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Existence of solutions for a nonlinear discrete problem with the $p(k)$-Laplacian-like operators

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Abstract In this article, we investigate the existence of nontrivial solutions of a nonlinear discrete problem arising from capillary phenomena, involving the $p(k)$-Laplacian-like operators with Dirichlet-type boundary conditions. Under appropriate assumptions on the function f and its primitive F near zero and infinity, using critical point theory and variational methods, we obtain the results.

1 Introduction

We consider the discrete anisotropic problem with Dirichlet-type boundary conditions as follows:

$$(P_\lambda):\begin{cases}-\Delta\Big(\big(1+\phi_c\left(|\Delta u(t-1)|^{p(t-1)}\right)\big)|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)\Big)=\lambda f(t,u(t)),\ t\in[1,N]_{\mathbb{N}},\\ u(0)=u(N+1)=0,\end{cases}$$

where $N\geq 2$ is a positive integer and $[1,N]_{\mathbb{N}}:=\{1,2,\dots,N\}$ is a discrete interval, $\Delta u(t)=u(t+1)-u(t)$ is the forward difference operator and $\phi_c(s):=\frac{s}{\sqrt{1+s^2}}$, for $s\in\mathbb{R}$ is called the mean curvature operator [23].
For every fixed $t\in[0,N]_{\mathbb{N}}$, $f(t,.):\mathbb{R}\to\mathbb{R}$ is a continuous function that checks some conditions mentioned below and $p:[0,N+1]_{\mathbb{N}}\to[2;+\infty)$ is a given bounded function. We say that a function $u:[0,N+1]_{\mathbb{N}}\to\mathbb{R}$ is a solution of problem (P_λ) if it satisfies both equations of (P_λ).

For any bounded function $h:[0,N+1]_{\mathbb{N}}\to\mathbb{R}$, we consider the symbols

$$h^+:=\max_{t\in[0,N+1]_{\mathbb{N}}}h(t),\qquad h^-:=\min_{t\in[0,N+1]_{\mathbb{N}}}h(t).$$

Recent years have received a great deal of attention in the study of partial difference equations with a variable exponent, which contributed to the development of research and studies related to the problems of differential equations. In [33], the authors have focused mainly on existence and multiplicity results for the following discrete problem :

$$(P):\begin{cases}-\Delta\left(|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)\right)=g(t,u(t)),t\in[1,N]_{\mathbb{Z}},\\ u(0)=u(N+1)=0,\end{cases}$$

where $N\geq 2$ is an integer, $[1,N]_{\mathbb{Z}}$ is the discrete interval $\{1,2,3\dots,N\},\Delta$ is the forward difference operator defined by $\Delta u(t)=u(t+1)-u(t)$ and $g:[1,N]_{\mathbb{Z}}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function, i.e. for any fixed $t\in[1,N]_{\mathbb{Z}}$, the function $g(t,.)$ is continuous. For further insight into the topic, we refer the reader to [1, 7, 8, 13, 20, 21, 25, 28, 33] and references therein.

Problems such as (P_λ) derive interest from their close connection with the study of capillary phenomena. These phenomena, which can be briefly explained by considering the effects of two opposing forces: cohesion, i.e. the attractive force between the molecules of the liquid; and adhesion, i.e. the attractive (or repulsive) force between the molecules of the liquid and those of the container. The study of this kind of problems possesses a solid background in physics and other areas of research such as combustible gas dynamics [31], image restoration [10], the analysis of capillary surfaces [6, 17] and economic systems processing [29]. For other applications, the reader can see [14, 15, 19, 33] and references therein.

Concerning the investigation of boundary value problems with $p(.)$-Laplacian-like operators in the continuous case, we mention, far from being exhaustive, the following recent papers:

In [26], M. Rodrigues studies the existence and multiplicity of solutions for the following problem involving the $p(x)$-Laplacian-like operators originated from a capillary phenomena:

$$(P_0):\begin{cases} -\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), & x\in\Omega,\\ u=0, \quad x\in\partial\Omega \end{cases}$$

where $\lambda>0$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p\in C(\overline{\Omega})$ and $p(x)>2$, $\forall x\in\Omega$. Based on the mountain pass theorem, the author showed that the problem has at least one nontrivial solution. The author proved also the existence of a sequence of solutions using the Fountain theorem.

