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EXTERNAL DIRECT PRODUCT OF GE-ALGEBRAS

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Abstract The concept of the direct product of a finite family of B-algebras is introduced by Lingcong and Endam [J. A. V. Lingcong and J. C. Endam, Direct product of B-algebras, Int. J. Algebra 10 (2016), no. 1, 33-40.]. In this paper, we introduce the concept of the direct product of an infinite family of GE-algebras, which we call the external direct product —a generalization of the direct product in the sense of Lingcong and Endam. To find the outcome of the external direct product of GE-algebras that are transitive, commutative, left exchangeable, and belligerent, as well as the outcome of the external direct product of GE-subalgebras, GE-filters, belligerent GE-filters, prominent GE-filters, and imploring GE-filters. Also, we introduce the concept of the weak direct product of GE-algebras. Finally, given the external direct product of GE-algebras, we provide several fundamental theorems of (anti-)GE-homomorphisms.

1 Introduction and Preliminaries

Imai and Iséki introduced two classes of abstract algebras called BCK-algebras and BCI-algebras, which are the two important classes of logical algebras and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras [16, 17]. Afterwards, the notion of BCK-algebras and BCI-algebras numbers in different types of logical algebras has been defined by numerous authors, and their distinct properties have been discussed in different articles in [4]. Henkin and Skolem introduced the notion of Hilbert algebras in the mid-20th century. Since then, several scholars have contributed to the study of Hilbert algebras; for example, the reader can refer to [10, 11, 12, 20, 21, 27]. In 1967, Diego [14] proved that Hilbert algebras form a locally finite variety. In 2002, Neggers and Kim [28] constructed a new algebraic structure. They took some properties from BCI and BCK algebras to be called B-algebras. Furthermore, Kim and Kim [23] introduced a new notion called BG-algebra, which is a generalization of B-algebra. They obtained several isomorphism theorems for BG-algebras and related properties.

The generalization process in studying algebraic structures is also a significant area of study. Bandaru et al. introduced and examined the idea of GE-algebras in 2021–2022, which is a generalization of Hilbert algebras [6, 8, 29, 32]. Researchers have extensively and continuously studied the notion of GE-algebras. For example, in 2021, Lee et al. [24] introduced the notions of (commutative, transitive, left exchangeable, belligerent, antisymmetric) interior GE-algebras and bordered interior GE-algebras. In the same year, Ahn et al. [2] introduced the concept of imploring interior GE-filters and investigated their properties. In 2022, Song et al. [33] introduced the notions of an interior GE-filter, a weak interior GE-filter, and a belligerent interior

GE-filter. In the same year, Jun et al. [19] introduced the concept of very true GE-algebra using very true operators, and its properties were studied to expand the scope of research on GE-algebras. Sambasiva Rao et al. [5, 34] studied the radical of filters and generalized lower sets in transitive BE-algebras. Later, Jun and Bandaru [18] introduced a new sub-structure called (vivid) deductive system and examined its properties. In 2024, Bandaru et al. [7] introduced the notion of pseudo GE-algebra as a non-commutative generalization of GE-algebra and studied its properties.

The concept of the direct product [30] was first introduced in the context of groups and has since been shown to possess specific properties. For example, a direct product of a group is also a group, and a direct product of an abelian group is also an abelian group. Then, direct product groups are applied to other algebraic structures. In 2016, Lingcong and Endam [25] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms and obtained related properties, one of which is a direct product of two B-algebras, which is also a B-algebra. Then, they extended the concept of the direct product of a B-algebra to a finite family of B-algebras, and some of the related properties were investigated. Also, they introduced two canonical mappings of the direct product of B-algebras, and we obtained some of their properties [26]. In the same year, Endam and Teves [15] defined the direct product of BF-algebras, 0-commutative BF-algebras, and BF-homomorphisms and obtained related properties. In 2018, Abebe [1] introduced the concept of the finite direct product of BRK-algebras and proved that the finite direct product of BRK-algebras is a BRK-algebra. Widianto et al. [35] defined the direct product of BG-algebras, BG-homomorphisms, and 0-commutative BG-algebras in 2019. They also discussed properties of BG-algebras related to these concepts. In 2020, Setiani et al. [30] defined the direct product of BP-algebras, which is equivalent to B-algebras. Find out the important property of the direct product of BP-algebras, and then describe how it works with finite sets of BP-algebras, finite families of 0-commutative BP-algebras, and finite families of BP-homomorphisms. In 2021, Kavitha and Gowri [22] defined the direct product of GK-algebra. They also derived the finite form of the direct product of a GK-algebra and a function. They investigated and applied the concept of the direct product of GK-algebras in GK-functions and GK-kernels, obtaining interesting results.

We talk about the external direct product in this paper. It is the direct product of an infinite family of GE-algebras. This is a general idea of the direct product in the sense of Lingcong and Endam [25]. To discover the result of the external direct product of GE-algebras: transitive, commutative, left exchangeable, belligerent, and find the result of the external direct product of special subsets of GE-algebras: GE-subalgebras, GE-filters, belligerent GE-filters, prominent GE-filters, and imploring GE-filters. Moreover, we introduce the concept of the weak direct product of GE-algebras. Finally, we discuss several (anti-)GE-homomorphism theorems given the external direct product of GE-algebras.

