)}\left(\tfrac{\lambda+2\omega}{3}\right)\right|^{q}\right)(dt)^{\varrho}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\left|\tfrac{5}{4}-t\right|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\left|\frac{5}{4}-t\right|^{\varrho}\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}+t^{\varrho}\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}\right)(dt)^{\varrho}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}\left|t-\frac{1}{4}\right|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}\left|t-\frac{1}{4}\right|^{\varrho}\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}+t^{\varrho}\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)(dt)^{\varrho}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\left|t-\frac{1}{4}\right|^{\varrho}(dt)^{\varrho}\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\left|t-\frac{1}{4}\right|^{\varrho}\left((1-t)^{\varrho}\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}+t^{\varrho}\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)(dt)^{\varrho}\right)^{\frac{1}{q}}\right\}$$

$$=\frac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{(\mathcal{M}_1)^{1-\frac{1}{q}}\left(\mathcal{M}_2\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+\mathcal{M}_3\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$+(\mathcal{M}_4)^{1-\frac{1}{q}}\left(\mathcal{M}_3\left|\Lambda^{(\varrho)}(\lambda)\right|^{q}+\mathcal{M}_5\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}\right)^{\frac{1}{q}}$$

$$+(\mathcal{N}_1)^{1-\frac{1}{q}}\left(\mathcal{N}_2\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}+\mathcal{N}_3\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}\right)^{\frac{1}{q}}$$

$$+(\mathcal{N}_4)^{1-\frac{1}{q}}\left(\mathcal{N}_3\left|\Lambda^{(\varrho)}\left(\frac{2\lambda+\omega}{3}\right)\right|^{q}+\mathcal{N}_5\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}\right)^{\frac{1}{q}}$$

$$+(\mathcal{K}_1)^{1-\frac{1}{q}}\left(\mathcal{K}_2\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}+\mathcal{K}_3\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)^{\frac{1}{q}}$$

$$\left.+(\mathcal{K}_4)^{1-\frac{1}{q}}\left(\mathcal{K}_3\left|\Lambda^{(\varrho)}\left(\frac{\lambda+2\omega}{3}\right)\right|^{q}+\mathcal{K}_5\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)^{\frac{1}{q}}\right\},$$

where we have used

$$\mathcal{M}_1=\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}t^{\varrho}(dt)^{\varrho}=\frac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}-\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}, \tag{4.7}$$

$$\mathcal{M}_2=\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{2\varrho}t^{\varrho}(dt)^{\varrho}=\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}-\frac{\Gamma(1+3\varrho)}{\Gamma(1+4\varrho)}, \tag{4.8}$$

$$\mathcal{M}_3=\frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}t^{2\varrho}(dt)^{\varrho}=\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}-\frac{\Gamma(1+3\varrho)}{\Gamma(1+4\varrho)}, \tag{4.9}$$

$$\mathcal{M}_4=\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{2\varrho}(dt)^{\varrho}=\frac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}, \tag{4.10}$$

$$\mathcal{M}_5=\frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{3\varrho}(dt)^{\varrho}=\frac{\Gamma(1+3\varrho)}{\Gamma(1+4\varrho)}, \tag{4.11}$$

$$\mathcal{N}_1 = \frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}\left|\tfrac{5}{4}-t\right|^{\varrho}(dt)^{\varrho} = \tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)} + \left(\tfrac{1}{4}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}, \tag{4.12}$$

$$\mathcal{N}_2 = \frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{2\varrho}\left|\tfrac{5}{4}-t\right|^{\varrho}(dt)^{\varrho} = \tfrac{\Gamma(1+3\varrho)}{\Gamma(1+4\varrho)} + \left(\tfrac{1}{4}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}, \tag{4.13}$$

$$\mathcal{N}_3 = \frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}(1-t)^{\varrho}\left|\tfrac{5}{4}-t\right|^{\varrho}(dt)^{\varrho} = \left(\tfrac{5}{4}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)} - \left(\tfrac{9}{4}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)} + \tfrac{\Gamma(1+3\varrho)}{\Gamma(1+4\varrho)}, \tag{4.14}$$