In [11], the authors consider the problem (P_0) where $p:\overline{\Omega}\to\mathbb{R}$ is a Lipschitz continuous function with

$$1<p^-:=\operatorname{essinf}_{x\in\Omega}p(x)\le p(x)\le p^+:=\operatorname{esssup}_{x\in\Omega}p(x)<N$$

and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is superlinear but does not satisfy the usual Ambrosetti-Rabinowitz type condition. Under some conditions on f at infinity, they proved that the above problem has at least one nontrivial solution. Moreover, the existence of infinite solutions is proved for odd nonlinearity and with some new trickes.

However, for the discrete setting, we'd mention that papers with ϕ_c and $p(k)$-Laplacian-like operators are relatively limited, but we can refer the readers to [32], where Zhou and Ling employed the critical point theory to establish the existence of positive solutions for the following prescribed mean curvature problem with Dirichlet boundary condition:

$$\begin{cases} -\triangle\left(\phi_c\left(\triangle u_{k-1}\right)\right)=\lambda f\left(k,u_k\right), & k\in Z(1,T),\\ u_0=u_{T+1}=0, \end{cases}$$

with T being a given positive integer, and $Z(a,b)=\{a,a+1,\dots,b\}$ when $a\le b$ is the discrete interval.

In this work, we are motivated by some results mentioned in the above papers and by taking advantage of certain techniques used in [24] and [30]. Specifically, we are interested to look for the existence of solutions for the problem (P_λ) which is considered as the discrete variant of (P_0), where some known tools from the critical point theory and variational methods are applied, such as the Mountain Pass theorem and Ekeland's variational principle.

Now, we introduce the following assumptions:

(F_0): There exists a bounded function $\beta:[1,N]_{\mathbb{N}}\to[2,\infty)$ such that

$$|f(t,s)|\le C_0\left(1+|s|^{\beta(t)-1}\right),$$

for all $(t,s)\in[1,N]_{\mathbb{N}}\times\mathbb{R}$, where C_0 is a positive constant.

(F_1): $\lim_{x\to 0}\frac{F(t,x)}{|x|^{p^+}}=0$, uniformly for all $t\in[1,N]_{\mathbb{N}}$, where $F(t,x)=\int_0^x f(t,s)ds$.

(F_2): There exists κ such that $\kappa>\frac{2^{p^++1}}{p^-}(N+1)^{\frac{p^+}{2}}$ and

$$\liminf_{|x|\to\infty}\frac{F(t,x)}{|x|^{p^+}}\ge\kappa,\ \text{ for all } t\in[1,N]_{\mathbb{N}}.$$

Many papers treat the existence of solutions for some similar discrete problem (see [30]) by considering the hypotheses (F_1), (F_2) and the following condition:

(AR): there exist $t_* > 0,\ \kappa > p^+$ such that

$$0 \leq \kappa F(t,s) \leq f(t,s)s,\ |s| \geq t_*, \quad \forall t \in [1,N]_{\mathbb{N}},$$

that is called Ambrosetti-Rabinowitz condition type [5].

It is well-known that the (AR)-condition has a crucial role in the application of variational methods, it is extensively used to ensure the boundedness of the Palais-Smale sequences of the energy functional. Despite this, it still restrictive and eliminates many nonlinearities. For example, the function

$$f(t,x) = |x|^{p^+-2}x\ln(1+|x|) + \frac{1}{p^+}\frac{|x|^{p^+-1}x}{1+|x|},\ t \in [1,N]_{\mathbb{N}},\ x \in \mathbb{R}$$

with a primitive

$$F(t,x) = \frac{1}{p^+}|x|^{p^+}\ln(1+|x|),\ t \in [1,N]_{\mathbb{N}},\ x \in \mathbb{R},$$

does not satisfy the (AR)-condition. However, it satisfies the hypothesis (F_2), which is weaker and encompasses a broader class of nonlinearities.

We organize the rest of this work as follows. In Section 2, we introduce some necessary preliminary results. Next, we set the variational setting associated to (P_λ), and in the last section, we give our main results with their proofs.