First of all, we start with the definitions and examples of GE-algebras, as well as other relevant definitions for the study in this paper, as follows:

Definition 1.1. [6] An algebra $X = (X; *, 0)$ of type $(2, 0)$ is called a *GE-algebra* if it satisfies the following axioms:

$$(\forall x \in X)(x * x = 0), \tag{GE-1}$$

$$(\forall x \in X)(0 * x = x), \tag{GE-2}$$

$$(\forall x, y, z \in X)(x * (y * z) = x * (y * (x * z))). \tag{GE-3}$$

In a GE-algebra $X = (X; *, 0)$, the binary relation $\leq$ on X is defined as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0).$$

Definition 1.2. [6, 8] A GE-algebra $X = (X; *, 0)$ is said to be

(i) *transitive* if it satisfies:

$$(\forall x, y, z \in X)((x * y) * ((z * x) * (z * y)) = 0), \tag{transitive}$$

(ii) *commutative* if it satisfies:

$$(\forall x, y \in X)((x * y) * y = (y * x) * x), \tag{commutative}$$

(iii) *left exchangeable* if it satisfies:

$$(\forall x, y, z \in X)(x * (y * z) = y * (x * z)), \qquad \text{(left exchangeable)}$$

(iv) *belligerent* if it satisfies:

$$(\forall x, y, z \in X)(x * (y * z) = (x * y) * (x * z)). \qquad \text{(belligerent)}$$

Definition 1.3. [6, 29, 31] A nonempty subset S of a GE-algebra $X = (X; *, 0)$ is called

(i) a *GE-subalgebra* of X if it satisfies the following condition:

$$(\forall x, y \in S)(x * y \in S), \tag{1.1}$$

(ii) a *GE-filter* of X if it satisfies the following conditions:

$$\text{the constant } 0 \text{ of } X \text{ is in } S, \tag{1.2}$$

$$(\forall x, y \in X)(x * y \in S, x \in S \Rightarrow y \in S), \tag{1.3}$$

(iii) a *belligerent GE-filter* of X if it satisfies the condition (1.2) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, x * y \in S \Rightarrow x * z \in S), \tag{1.4}$$

(iv) a *prominent GE-filter* of X if it satisfies the condition (1.2) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, x \in S \Rightarrow ((z * y) * y) * z \in S), \tag{1.5}$$

(v) an *imploring GE-filter* of X if it satisfies the condition (1.2) and the following condition:

$$(\forall x, y, z \in X)(x * ((y * z) * y) \in S, x \in S \Rightarrow y \in S). \tag{1.6}$$

Example 1.4. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	3	0	5
2	0	0	0	0	0	5
3	0	0	0	0	4	5
4	0	0	0	0	0	0
5	0	1	2	3	4	0

Then $X = (X; *, 0)$ is a GE-algebra.

The concept of GE-homomorphisms was also introduced by Rezaei et al. [29].

Let $A = (A; *_A, 0_A)$ and $B = (B; *_B, 0_B)$ be GE-algebras. A map $\varphi : A \to B$ is called a *GE-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(x) *_B \varphi(y))$$

and an *anti-GE-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(y) *_B \varphi(x)).$$

The *kernel* of φ, denoted by $\ker \varphi$, is defined to be the $\{x \in A \mid \varphi(x) = 0_B\}$. The $\ker \varphi$ is a GE-filter of A, and $\ker \varphi = \{0_A\}$ if and only if φ is injective. A (anti-)GE-homomorphism φ is called a (anti-)GE-monomorphism, (anti-)GE-epimorphism, or (anti-)GE-isomorphism if φ is injective, surjective, or bijective, respectively.

In a GE-algebra $X = (X; *, 0)$, the following assertions are valid (see [6]).

$$(\forall x \in X)(x * 0 = 0), \tag{1.7}$$
$$(\forall x, y \in X)(x * (x * y) = x * y), \tag{1.8}$$
$$(\forall x \in X)(0 * x = 0 \Rightarrow x = 0), \tag{1.9}$$
$$(\forall x, y \in X)(x * (y * x) = 0), \tag{1.10}$$
$$(\forall x, y \in X)(x * ((x * y) * y) = 0), \tag{1.11}$$
$$(\forall x, y \in X)(x * ((y * x) * x) = 0), \tag{1.12}$$
$$(\forall x, y \in X)(x * ((x * y) * x) = 0), \tag{1.13}$$
$$(\forall x, y \in X)(x * (y * (y * x)) = 0), \tag{1.14}$$
$$(\forall x, y, z \in X)((x * (y * z)) * (y * (x * z)) = 0), \tag{1.15}$$
$$(\forall u, x, y, z \in X)(x * (y * z) = 0 \Leftrightarrow y * (x * z) = 0). \tag{1.16}$$

2 Main Results

Lingcong and Endam [25] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms and obtained related properties, one of which is a direct product of two B-algebras, which is also a B-algebra. Then, they extended the concept of the direct product of B-algebras to a finite family of B-algebras, and some of the related properties were investigated as follows:

Definition 2.1. [25] Let $(X_i; *_i)$ be an algebra for each $i \in \{1, 2, ..., k\}$. Define the *direct product* of algebras $X_1, X_2, ..., X_k$ to be the structure $(\prod_{i=1}^{k} X_i; \otimes)$, where

$$\prod_{i=1}^{k} X_i = X_1 \times X_2 \times ... \times X_k = \{(x_1, x_2, ..., x_k) \mid x_i \in X_i \ \forall i = 1, 2, ..., k\}$$

and whose operation $\otimes$ is given by

$$(x_1, x_2, ..., x_k) \otimes (y_1, y_2, ..., y_k) = (x_1 *_1 y_1, x_2 *_2 y_2, ..., x_k *_k y_k)$$

for all $(x_1, x_2, ..., x_k), (y_1, y_2, ..., y_k) \in \prod_{i=1}^{k} X_i$.