$$\mathcal{N}_4 = \frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\left|\tfrac{5}{4}-t\right|^{\varrho}(dt)^{\varrho} = \left(\tfrac{5}{4}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)} - \tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}, \tag{4.15}$$

$$\mathcal{N}_5 = \frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{2\varrho}\left|\tfrac{5}{4}-t\right|^{\varrho}(dt)^{\varrho} = \left(\tfrac{5}{4}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)} - \tfrac{\Gamma(1+3\varrho)}{\Gamma(1+4\varrho)}, \tag{4.16}$$

$$\mathcal{K}_1 = \frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{\varrho}\left|t-\tfrac{1}{4}\right|^{\varrho}(dt)^{\varrho} = \left(\tfrac{5}{32}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)} + \left(\tfrac{3}{32}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}, \tag{4.17}$$

$$\mathcal{K}_2 = \frac{1}{\Gamma(\varrho+1)}\int_0^1 (1-t)^{2\varrho}\left|t-\tfrac{1}{4}\right|^{\varrho}(dt)^{\varrho} = \left(\tfrac{47}{128}\right)^{\varrho}\tfrac{\Gamma(1+3\varrho)}{\Gamma(1+4\varrho)} - \left(\tfrac{15}{128}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}, \tag{4.18}$$

$$\mathcal{K}_3 = \frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}(1-t)^{\varrho}\left|t-\tfrac{1}{4}\right|^{\varrho}(dt)^{\varrho} = \tfrac{\Gamma(1+3\varrho)}{\Gamma(1+4\varrho)} + \left(\tfrac{155}{128}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)} - \left(\tfrac{7}{32}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)}, \tag{4.19}$$

$$\mathcal{K}_4 = \frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{\varrho}\left|t-\tfrac{1}{4}\right|^{\varrho}(dt)^{\varrho} = \left(\tfrac{31}{32}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)} - \left(\tfrac{7}{32}\right)^{\varrho}\tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)} \tag{4.20}$$

and

$$\mathcal{K}_5 = \frac{1}{\Gamma(\varrho+1)}\int_0^1 t^{2\varrho}\left|t-\tfrac{1}{4}\right|^{\varrho}(dt)^{\varrho} = \left(\tfrac{127}{128}\right)^{\varrho}\tfrac{\Gamma(1+3\varrho)}{\Gamma(1+4\varrho)} - \left(\tfrac{31}{128}\right)^{\varrho}\tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)}. \tag{4.21}$$

The proof is completed. □

Corollary 4.7. *In Theorem 4.6, using the generalized convexity of* $\left|\Lambda^{(\varrho)}\right|^{q}$, *we get*