2 Preliminaries

Consider E the N-dimensional Hilbert space [4]

$$E = \Big\{u : [0,N+1]_{\mathbb{N}} \to \mathbb{R} \mid\ u(0) = u(N+1) = 0\Big\},$$

equipped with the inner product

$$< u,v >= \sum_{t=0}^{N} \Delta u(t-1)\Delta v(t-1),$$

which is associated the norm defined by

$$\|u\| = \left(\sum_{t=0}^{N} |\Delta u(t-1)|^2\right)^{\frac{1}{2}}.$$

Also, for $m \geq 2$, we define the norm

$$|u|_m = \left(\sum_{t=1}^{N+1} |u(t)|^m\right)^{\frac{1}{m}},\quad \forall u \in E.$$

Since E is a finite dimensional space, all norms are equivalent.

For our purpose, it is useful to use the following inequalities.

Proposition 2.1. *([24, 18])*

(a) Let $u \in E$ and $\|u\| > 1$. Then

$$\sum_{t=1}^{N+1} \frac{|\Delta u(t-1)|^{p(t-1)}}{p(t-1)} \geq N^{\frac{2-p^-}{2}}\|u\|^{p^-} - N.$$

(b) *Let* $u \in E$ *and* $\|u\| < 1$. *Then*

$$\sum_{t=1}^{N+1} \frac{|\Delta u(t-1)|^{p(t-1)}}{p(t-1)} \geq N^{\frac{p^+-2}{2}} \|u\|^{p^+}.$$

(c) *For all* $u \in E$, *we have*

$$\|u\|_\infty := \max_{t\in[1,N]_{\mathbb{N}}} |u(t)| \leq \sqrt{N+1}\|u\|.$$

(d) *For all* $u \in E$, *we have*

$$\sum_{t=1}^{N+1} |\Delta u(t-1)|^{p^+} \leq (N+1)\|u\|^{p^+}.$$

(e) *For all* $u \in E$, $m \geq 2$ *we have*

$$\sum_{t=1}^{N} |u(t)|^m \geq 2^{-m} (N+1)^{\frac{2-m}{2}} \|u\|^m.$$

3 Variational framework

Let $T_\lambda : E \to \mathbb{R}$ be the functional associated to the problem (P_λ) defined in the following way

$$T_\lambda(u) = \Phi(u) - \lambda\Psi(u), \quad \forall u \in E, \tag{3.1}$$

with

$$\Phi(u) = \sum_{t=1}^{N+1} \frac{1}{p(t-1)} \left(|\Delta u(t-1)|^{p(t-1)} + \sqrt{1 + |\Delta u(t-1)\ |^{2p(t-1)}} - 1 \right),$$

$$\Psi(u) = \sum_{t=1}^{N} F(t,u(t)) \quad \text{and} \quad F(t,x) = \int_0^x f(t,s)ds, \ \text{for all} \ \ t \in [0,N]_{\mathbb{N}}.$$

By a standard argument, using summation by parts, it can be shown that the functional T_λ is well defined, of class C^1 on E and its Gâteaux derivative is given by

$$\begin{aligned}(T'_\lambda(u),v) = &\sum_{t=1}^{N+1} \left(1 + \phi_c\left(|\Delta u(t-1)|^{p(t-1)}\right)\right) |\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)\Delta v(t-1) \\ &- \lambda \sum_{t=1}^{N} f(t,u(t))v(t),\end{aligned} \tag{3.2}$$

for all $u, v \in E$.
Moreover, the critical points of the functional T_λ are exactly the solutions of problem (P_λ).
Indeed, for short, let us set

$$A(t-1) = \left(\left(1 + \phi_c\left(|\Delta u(t-1)|^{p(t-1)}\right)\right) |\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1)\right),$$

then for all $v \in E$, we have

$$\begin{aligned}(T'_\lambda(u),v) &= \sum_{t=1}^{N+1} A(t-1)\Delta v(t-1) - \lambda \sum_{t=1}^{N} f(t,u(t))v(t) \\ &= \Big[A(t-1)v(t-1)\Big]_1^{N+2} - \sum_{t=1}^{N+1} \Delta(A(t-1))v(t) - \lambda \sum_{t=1}^{N} f(t,u(t))v(t)\end{aligned}$$

Taking into account the boundary conditions, we obtain

$$(T'_\lambda(u), v) = -\sum_{t=1}^{N} \Delta(A(t-1))v(t) - \lambda \sum_{t=1}^{N} f(t, u(t))v(t)$$
$$= \sum_{t=1}^{N} \Big(-\Delta \Big((1 + \phi_c(|\Delta u(t-1)|^{p(t-1)}))|\Delta u(t-1)|^{p(t-1)-2}\Delta u(t-1) \Big) - \lambda f(t, u(t)) \Big) v(t)$$

Since $v \in E$ is arbitrary, we conclude that $u \in E$ is a critical point of T_λ in E (i.e., $(T'_\lambda(u), v) = 0$ for all $v \in E$) if and only if u is a solution of (P_λ).