Now, we extend the concept of the direct product to an infinite family of GE-algebras and provide some of its properties.

Definition 2.2. Let X_i be a nonempty set for each $i \in I$. Define the *external direct product* of sets X_i for all $i \in I$ to be the set $\prod_{i \in I} X_i$, where

$$\prod_{i \in I} X_i = \{f : I \to \bigcup_{i \in I} X_i \mid f(i) \in X_i \ \forall i \in I\}.$$

For convenience, we define an element of $\prod_{i \in I} X_i$ with a function $(x_i)_{i \in I} : I \to \bigcup_{i \in I} X_i$, where $i \mapsto x_i \in X_i$ for all $i \in I$.

Remark 2.3. Let X_i be a nonempty set and S_i a subset of X_i for all $i \in I$. Then $\prod_{i \in I} S_i$ is a nonempty subset of the external direct product $\prod_{i \in I} X_i$ if and only if S_i is a nonempty subset of X_i for all $i \in I$.

Definition 2.4. Let $X_i = (X_i; *_i)$ be an algebra for all $i \in I$. Define the binary operation $\otimes$ on the external direct product $\prod_{i \in I} X_i = (\prod_{i \in I} X_i; \otimes)$ as follows:

$$(\forall (x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} X_i)((x_i)_{i \in I} \otimes (y_i)_{i \in I} = (x_i *_i y_i)_{i \in I}). \tag{2.1}$$

We shall show that $\otimes$ is a binary operation on $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$. Since $*_i$ is a binary operation on X_i for all $i \in I$, we have $x_i *_i y_i \in X_i$ for all $i \in I$. Then $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} X_i$ such that

$$(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I}.$$

Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (x'_i)_{i\in I}, (y'_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} = (y_i)_{i\in I}$ and $(x'_i)_{i\in I} = (y'_i)_{i\in I}$. We shall show that $(x_i)_{i\in I} \otimes (x'_i)_{i\in I} = (y_i)_{i\in I} \otimes (y'_i)_{i\in I}$. Then

$$x_i = y_i \text{ for all } i \in I \text{ and } x'_i = y'_i \text{ for all } i \in I.$$

Since $*_i$ is a binary operation on X_i for all $i \in I$, we have $x_i *_i x'_i = y_i *_i y'_i$ for all $i \in I$. Thus,

$$\begin{aligned}(x_i)_{i\in I} \otimes (x'_i)_{i\in I} &= (x_i *_i x'_i)_{i\in I}\\ &= (y_i *_i y'_i)_{i\in I}\\ &= (y_i)_{i\in I} \otimes (y'_i)_{i\in I}.\end{aligned}$$

Hence, $\otimes$ is a binary operation on $\prod_{i\in I} X_i$.

Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra for all $i \in I$. For $i \in I$, let $x_i \in X_i$. We define the function $f_{x_i} : I \to \bigcup_{i\in I} X_i$ as follows:

$$(\forall j \in I)\left(f_{x_i}(j) = \begin{cases} x_i & \text{if } j = i\\ 0_j & \text{otherwise}\end{cases}\right). \tag{2.2}$$

Then $f_{x_i} \in \prod_{i\in I} X_i$.

Remark 2.5. Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra for all $i \in I$. For $i \in I$, we have $f_{0_i} = (0_i)_{i\in I}$.

Lemma 2.6. *Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra for all $i \in I$. For $i \in I$, let $x_i, y_i \in X_i$. Then $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i}$.*

Proof. Now,

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i\\ 0_j *_j 0_j & \text{otherwise}\end{cases}\right).$$

By (GE-1), we have

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i\\ 0_j & \text{otherwise}\end{cases}\right).$$

By (2.2), we have $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i}$. □

The following theorem shows that the direct product of GE-algebras in terms of an infinite family of GE-algebras is also a GE-algebra.

Theorem 2.7. *$X_i = (X_i; *_i, 0_i)$ is a GE-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a GE-algebra, where the binary operation $\otimes$ is defined in Definition 2.4.*

Proof. Assume that $X_i = (X_i; *_i, 0_i)$ is a GE-algebra for all $i \in I$.