$$\begin{aligned}
&\left|\left(\tfrac{1}{4}\right)^{\varrho}\left(3^{\varrho}\Lambda\left(\tfrac{2\lambda+\omega}{3}\right)+\Lambda(\omega)\right) - \tfrac{\Gamma(\varrho+1)}{(\omega-\lambda)^{\varrho}}\,{}_{\lambda}I_{\omega}^{\varrho}\Lambda(u)\right| \\
&\le \tfrac{(\omega-\lambda)^{\varrho}}{9^{\varrho}}\left\{(\mathcal{M}_1)^{1-\frac{1}{q}}\left(\tfrac{3^{\varrho}\mathcal{M}_2+2^{\varrho}\mathcal{M}_3}{3^{\varrho}}\left|\Lambda^{(\varrho)}(\lambda)\right|^{q} + \tfrac{\mathcal{M}_3}{3^{\varrho}}\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)^{\frac{1}{q}}\right. \\
&\quad + (\mathcal{M}_4)^{1-\frac{1}{q}}\left(\tfrac{3^{\varrho}\mathcal{M}_3+2^{\varrho}\mathcal{M}_5}{3^{\varrho}}\left|\Lambda^{(\varrho)}(\lambda)\right|^{q} + \tfrac{\mathcal{M}_5}{3^{\varrho}}\left|\Lambda^{(\varrho)}(\omega)\right|^{q}\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+ (\mathcal{N}_1)^{1-\frac{1}{q}} \left(\tfrac{2^{\varrho}\mathcal{N}_2+\mathcal{N}_3}{3^{\varrho}} \left| \Lambda^{(\varrho)}(\lambda) \right|^q + \tfrac{\mathcal{N}_2+2^{\varrho}\mathcal{N}_3}{3^{\varrho}} \left| \Lambda^{(\varrho)}(\omega) \right|^q \right)^{\frac{1}{q}} \\
&+ (\mathcal{N}_4)^{1-\frac{1}{q}} \left(\tfrac{2^{\varrho}\mathcal{N}_3+\mathcal{N}_5}{3^{\varrho}} \left| \Lambda^{(\varrho)}(\lambda) \right|^q + \tfrac{\mathcal{N}_3+2^{\varrho}\mathcal{N}_5}{3^{\varrho}} \left| \Lambda^{(\varrho)}(\omega) \right|^q \right)^{\frac{1}{q}} \\
&+ (\mathcal{K}_1)^{1-\frac{1}{q}} \left(\tfrac{\mathcal{K}_2}{3^{\varrho}} \left| \Lambda^{(\varrho)}(\lambda) \right|^q + \tfrac{2^{\varrho}\mathcal{K}_2+3^{\varrho}\mathcal{K}_3}{3^{\varrho}} \left| \Lambda^{(\varrho)}(\omega) \right|^q \right)^{\frac{1}{q}} \\
&+ (\mathcal{K}_4)^{1-\frac{1}{q}} \left(\tfrac{\mathcal{K}_3}{3^{\varrho}} \left| \Lambda^{(\varrho)}(\lambda) \right|^q + \tfrac{2^{\varrho}\mathcal{K}_3+3^{\varrho}\mathcal{K}_5}{3^{\varrho}} \left| \Lambda^{(\varrho)}(\omega) \right|^q \right)^{\frac{1}{q}} \Bigg\}.
\end{aligned}
$$

5 Application to special means

For arbitrary real numbers λ, ω we have:

The generalized Arithmetic mean: $A(\lambda, \omega) = \frac{\lambda^{\varrho}+\omega^{\varrho}}{2^{\varrho}}$.

The generalized p-Logarithmic mean: $L_p(\lambda, \omega) = \left[\frac{\Gamma(1+p\varrho)}{\Gamma(1+(1+p)\varrho)} \frac{\left(\lambda^{(p+1)\varrho}-\omega^{(p+1)\varrho}\right)}{(\lambda-\omega)^{\varrho}} \right]^{\frac{1}{p}}, \lambda \neq \omega$ and $p \in \mathbb{Z} \backslash \{-1, 0\}$.

Proposition 5.1. *Let $\lambda, \omega \in \mathbb{R}$ with $0 < \lambda < \omega$ and $n \geq 2$, then we have*

$$
\begin{aligned}
&\left| 3^{\varrho} A^n(\lambda, \lambda, \omega) + \omega^{n\varrho} - 4^{\varrho}\Gamma(\varrho+1) L_n^n(\lambda, \omega) \right| \\
\leq & \tfrac{4^{\varrho}(\omega-\lambda)^{\varrho}}{27^{\gamma}} \left(\tfrac{\Gamma(1+n\varrho)}{\Gamma(1+(n-1)\varrho)} \right) \left[\left(\left(\tfrac{155}{32}\right)^{\gamma} \tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)} + \left(\tfrac{5}{32}\right)^{\gamma} \tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)} \right) \lambda^{(n-1)\varrho} \right. \\
&\left. + \left(\left(\tfrac{73}{32}\right)^{\gamma} \tfrac{\Gamma(1+\varrho)}{\Gamma(1+2\varrho)} + \left(\tfrac{103}{32}\right)^{\gamma} \tfrac{\Gamma(1+2\varrho)}{\Gamma(1+3\varrho)} \right) \omega^{(n-1)\varrho} \right].
\end{aligned}
$$