Now, we recall this important critical point theorem due to A. Ambrosetti and H. Rabinowitz.

Theorem 3.1. *(Mountain Pass Theorem, see [5]) Let $(X, \|.\|)$ be a real Banach space, $\Psi \in C^1(X, \mathbb{R})$ satisfies (PS)-condition, i.e., any sequence $(u_n) \subset X$ such that $(\Psi(u_n))$ is bounded and $(\Psi'(u_n)) \longrightarrow 0$ as $n \to +\infty$ has a subsequence which converges in X. Moreover, if $\Psi(0) = 0$ and the following conditions hold:*

(i) There exist positive constant ρ and α such that $\Psi(u) \geq \alpha$ for any $u \in X$ with $\|u\| = \rho$.

(ii) There exists a function $e \in X$ such that $\|e\| > \rho$ and $\Psi(e) \leq 0$.

Then, the functional Ψ possesses a critical value $c \geq \alpha$. Moreover, the critical value c is characterised by

$$c = \inf_{g \in \Gamma} \max_{t \in [0,1]} \Psi(g(t)),$$

where

$$\Gamma = \{g \in C([0, 1], X) \,/ g(0) = 0, g(1) = e\}.$$

4 Main results and proofs

Theorem 4.1. *Assume that the assumption (F_0) is fulfilled, and*

$$\beta^+ < p^-.$$

Then, for all $\lambda > 0$ the problem (P_λ) has a solution $u_\lambda \in E$.
Moreover, if there exist $k^* \in [1, N]_{\mathbb{N}}$ and $d \geq 1$ such that $F(k^*, d) > 0$, then there exists $\lambda^* > 0$ such that for any $\lambda \geq \lambda^*$, the solution u_λ is not trivial.

Proof. (of Theorem 4.1) Let $u \in E$ such that $\|u\| > 1$. We point out that

$$|u(t)|^{\beta(t)} \leq |u(t)|^{\beta^-} + |u(t)|^{\beta^+}, \quad \forall (t, u) \in [1, N]_{\mathbb{N}} \times E. \tag{4.1}$$

On the other hand, the relation (c) implies that for $m \geq 2$

$$|u(t)|^m \leq (N + 1)^{\frac{m}{2}} \|u\|^m, \text{ for all } t \in [1, N]_{\mathbb{N}},$$

and thus

$$\sum_{t=1}^{N} |u(t)|^m \leq N(N + 1)^{\frac{m}{2}} \|u\|^m, \tag{4.2}$$

this combined with (4.1), gives

$$\sum_{t=1}^{N} |u(t)|^{\beta(t)} \leq N(N + 1)^{\frac{\beta^+}{2}} \|u\|^{\beta^+} + N(N + 1)^{\frac{\beta^-}{2}} \|u\|^{\beta^-}$$
$$\leq 2N(N + 1)^{\frac{\beta^+}{2}} \|u\|^{\beta^+}. \tag{4.3}$$

Now, from (F_0) there exists $C_1 > 0$ such that

$$|F(t, s)| \leq C_0 \frac{|s|^{\beta(t)}}{\beta(t)} + C_1, \text{ for all } t \in [1, N]_{\mathbb{N}}, \; s \in \mathbb{R},$$

which implies that

$$\Psi(u)=\sum_{t=1}^{N}F(t,u(t))\leq\sum_{t=1}^{N}C_0\frac{|u(t)|^{\beta(t)}}{\beta(t)}+NC_1$$
$$\leq\frac{2NC_0(N+1)^{\frac{\beta^+}{2}}}{\beta^-}\|u\|^{\beta^+}+NC_1. \tag{4.4}$$