(GE-1) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (GE-1), we have $x_i *_i x_i = 0_i$ for all $i \in I$. Thus,

$$(x_i)_{i\in I} \otimes (x_i)_{i\in I} = (x_i *_i x_i)_{i\in I} = (0_i)_{i\in I}.$$

(GE-2) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (GE-2), we have $0_i *_i x_i = x_i$ for all $i \in I$. Thus,

$$(0_i)_{i\in I} \otimes (x_i)_{i\in I} = (0_i *_i x_i)_{i\in I} = (x_i)_{i\in I}.$$

(GE-3) Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (GE-3), we have $x_i *_i (y_i *_i z_i) = x_i *_i (y_i *_i (x_i *_i z_i))$ for all $i \in I$. Thus,

$$\begin{aligned}(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) &= (x_i)_{i\in I} \otimes (y_i *_i z_i)_{i\in I}\\ &= (x_i *_i (y_i *_i z_i))_{i\in I}\\ &= (x_i *_i (y_i *_i (x_i *_i z_i)))_{i\in I}\\ &= (x_i)_{i\in I} \otimes (y_i *_i (x_i *_i z_i))_{i\in I}\\ &= (x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (x_i *_i z_i)_{i\in I})\\ &= (x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes ((x_i)_{i\in I} \otimes (z_i)_{i\in I})).\end{aligned}$$

Hence, $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a GE-algebra.

Conversely, assume that $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a GE-algebra, where the binary operation $\otimes$ is defined in Definition 2.4. Let $i \in I$.

(GE-1) Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (2.2). Since $\prod_{i\in I} X_i$ satisfies (GE-1), we have $f_{x_i} \otimes f_{x_i} = (0_i)_{i\in I}$. By Lemma 2.6 and (2.2), we have $x_i *_i x_i = 0_i$.

(GE-2) Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (2.2). Since $\prod_{i\in I} X_i$ satisfies (GE-2), we have $(0_i)_{i\in I} \otimes f_{x_i} = f_{x_i}$. By Lemma 2.6, Remark 2.5, and (2.2), we have $0_i *_i x_i = x_i$.

(GE-3) Let $x_i, y_i, z_i \in X_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$, which are defined by (2.2). Since $\prod_{i\in I} X_i$ satisfies (GE-3), we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = f_{x_i} \otimes (f_{y_i} \otimes (f_{x_i} \otimes f_{z_i}))$. By Lemma 2.6 and (2.2), we have $x_i * (y_i * z_i) = x_i * (y_i * (x_i * z_i))$.

Hence, $X_i = (X_i; *_i, 0_i)$ is a GE-algebra for all $i \in I$. □

Example 2.8. Let $X_1 = \{0,1,2\}, X_2 = X_3 = \{0,1,2,3\}$, and $X_4 = X_5 = \{0,1,2,3,4\}$ be sets with the Cayley tables, respectively, as follows:

$*_1$	0	1	2
0	0	1	2
1	0	0	2
2	0	1	0

$*_2$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	1
3	0	0	0	0

$*_3$	0	1	2	3
0	0	1	2	3
1	0	0	0	3
2	0	0	0	3
3	0	0	0	0

$*_4$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	4	0	4
2	0	0	0	0	0
3	0	1	4	0	4
4	0	1	3	3	0

$*_5$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	4	3	4
2	0	4	0	3	4
3	0	4	4	0	4
4	0	0	0	0	0

Then $X_i = (X_i; *_i, 0)$ is a GE-algebra for all $i \in I = \{1,2,3,4,5\}$. Then $X_1 \times X_2 \times X_3 \times X_4 \times X_5 = \prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0,0,0,0,0))$ is a GE-algebra, where the binary operation $\otimes$ is defined in Definition 2.4.

We call the GE-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ in Theorem 2.7 the external direct product GE-algebra induced by a GE-algebra $X_i = (X_i; *_i, 0_i)$ for all $i \in I$.

Theorem 2.9. *Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra for all $i \in I$. Then X_i is transitive for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is transitive, where the binary operation $\otimes$ is defined in Definition 2.4.*

Proof. By Theorem 2.7, we have $X_i = (X_i; *_i, 0_i)$ is a GE-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a GE-algebra, where the binary operation $\otimes$ is defined in Definition 2.4. We are left to prove that X_i is transitive for all $i \in I$ if and only if $\prod_{i\in I} X_i$ is transitive.

Assume that X_i is transitive for all $i \in I$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i is transitive for all $i \in I$, we have $(x_i *_i y_i) *_i ((z_i *_i x_i) *_i (z_i *_i y_i)) = 0_i$ for all $i \in I$. Thus,

$$\begin{aligned}
&((x_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes (((z_i)_{i\in I} \otimes (x_i)_{i\in I}) \otimes ((z_i)_{i\in I} \otimes (y_i)_{i\in I}))\\
&= (x_i *_i y_i)_{i\in I} \otimes ((z_i *_i x_i)_{i\in I} \otimes (z_i *_i y_i)_{i\in I})\\
&= (x_i *_i y_i)_{i\in I} \otimes ((z_i *_i x_i) *_i (z_i *_i y_i))_{i\in I}\\
&= ((x_i *_i y_i) *_i ((z_i *_i x_i) *_i (z_i *_i y_i)))_{i\in I}\\
&= (0_i)_{i\in I}.
\end{aligned}$$

Hence, $\prod_{i\in I} X_i$ is transitive.