Proof. The assertion follows from Corollary 3.3 applied to the function $\Lambda(u) = u^{n\varrho}$ where $\Lambda : (0, +\infty) \to \mathbb{R}^{\varrho}$. □

6 Conclusion

In this study, we have successfully extended the classical 2-point right-Radau inequality to the realm of fractal sets by employing local fractional integrals. The introduction of a novel local fractional integral identity has played a pivotal role in deriving these generalized inequalities, demonstrating the power of this identity as a foundational tool in fractal analysis. By leveraging the concept of generalized convexity, we have broadened the scope of traditional Radau-type inequalities, enabling their application to functions defined on fractal domains. Moreover, the results obtained in this work represent a refinement of several existing results in the literature, offering improved precision and broader applicability. The theoretical advancements presented here not only enrich the framework of fractal calculus but also highlight their practical relevance through several illustrative applications.

This work lays the groundwork for numerous future research directions. One promising avenue is the exploration of Radau-type inequalities using other forms of generalized convexity, such as h-convexity, m-convexity, or harmonic convexity. Such investigations could further expand the applicability of these inequalities to a wider range of problems, particularly in the context of local fractional integrals.

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References

[1] T. Abdeljawad, S. Rashid, Z. Hammouch and Y.-M. Chu, *Some new local fractional inequalities associated with generalized (s,m)-convex functions and applications*, Adv. Difference Equ. **2020**, Paper No. 406, 27 pp.

[2] D. C. Benchettah, A. Lakhdari and B. Meftah, *Refinement of the general form of the two-point quadrature formulas via convexity*, Journal of Applied Mathematics, Statistics and Informatics, **19(1)**, 93-101, (2023).

[3] A. Berkane, B. Meftah and A. Lakhdari, *Right-Radau-type inequalities for multiplicative differentiable s-convex functions*, Journal of applied mathematics & informatics, **42(4)**, 785-800, (2024).

[4] B. Bin-Mohsin, A. Lakhdari, N. E. I. Karabadji, M. U. Awan, A. Ben Makhlouf, B. Meftah and S. S. Dragomir, *An extension of left radau type inequalities to fractal spaces and applications*, Axioms, **13(9)**, 653, (2024).

[5] G.-S. Chen, *Generalizations of Hölder's and some related integral inequalities on fractal space*, J. Funct. Spaces Appl. **2013**, Art. ID 198405, 9 pp.

[6] T. Du, and X. Yuan, *On the parameterized fractal integral inequalities and related applications*, Chaos, Solitons & Fractals **170**, 113375, (2023).

[7] F. Gao, W. P. Zhong and X. M. Shen, *Applications of Yang-Fourier transform to local fractional equations with local fractional derivative and local fractional integral*, Adv. Mater. Res. **461**, 306–310, (2012).

[8] A. Lakhdari, B. Meftah and W. Saleh, *On corrected Simpson-type inequalities via local fractional integrals*, Georgian Math. J. **32(1)**, 111–121, (2025).

[9] A. Lakhdari, M. U. Awan, S. S. Dragomir, H. Budak and B. Meftah, *Exploring error estimates of Newton-Cotes quadrature rules across diverse function classes*, J. Inequal. Appl. **2025**, Paper No. 6.

[10] A. Lakhdari, H. Budak, N. Mlaiki, B. Meftah and T. Abdeljawad, *New insights on fractal–fractional integral inequalities: Hermite–Hadamard and Milne estimates*, Chaos Solitons Fractals **193**, Paper No. 116087, (2025).

[11] H. Li, A. Lakhdari, F. Jarad, H. Xu and B. Meftah, *An expanded analysis of local fractional integral inequalities via generalized (s,P)-convexity*, J. Inequal. Appl. **2024**, Paper No. 78, 22 pp.

[12] Y. S. Liang and W. Su, *A geometric based connection between fractional calculus and fractal functions*, Acta Math. Sin. (Engl. Ser.) **40(2)**, 537–567, (2024).