We recall that for all $s\geq 0$, we have $\max(0,s-1)\leq\sqrt{1+s^2}-1\leq s$. Then, for $u\in E$ with a sufficiently large norm such that $|\Delta u(t-1)|\geq 1$ for all $t\in[1,N]_{\mathbb{N}}$, by proposition 2.1 (a), we have

$$\Phi(u)=\sum_{t=1}^{N+1}\frac{1}{p(t-1)}\left(|\Delta u(t-1)|^{p(t-1)}+\sqrt{1+|\Delta u(t-1)|^{2p(t-1)}}-1\right)$$
$$\geq\sum_{t=1}^{N+1}\frac{2}{p(t-1)}|\Delta u(t-1)|^{p(t-1)}-\sum_{t=1}^{N+1}\frac{1}{p(t-1)}$$
$$\geq\frac{2}{p^+(\sqrt{N})^{p^--2}}\|u\|^{p^-}-2N-\frac{N+1}{p^-}. \tag{4.5}$$

In addition, the above inequality combined with (4.4), leads to us to

$$T_\lambda(u)\geqslant\frac{2}{p^+(\sqrt{N})^{p^--2}}\|u\|^{p^-}-2N-\frac{N+1}{p^-}-\lambda\frac{2NC_0(N+1)^{\frac{\beta^+}{2}}}{\beta^-}\|u\|^{\beta^+}-\lambda NC_1.$$

Since $p^->\beta^+$, the last inequality means that T_λ is coercive.

T_λ is a continuous coercive functional and it is also weakly lower semicontinuous on E. We recall that E is a finite dimensional space, which allows us to say that there exists $u_\lambda\in E$ a global minimizer of T_λ and thus, a solution of (P_λ).

In order to finish the proof, it remains to suppose that there exist $k^*\in[1,N]_{\mathbb{N}}$ and $d\geq 1$ such that $F(k^*,d)>0$, and show that the solution u_λ is not trivial. For this purpose, we choose a function $\omega\in E$ defined as

$$\begin{cases}\omega(t)=0, & \textit{for all } t\in[1,N]_{\mathbb{N}}\backslash\{k^*\}\\ \omega(k^*)=d.\end{cases} \tag{4.6}$$

obviously, we have

$$T_\lambda(\omega)=\Phi(\omega)-\lambda\Psi(\omega)$$
$$=\sum_{t=1}^{N+1}\frac{1}{p(t-1)}\left(|\Delta\omega(t-1)|^{p(t-1)}+\sqrt{1+|\Delta\omega(t-1)|^{2p(t-1)}}-1\right)-\lambda\sum_{t=1}^{N}F(t,\omega(t))$$
$$=\frac{1}{p(k^*-1)}\left(d^{p(k^*-1)}+\sqrt{1+d^{2p(k^*-1)}}-1\right)+\frac{1}{p(k^*)}\left(d^{p(k^*)}+\sqrt{1+d^{2p(k^*)}}-1\right)$$
$$-\lambda F(k^*,d)$$
$$\leq\frac{1}{p^-}\left(2d^{p^+}+2\sqrt{1+d^{2p^+}}-2\right)-\lambda F(k^*,d)$$
$$\leq\frac{4d^{p^+}}{p^-}-\lambda F(k^*,d). \tag{4.7}$$

Then, if we put

$$\lambda^*:=\frac{4d^{p^+}}{p^-F(k^*,d)},$$

it follows that $T_\lambda(\omega)<0$ and thus $T_\lambda(u_\lambda)<0=T_\lambda(0)$ for any $\lambda\geq\lambda^*$. Therefore, u_λ is a non trivial solution of (P_λ). □

Theorem 4.2. *Assume that hypotheses (F_1) and (F_2) hold.*
Then, for any $\lambda \in (0, +\infty)$ the problem (P_λ) admits at least a nontrivial solution.

To give the proof, we will apply a Mountain Pass Theorem (Theorem 3.1). So, we should prove that the functional T_λ has a Mountain pass geometry. For this reason, we give these two lemmas.

Lemma 4.3. *Suppose that the hypothesis (F_1) holds. Then, there exist two constant $\gamma, \rho > 0$ such that*

$$T_\lambda(u) \geq \gamma > 0, \ u \in E \text{ with } \|u\| = \rho.$$

Proof. By (F_1), for any $\epsilon > 0$, there exists $K > 0$ such that

$$|F(t,s)| \leq \epsilon |s|^{p^+}, \forall (t, |s|) \in [1, N]_{\mathbb{N}} \times [0, K].$$