Conversely, assume that $\prod_{i\in I} X_i$ is transitive. Let $i \in I$. Let $x_i, y_i, z_i \in X_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$, which are defined by (2.2). Since $\prod_{i\in I} X_i$ is transitive, we have $(f_{x_i} \otimes f_{y_i}) \otimes ((f_{z_i} \otimes f_{x_i}) \otimes (f_{z_i} \otimes f_{y_i})) = (0_i)_{i\in I}$. By Lemma 2.6 and (2.2), we have $(x_i *_i y_i) *_i ((z_i *_i x_i) *_i (z_i *_i y_i)) = 0_i$. Hence, X_i is transitive for all $i \in I$. □

Theorem 2.10. *Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra for all $i \in I$. Then X_i is commutative for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is commutative, where the binary operation $\otimes$ is defined in Definition 2.4.*

Proof. By Theorem 2.7, we have $X_i = (X_i; *_i, 0_i)$ is a GE-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a GE-algebra, where the binary operation $\otimes$ is defined in Definition 2.4. We are left to prove that X_i is commutative for all $i \in I$ if and only if $\prod_{i\in I} X_i$ is commutative.

Assume that X_i is commutative for all $i \in I$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i is commutative for all $i \in I$, we have $(x_i *_i y_i) *_i y_i = (y_i *_i x_i) *_i x_i$ for all $i \in I$. Thus,

$$\begin{aligned}
((x_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes (y_i)_{i\in I} &= (x_i *_i y_i)_{i\in I} \otimes (y_i)_{i\in I}\\
&= ((x_i *_i y_i) *_i y_i)_{i\in I}\\
&= ((y_i *_i x_i) *_i x_i)_{i\in I}\\
&= (y_i *_i x_i)_{i\in I} \otimes (x_i)_{i\in I}\\
&= ((y_i)_{i\in I} \otimes (x_i)_{i\in I}) \otimes (x_i)_{i\in I}.
\end{aligned}$$

Hence, $\prod_{i\in I} X_i$ is commutative.

Conversely, assume that $\prod_{i\in I} X_i$ is commutative. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (2.2). Since $\prod_{i\in I} X_i$ is commutative, we have $(f_{x_i} \otimes f_{y_i}) \otimes f_{y_i} = (f_{y_i} \otimes f_{x_i}) \otimes f_{x_i}$. By Lemma 2.6 and (2.2), we have $(x_i *_i y_i) *_i y_i = (y_i *_i x_i) *_i x_i$. Hence, X_i is commutative for all $i \in I$. □

Theorem 2.11. *Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra for all $i \in I$. Then X_i is left exchangeable for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is left exchangeable, where the binary operation $\otimes$ is defined in Definition 2.4.*

Proof. By Theorem 2.7, we have $X_i = (X_i; *_i, 0_i)$ is a GE-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a GE-algebra, where the binary operation $\otimes$ is defined in Definition 2.4. We are left to prove that X_i is left exchangeable for all $i \in I$ if and only if $\prod_{i\in I} X_i$ is left exchangeable.

Assume that X_i is left exchangeable for all $i \in I$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i is left exchangeable for all $i \in I$, we have $x_i *_i (y_i *_i z_i) = y_i *_i (x_i *_i z_i)$ for all $i \in I$. Thus,

$$\begin{aligned}
(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) &= (x_i)_{i\in I} \otimes (y_i *_i z_i)_{i\in I}\\
&= (x_i *_i (y_i *_i z_i))_{i\in I}\\
&= (y_i *_i (x_i *_i z_i))_{i\in I}\\
&= (y_i)_{i\in I} \otimes (x_i *_i z_i)_{i\in I}\\
&= (y_i)_{i\in I} \otimes ((x_i)_{i\in I} \otimes (z_i)_{i\in I}).
\end{aligned}$$

Hence, $\prod_{i\in I} X_i$ is left exchangeable.

Conversely, assume that $\prod_{i\in I} X_i$ is left exchangeable. Let $i \in I$. Let $x_i, y_i, z_i \in X_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$, which are defined by (2.2). Since $\prod_{i\in I} X_i$ is left exchangeable, we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = f_{y_i} \otimes (f_{x_i} \otimes f_{z_i})$. By Lemma 2.6 and (2.2), we have $x_i *_i (y_i *_i z_i) = y_i *_i (x_i *_i z_i)$. Hence, X_i is left exchangeable for all $i \in I$. □

Theorem 2.12. *Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra for all $i \in I$. Then X_i is belligerent for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is belligerent, where the binary operation $\otimes$ is defined in Definition 2.4.*

Proof. By Theorem 2.7, we have $X_i = (X_i; *_i, 0_i)$ is a GE-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a GE-algebra, where the binary operation $\otimes$ is defined in Definition 2.4. We are left to prove that X_i is belligerent for all $i \in I$ if and only if $\prod_{i\in I} X_i$ is belligerent.

Assume that X_i is belligerent for all $i \in I$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i is belligerent for all $i \in I$, we have $x_i *_i (y_i *_i z_i) = (x_i *_i y_i) *_i (x_i *_i z_i)$ for all $i \in I$. Thus,

$$\begin{aligned}(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) &= (x_i)_{i\in I} \otimes (y_i *_i z_i)_{i\in I}\\ &= (x_i *_i (y_i *_i z_i))_{i\in I}\\ &= ((x_i *_i y_i) *_i (x_i *_i z_i))_{i\in I}\\ &= (x_i *_i y_i)_{i\in I} \otimes (x_i *_i z_i)_{i\in I}\\ &= ((x_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes ((x_i)_{i\in I} \otimes (z_i)_{i\in I}).\end{aligned}$$

Hence, $\prod_{i\in I} X_i$ is belligerent.