[13] X. L. Liu, H. Y. Xu, A. Shokri, A. Lakhdari and B. Meftah, *Some error bounds for 2-point right Radau formula in the setting of fractional calculus via s-convexity*, J. Math. **2024**, Art. ID 6709056, 14 pp.

[14] B. Meftah, A. Lakhdari and W. Saleh, *2-point left Radau-type inequalities via s-convexity*, J. Appl. Anal. **29(2)**, 341–346, (2023).

[15] M. Muawwaz, M. Maaz, A. Qayyum, M. Ahmad, M. D. Faiz and A. Mehboob, *Innovative Ostrowski's type inequalities based on linear kernel and applications*, Palestine Journal of Mathematics, **14(1)**, 802–812, (2025).

[16] A. Munir, H. Budak, I. Faiz, L. Rathour, H. Kara and A. Kashuri, *A study on Newton-type inequalities for fractional operator and application*, Palestine Journal of Mathematics, **14(1)**, 298–311, (2025).

[17] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, **187**. Academic Press, Inc., Boston, MA, 1992.

[18] A. Qayyum, H. Budak, U. Bat, H. Kara, A. Munir and L. Rathour, *A new extension of Hermite Hadamard inequalities associating Ψ-Hilfer fractional integral*, Palestine Journal of Mathematics, **14(1)**, 578–586, (2025).

[19] R. Radau, *Etude sur les formules d'approximation qui servent à calculer la valeur numérique d'une intégrale définie*, J. Math. Pures Appl. **6**, 283–336, (1880).

[20] G. Rebiai and B. Meftah, *2-point right Radau inequalities for differentiable s-convex functions*, J. Interdiscip. Math, **27**, 1243-1255, (2024).

[21] W. Saleh, A. Lakhdari, O. Almutairi and A. Kiliçman, *Some remarks on local fractional integral inequalities involving Mittag–Leffler kernel using generalized (E,h)-convexity*, Mathematics, **11(6)**, 1373, (2023).

[22] W. Saleh, B. Meftah, A. Lakhdari and A. Kiliçman, *Exploring the companion of Ostrowski's inequalities via local fractional integrals*, European Journal of Pure and Applied Mathematics, **16(3)**, 1359-1380, (2023).

[23] X. J. Yang, *Advanced local fractional calculus and its applications*, World Science Publisher, New York, **2012**.

[24] Y. J. Yang, D. Baleanu and X.-J. Yang, *Analysis of fractal wave equations by local fractional Fourier series method*, Adv. Math. Phys. **2013**, Art. ID 632309, 6 pp.

[25] A. M. Yang, Z. S. Chen, H. M. Srivastava and X. J. Yang, *Application of the local fractional series expansion method and the variational iteration method to the Helmholtz equation involving local fractional derivative operators*, Abstr. Appl. Anal. **(2013)**, Article ID 259125

[26] Y. Yu, J. Liu and T. Du, *Certain error bounds on the parameterized integral inequalities in the sense of fractal sets*, Chaos, Solitons & Fractals, **161**, 112328, (2022).

[27] S. H. Yu, P. O. Mohammed, Lei Xu, T. S. Du, *An improvement of the power-mean integral inequality in the frame of fractal space and certain related midpoint-type integral inequalities*, Fractals, **30(4)**, 2250085, (2022).

[28] H. Xu, A. Lakhdari, W. Saleh and B. Meftah, *Some new parametrized inequalities on Fractal set*, Fractals, **32(3)**, 2450063, (2024).

[29] H. Xu, A. Lakhdari, F. Jarad, T. Abdeljawad and B. Meftah, *On multiparametrized integral inequalities via generalized α-convexity on fractal set*, Math. Methods Appl. Sci. **48(1)**, 980–1002, (2025).

[30] Z. Zhou and T. S. Du, *Analytical properties and related inequalities derived from multiplicative Hadamard k-fractional integrals*, Chaos Solitons Fractals, **189**, part 2, Paper No. 115715, 36 pp, (2024).

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