Let $u \in E$, such that $\|u\| \leq \xi$ with $\xi = \min\left\{1, \frac{K}{\sqrt{N+1}}\right\}$.
By inequality (c) it follows that

$$|u(t)| \leq \max_{r\in[1,N]_{\mathbb{N}}} |u(r)| \leq K, \forall t \in [1, N]_{\mathbb{N}}.$$

So, we conclude that

$$|F(t, u(t))| \leq \epsilon |u(t)|^{p^+}, \quad \text{for all } t \in [1, N]_{\mathbb{N}}.$$

Also by (4.2), we get

$$\Psi(u) = \sum_{t=1}^{N} F(t, u(t)) \leq \epsilon N (N+1)^{\frac{p^+}{2}} \|u\|^{p^+}.$$

Taking $u \in E$ such that $\|u\| < 1$, by inequality (b) in proposition 2.1, it follows that

$$\begin{aligned} T_\lambda(u) &= \sum_{t=1}^{N+1} \frac{1}{p(t-1)} \left(|\Delta u(t-1)|^{p(t-1)} + \sqrt{1 + |\Delta u(t-1)|^{2p(t-1)}} - 1 \right) - \lambda \sum_{t=1}^{N} F(t, u(t)) \\ &\geq \sum_{t=1}^{N+1} \frac{1}{p(t-1)} |\Delta u(t-1)|^{p(t-1)} - \lambda \epsilon N (N+1)^{\frac{p^+}{2}} \|u\|^{p^+} \\ &\geq \frac{1}{p^+ (\sqrt{N})^{2-p^+}} \|u\|^{p^+} - \lambda \epsilon N (N+1)^{\frac{p^+}{2}} \|u\|^{p^+}. \end{aligned}$$

Let us now take $\varepsilon > 0$ sufficiently small such that $\lambda \varepsilon < \frac{1}{2p^+} \frac{N^{\frac{p^+-2}{2}}}{N(N+1)^{\frac{p^+}{2}}}$. Then, we get

$$T_\lambda(u) \geq \frac{N^{\frac{p^+-2}{2}}}{2p^+} \|u\|^{p^+}.$$

Also, if we take $\gamma = \frac{N^{\frac{p^+-2}{2}}}{2p^+} \rho^{p^+} > 0$, it follows that

$$T_\lambda(u) \geq \gamma > 0, \forall u \in E \text{ with } \|u\| = \rho. \tag{4.8}$$

□

Lemma 4.4. *Suppose that condition (F_2) is verified. Then,*

- *the functional T_λ is unbounded from below and satisfies the (PS)-condition,*
- *there exists $e \in E$ such that $\|e\| > \rho$ and $T_\lambda(e) < 0$, where ρ is given in (4.8).*

Proof.
According to the hypothesis (F_2), for any $\varepsilon > 0$ there exists $R_0 > 0$ such that

$$\frac{F(t,x)}{|x|^{p^+}} \geq \kappa - \varepsilon \text{ for all } (t,|x|) \in [1,N]_{\mathbb{N}} \times]R_0,+\infty[,$$

so,

$$F(t,x) \geq (\kappa - \varepsilon)|x|^{p^+} \text{ for } (t,|x|) \in [1,N]_{\mathbb{N}} \times]R_0,+\infty[.$$

By continuity of the function $s \to F(t,s)$, there exists $K > 0$ such that

$$F(t,s) \geq (\kappa - \varepsilon)|s|^{p^+} - K, \quad \forall (t,s) \in [1,N]_{\mathbb{N}} \times \mathbb{R}.$$

Then,

$$F(t,u(t)) \geq (\kappa - \varepsilon)|u(t)|^{p^+} - K, \quad \forall t \in [1,N]_{\mathbb{N}},$$

and by relation (e), we get

$$\begin{aligned}\Psi(u) &\geq (\kappa - \varepsilon)\sum_{t=1}^{N}|u(t)|^{p^+} - KN \\ &\geq (\kappa - \varepsilon)2^{-p^+}(N+1)^{\frac{2-p^+}{2}}\|u\|^{p^+} - KN, \quad \forall u \in E.\end{aligned}$$