Conversely, assume that $\prod_{i\in I} X_i$ is belligerent. Let $i \in I$. Let $x_i, y_i, z_i \in X_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$, which is defined by (2.2). Since $\prod_{i\in I} X_i$ is belligerent, we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = (f_{x_i} \otimes f_{y_i}) \otimes (f_{x_i} \otimes f_{z_i})$. By Lemma 2.6 and (2.2), we have $x_i *_i (y_i *_i z_i) = (x_i *_i y_i) *_i (x_i *_i z_i)$. Hence, X_i is belligerent for all $i \in I$. □

Next, we introduce the concept of the weak direct product of an infinite family of GE-algebras and obtain some of its properties as follows:

Definition 2.13. Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra for all $i \in I$. Define the *weak direct product* of a GE-algebra X_i for all $i \in I$ to be the structure $\prod^{\mathrm{w}}_{i\in I} X_i = (\prod^{\mathrm{w}}_{i\in I} X_i; \otimes)$, where

$$\prod^{\mathrm{w}}_{i\in I} X_i = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid x_i \neq 0_i, \text{ where the number of such } i \text{ is finite}\}.$$

Then $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i \subseteq \prod_{i\in I} X_i$.

Theorem 2.14. *Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra for all $i \in I$. Then $\prod^{\mathrm{w}}_{i\in I} X_i$ is a GE-subalgebra of the external direct product GE-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. We see that $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$, where $I_1 = \{i \in I \mid x_i \neq 0_i\}$ and $I_2 = \{i \in I \mid y_i \neq 0_i\}$ are finite. Then $|I_1 \cup I_2|$ is finite. Thus,

$$(\forall j \in I)\left(((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) = \begin{cases} x_j *_j 0_j & \text{if } j \in I_1 - I_2\\ x_j *_j y_j & \text{if } j \in I_1 \cap I_2\\ 0_j *_j y_j & \text{if } j \in I_2 - I_1\\ 0_j *_j 0_j & \text{otherwise}\end{cases}\right).$$

By (GE-2) and (1.7), we have

$$(\forall j \in I)\left(((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) = \begin{cases} 0_j & \text{if } j \in I_1 - I_2\\ x_j *_j y_j & \text{if } j \in I_1 \cap I_2\\ y_j & \text{if } j \in I_2 - I_1\\ 0_j & \text{otherwise}\end{cases}\right).$$

This implies that the number of such $((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j)$ is not more than $|I_1 \cup I_2|$, that is, it is finite. Thus, $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$. Hence, $\prod^{\mathrm{w}}_{i\in I} X_i$ is a GE-subalgebra of $\prod_{i\in I} X_i$. □

Theorem 2.15. *Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra for all $i \in I$. Then $\prod^{\text{w}}_{i\in I} X_i$ is a GE-filter of the external direct product GE-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. We see that $(0_i)_{i\in I} \in \prod^{\text{w}}_{i\in I} X_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod^{\text{w}}_{i\in I} X_i$, where $I_1 = \{i \in I \mid x_i \neq 0_i\}$ and $I_2 = \{i \in I \mid y_i \neq 0_i\}$ are finite. Then $|I_1 \cup I_2|$ is finite. Thus,

$$(\forall j \in I)\left(((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) = \begin{cases} x_j *_j 0_j & \text{if } j \in I_1 - I_2 \\ x_j *_j y_j & \text{if } j \in I_1 \cap I_2 \\ 0_j *_j y_j & \text{if } j \in I_2 - I_1 \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By (GE-2) and (1.7), we have

$$(\forall j \in I)\left(((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) = \begin{cases} 0_j & \text{if } j \in I_1 - I_2 \\ x_j *_j y_j & \text{if } j \in I_1 \cap I_2 \\ y_j & \text{if } j \in I_2 - I_1 \\ 0_j & \text{otherwise} \end{cases}\right).$$

This implies that the number of such $((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) \neq 0_j$ is not more than $|I_1 \cup I_2|$, that is, it is finite. Thus, $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod^{\text{w}}_{i\in I} X_i$. Hence, $\prod^{\text{w}}_{i\in I} X_i$ is a GE-filter of $\prod_{i\in I} X_i$. □

Theorem 2.16. *Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a GE-subalgebra of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a GE-subalgebra of the external direct product GE-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a GE-subalgebra of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 2.3, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $x_i, y_i \in S_i$ for all $i \in I$. By (1.1), we have $x_i *_i y_i \in S_i$ for all $i \in I$ and so $(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a GE-subalgebra of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a GE-subalgebra of $\prod_{i\in I} X_i$. Since $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 2.3, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By (1.1) and Lemma 2.6, we have $f_{x_i *_i y_i} = f_{x_i} \otimes f_{y_i} \in \prod_{i\in I} S_i$. By (2.2), we have $x_i *_i y_i \in S_i$. Hence, S_i is a GE-subalgebra of X_i for all $i \in I$. □

Theorem 2.17. *Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a GE-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a GE-filter of the external direct product GE-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a GE-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus, $x_i *_i y_i \in S_i$ and $x_i \in S_i$, it follows from (1.3) that $y_i \in S_i$ for all $i \in I$. Thus, $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a GE-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a GE-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $x_i *_i y_i \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.6, we have $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (1.3), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (2.2), we have $y_i \in S_i$. Hence, S_i is a GE-filter of X_i for all $i \in I$. □