On the other hand, without loss of generality, let $u \in E$ with $\|u\| > 1$ such that $|\Delta u(t-1)| > 1$ for all $t \in [1,N]_{\mathbb{N}}$, by relation (d) we have

$$\begin{aligned}\Phi(u) &\leq \frac{2}{p^-}\sum_{t=1}^{N+1}|\Delta u(t-1)|^{p(t-1)} \\ &\leq \frac{2}{p^-}\sum_{t=1}^{N+1}|\Delta u(t-1)|^{p^+} \\ &\leq \frac{2}{p^-}(N+1)\|u\|^{p^+},\end{aligned}$$

it follows that

$$\begin{aligned}T_\lambda(u) &\leq \frac{2}{p^-}(N+1)\|u\|^{p^+} - 2^{-p^+}(N+1)^{\frac{2-p^+}{2}}\lambda(\kappa-\varepsilon)\|u\|^{p^+} + \lambda KN \\ &\leq \left[\frac{2}{p^-}(N+1) - 2^{-p^+}(N+1)^{\frac{2-p^+}{2}}\lambda(\kappa-\varepsilon)\right]\|u\|^{p^+} + \lambda KN.\end{aligned}$$

Consequently, for $\varepsilon < \kappa - \frac{2^{p^+ +1}}{\lambda p^-}(N+1)^{\frac{p^+}{2}}$, we have

$$T_\lambda(u) \to -\infty, \text{ as } \|u\| \to \infty.$$

Hence T_λ is anti-coercive on E.
Therefore, the functional T_λ is unbounded from below and any (PS) sequence (u_n) associated to T_λ will be bounded in E which is a finite dimensional space. So, the functional T_λ satisfies the (PS)-condition.
Furthermore, we can find $e \in E$ such that $\|e\| > \rho$ and $T_\lambda(e) < 0$. □

Proof. (of the Theorem 4.2) Since $T_\lambda(0) = 0$ and from the above lemmas, according to the Mountain Pass Theorem (Theorem 3.1), we conclude that the problem (P_λ) admits at least a nontrivial solution. □

Theorem 4.5. *Suppose that assumption (F_1) is satisfied. Then, there exists $\lambda_* > 0$ such that for any $\lambda > \lambda_*$, the problem (P_λ) has at least a nontrivial solution in E.*

Proof. (of Theorem 4.5) In this proof, we shall apply the Ekeland's variational principle [16].

As in Lemma 4.3, hypothesis (F_1) implies that for all $\lambda > 0$, there exist two constant $\gamma, \rho > 0$ such that

$$T_\lambda(u) \geq \gamma > 0, \; u \in E \text{ with } \|u\| = \rho,$$

which means that on $\partial B_\rho(0)$; the boundary of the ball centered at the origin and of radius ρ in E

$$\inf_{\partial B_\rho(0)} T_\lambda(u) > 0.$$

Furthermore, taking a function ω in E defined as

$$\omega(t) = \delta, \quad for\ all\ t \in [1, N]_{\mathbb{N}},$$

where δ is a positive real number small enough such that

$$\delta < \frac{\rho^2}{2},$$

then $\omega \in B_\rho(0)$. We put

$$\lambda_* = \frac{4\delta^{p^+}}{p^- N F_\delta^-} \text{ and } F_\delta^- = \min_{t\in[1,N]_{\mathbb{N}}} F(t,\delta).$$

Then, we have

$$\begin{aligned}
T_\lambda(\omega) &= \sum_{t=1}^{N+1} \frac{1}{p(t-1)} \left(|\Delta\omega(t-1)|^{p(t-1)} + \sqrt{1 + |\Delta\omega(t-1)|^{2p(t-1)}} - 1 \right) - \lambda \sum_{t=1}^{N} F(t, \omega(t)) \\
&= \frac{1}{p(0)} \left(\delta^{p(0)} + \sqrt{1 + \delta^{2p(0)}} - 1 \right) + \frac{1}{p(N)} \left(\delta^{p(N)} + \sqrt{1 + \delta^{2p(N)}} - 1 \right) - \lambda \sum_{t=1}^{N} F(t, \delta) \\
&\leq \frac{1}{p^-} \left(2\delta^{p^+} + 2\sqrt{1 + \delta^{2p^+}} - 2 \right) - \lambda N F_\delta^- \\
&\leq \frac{4\delta^{p^+}}{p^-} - \lambda N F_\delta^- .
\end{aligned} \tag{4.9}$$

So, for all $\lambda > \lambda_*$, $T_\lambda(\omega) < 0$, thus

$$-\infty < \bar{c} := \inf_{B_\rho(0)} T_\lambda(u) \leq T_\lambda(\omega) < 0.$$