Theorem 2.18. *Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a belligerent GE-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a belligerent GE-filter of the external direct product GE-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a belligerent GE-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$ and

$(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus, $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i *_i y_i \in S_i$, it follows from (1.4) that $x_i *_i z_i \in S_i$ for all $i \in I$. Thus, $(x_i)_{i\in I} \otimes (z_i)_{i\in I} = (x_i *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a belligerent GE-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a belligerent GE-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i *_i y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.6, we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (1.4) and Lemma 2.6, we have $f_{x_i *_i z_i} = f_{x_i} \otimes f_{z_i} \in \prod_{i\in I} S_i$. By (2.2), we have $x_i *_i z_i \in S_i$. Hence, S_i is a belligerent GE-filter of X_i for all $i \in I$. □

Theorem 2.19. *Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a prominent GE-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a prominent GE-filter of the external direct product GE-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a prominent GE-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus, $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i \in S_i$, it follows from (1.5) that $((z_i *_i y_i) *_i y_i) *_i z_i \in S_i$ for all $i \in I$. Thus, $(((z_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes (y_i)_{i\in I}) \otimes (z_i)_{i\in I} = (((z_i *_i y_i) *_i y_i) *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a prominent GE-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a prominent GE-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.6, we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$. By (1.5) and Lemma 2.6, we have $f_{((z_i *_i y_i) *_i y_i) *_i z_i} = ((f_{z_i} \otimes f_{y_i}) \otimes f_{y_i}) \otimes f_{z_i} \in \prod_{i\in I} S_i$. By (2.2), we have $((z_i *_i y_i) *_i y_i) *_i z_i \in S_i$. Hence, S_i is a prominent GE-filter of X_i for all $i \in I$. □

Theorem 2.20. *Let $X_i = (X_i; *_i, 0_i)$ be a GE-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is an imploring GE-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is an imploring GE-filter of the external direct product GE-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is an imploring GE-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes (((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \otimes (y_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x * ((y * z) * y))_{i\in I} \in \prod_{i\in I} S_i$. Thus, $x * ((y * z) * y) \in S_i$ and $x_i \in S_i$, it follows from (1.6) that $y_i \in S_i$ for all $i \in I$. Thus, $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is an imploring GE-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is an imploring GE-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x * ((y * z) * y) \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x*((y*z)*y)} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.6, we have $f_{x_i} \otimes ((f_{y_i} \otimes f_{z_i}) \otimes f_{y_i}) = f_{x*((y*z)*y)} \in \prod_{i\in I} S_i$. By (1.6), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (2.2), we have $y_i \in S_i$. Hence, S_i is an imploring GE-filter of X_i for all $i \in I$. □

Moreover, we discuss several homomorphism theorems given the external direct product of GE-algebras.

Definition 2.21. [13] Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Define the function $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ given by

$$(\forall (x_i)_{i\in I} \in \prod_{i\in I} X_i)(\psi(x_i)_{i\in I} = (\psi_i(x_i))_{i\in I}). \tag{2.3}$$

Then $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ is a function (see [13]).

Theorem 2.22. *[13] Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$.*

(i) ψ_i is injective for all $i \in I$ if and only if ψ is injective which is defined in Definition 2.21,

(ii) ψ_i is surjective for all $i \in I$ if and only if ψ is surjective,

(iii) *ψ_i is bijective for all $i \in I$ if and only if ψ is bijective.*

Theorem 2.23. *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be GE-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

(i) *ψ_i is a GE-homomorphism for all $i \in I$ if and only if ψ is a GE-homomorphism which is defined in Definition 2.21,*

(ii) *ψ_i is a GE-monomorphism for all $i \in I$ if and only if ψ is a GE-monomorphism,*

(iii) *ψ_i is a GE-epimorphism for all $i \in I$ if and only if ψ is a GE-epimorphism,*

(iv) *ψ_i is a GE-isomorphism for all $i \in I$ if and only if ψ is a GE-isomorphism,*

(v) $\ker \psi = \prod_{i\in I} \ker \psi_i$ *and* $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$.

Proof. (i) Assume that ψ_i is a GE-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x_i')_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned} \psi((x_i)_{i\in I} \otimes (x_i')_{i\in I}) &= \psi(x_i *_i x_i')_{i\in I} \\ &= (\psi_i(x_i *_i x_i'))_{i\in I} \\ &= (\psi_i(x_i) *_i \psi_i(x_i'))_{i\in I} \\ &= (\psi_i(x_i))_{i\in I} \otimes (\psi_i(x_i'))_{i\in I} \\ &= \psi(x_i)_{i\in I} \otimes \psi(x_i')_{i\in I}. \end{aligned}$$

Hence, ψ is a GE-homomorphism.

Conversely, assume that ψ is a GE-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which is defined by (2.2). Since ψ is a GE-homomorphism, we have $\psi(f_{x_i} \otimes f_{y_i}) = \psi(f_{x_i}) \otimes \psi(f_{y_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{x_i} \otimes f_{y_i})(j) = \begin{cases} \psi_i(x_i *_i y_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise} \end{cases}\right). \tag{2.4}$$

Since

$$(\forall j \in I)\left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right)$$

and

$$(\forall j \in I)\left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{x_i}) \otimes \psi(f_{y_i}))(j) = \begin{cases} \psi_i(x_i) \circ_i \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases}\right). \tag{2.5}$$

By (2.4) and (2.5), we have $\psi_i(x_i *_i y_i) = \psi_i(x_i) \circ_i \psi_i(y_i)$. Hence, ψ_i is a GE-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 2.22 (i).