Let $\epsilon > 0$ such that $\epsilon < \inf_{\partial B_\rho(0)} T_\lambda - \inf_{B_\rho(0)} T_\lambda$ and let's use the Ekeland variational principle [16] to the functional $T_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, there exists $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$T_\lambda(u_\epsilon) < \inf_{\overline{B_\rho(0)}} T_\lambda + \epsilon$$

$$T_\lambda(u) > T_\lambda(u_\epsilon) - \epsilon \|u - u_\epsilon\|, \quad u \neq u_\epsilon.$$

Since

$$T_\lambda(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} T_\lambda + \epsilon \leq \inf_{B_\rho(0)} T_\lambda + \epsilon < \inf_{\partial B_\rho(0)} T_\lambda,$$

we deduce that $u_\epsilon \in B_\rho(0)$.

Let define now $L_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $L_\lambda(u) = T_\lambda(u) + \epsilon \|u - u_\epsilon\|$. It is obvious that u_ϵ is a minimum point of L_λ and thus

$$\frac{L_\lambda(u_\epsilon + t\varphi) - L_\lambda(u_\epsilon)}{t} \geq 0, \tag{4.10}$$

for $t > 0$ sufficiently small and $\varphi \in B_\rho(0)$. The last inequality leads us to

$$\frac{T_\lambda(u_\epsilon + t\varphi) - T_\lambda(u_\epsilon)}{t} + \epsilon\|\varphi\| \geq 0.$$

Taking limit when $t \to 0$, we infer that $\langle T'_\lambda(u_\epsilon), \varphi\rangle + \epsilon\|\varphi\| > 0$. Then, $\|T'_\lambda(u_\epsilon)\| \leq \epsilon$.
So, there exists a sequence $(\varphi_n) \subset B_\rho(0)$ and $\bar{c} \in \mathbb{R}$ such that

$$T_\lambda(\varphi_n) \to \bar{c}, \quad T'_\lambda(\varphi_n) \to 0 \text{ where } n \to \infty.$$

It is clear that (φ_n) is bounded in E. Thus, there exists $\varphi_0 \in E$ and a subsequence, still denoted (φ_n) converges to φ_0 in E.

$$T_\lambda(\varphi_0) = \bar{c} < 0, \quad T'_\lambda(\varphi_0) = 0.$$

Consequently, the problem (P_λ) possesses a nontrivial solution. □

Example

Now, we present an example to illustrate the result of Theorem 4.2.
Let us take $N = 10$ and $p(t) = \frac{2}{11} + 3$. By simple calculations, we obtain

$$p^+ = 5 \text{ and } p^- = 3.$$

Let f be the function defined as follows:

$$\begin{cases} f(t,x) = 30e^{9t}x^5, & t \in [1,10]_{\mathbb{N}},\ |x| \leq 1, \\ f(t,x) = 30e^{9t}|x|^3x, & t \in [1,10]_{\mathbb{N}},\ |x| > 1 \end{cases}$$

and

$$\begin{cases} F(t,x) = 5e^{9t}x^6, & t \in [1,10]_{\mathbb{N}}, |x| \leq 1 \\ F(t,x) = 6e^{9t}|x|^5 - e^{9t}, & t \in [1,10]_{\mathbb{N}}, |x| > 1. \end{cases}$$

We have

$$\lim_{x\to 0} \frac{F(t,x)}{|x|^{p^+}} = 0 \text{ and } \liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^{p^+}} \geq 6e^9 \geq \kappa$$

where κ is a constant such that

$$\kappa > \frac{2^{p^+ + 1}}{p^-}(N+1)^{\frac{p^+}{2}} = \frac{2^6 \times 11^3}{3}.$$

Then, conditions (F_1) and (F_2) hold. Therefore, by virtue of Theorem 4.2, the problem (P_λ) has at least one nontrivial solution.

Conclusion

In this work, we used critical point theory, including the Mountain Pass Theorem and Ekeland's variational principle, to establish the existence of nontrivial solutions for the nonlinear discrete problem (P_λ), which involves $p(k)$-Laplacian-like operators. The obtained results highlight the strong theoretical significance and wide potential applications of discrete problems of this type, especially due to their relation to capillary phenomena. However, several questions remain to be addressed in this context, such as uniqueness, the existence of infinitely many solutions, and non-existence of solutions. We intend to explore these aspects in future work.

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