(iii) It is straightforward from (i) and Theorem 2.22 (ii).

(iv) It is straightforward from (i) and Theorem 2.22 (iii).

(v) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}(x_i)_{i\in I} \in \ker\psi &\Leftrightarrow \psi(x_i)_{i\in I} = (1_i)_{i\in I}\\ &\Leftrightarrow (\psi_i(x_i))_{i\in I} = (1_i)_{i\in I}\\ &\Leftrightarrow \psi_i(x_i) = 1_i \ \forall i \in I\\ &\Leftrightarrow x_i \in \ker\psi_i \ \forall i \in I\\ &\Leftrightarrow (x_i)_{i\in I} \in \prod_{i\in I} \ker\psi_i.\end{aligned}$$

Hence, $\ker\psi = \prod_{i\in I} \ker\psi_i$. Now,

$$\begin{aligned}(y_i)_{i\in I} \in \psi(\prod_{i\in I} X_i) &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = \psi(x_i)_{i\in I}\\ &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = (\psi_i(x_i))_{i\in I}\\ &\Leftrightarrow \exists x_i \in X_i \text{ s.t. } y_i = \psi_i(x_i) \in \psi(X_i) \ \forall i \in I\\ &\Leftrightarrow (y_i)_{i\in I} \in \prod_{i\in I} \psi_i(X_i).\end{aligned}$$

Hence, $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$. □

Finally, we discuss several anti-GE-homomorphism theorems given the external direct product of GE-algebras.

Theorem 2.24. *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be GE-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

- (i) *ψ_i is an anti-GE-homomorphism for all $i \in I$ if and only if ψ is an anti-GE-homomorphism which is defined in Definition 2.21,*
- (ii) *ψ_i is an anti-GE-monomorphism for all $i \in I$ if and only if ψ is an anti-GE-monomorphism,*
- (iii) *ψ_i is an anti-GE-epimorphism for all $i \in I$ if and only if ψ is an anti-GE-epimorphism,*
- (iv) *ψ_i is an anti-GE-isomorphism for all $i \in I$ if and only if ψ is an anti-GE-isomorphism.*

Proof. (i) Assume that ψ_i is an anti-GE-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}\psi((x_i)_{i\in I} \otimes (x'_i)_{i\in I}) &= \psi(x_i *_i x'_i)_{i\in I}\\ &= (\psi_i(x_i *_i x'_i))_{i\in I}\\ &= (\psi_i(x'_i) *_i \psi_i(x_i))_{i\in I}\\ &= (\psi_i(x'_i))_{i\in I} \otimes (\psi_i(x_i))_{i\in I}\\ &= \psi(x'_i)_{i\in I} \otimes \psi(x_i)_{i\in I}.\end{aligned}$$

Hence, ψ is an anti-GE-homomorphism.

Conversely, assume that ψ is an anti-GE-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (2.2). Since ψ is an anti-GE-homomorphism, we have $\psi(f_{x_i} \otimes f_{y_i}) = \psi(f_{y_i}) \otimes \psi(f_{x_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i\\ 0_j *_j 0_j & \text{otherwise}\end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{x_i} \otimes f_{y_i})(j) = \begin{cases} \psi_i(x_i *_i y_i) & \text{if } j = i\\ \psi_j(0_j *_j 0_j) & \text{otherwise}\end{cases}\right). \tag{2.6}$$

Since

$$(\forall j \in I) \left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases} \right)$$

and

$$(\forall j \in I) \left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases} \right),$$

we have

$$(\forall j \in I) \left((\psi(f_{y_i}) \otimes \psi(f_{x_i}))(j) = \begin{cases} \psi_i(y_i) \circ_i \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases} \right). \tag{2.7}$$

By (2.6) and (2.7), we have $\psi_i(x_i *_i y_i) = \psi_i(y_i) \circ_i \psi_i(x_i)$. Hence, ψ_i is an anti-GE-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 2.22 (i).

(iii) It is straightforward from (i) and Theorem 2.22 (ii).

(iv) It is straightforward from (i) and Theorem 2.22 (iii). □

3 Conclusions and Future Work

We talked about the external direct product in this paper. It is the direct product of an infinite family of GE-algebras. This is a general idea of the direct product in the sense of Lingcong and Endam [25]. We showed that the external direct product of GE-algebras (transitions, commutative, left exchangeable, and belligerent) is also a GE-algebra. Also, we have introduced the concept of the weak direct product of GE-algebras. Our proof shows that the weak direct product of GE-algebras is a GE-subalgebra. Also, the external direct product of GE-subalgebras (i.e., GE-filters, belligerent GE-filters, prominent GE-filters, imploring GE-filters) is a GE-subalgebra of the external direct product of GE-algebras (i.e., GE-filters, belligerent GE-filters, prominent GE-filters, imploring GE-filters). Finally, we have given several basic theorems about (anti-)GE-homomorphisms regarding external direct product GE-algebras.

Based on the concept of the external direct product of GE-algebras in this article, we can apply it to studying the external direct product in other algebraic systems. In the future, we will look into the external and weak direct sums of semi-nil clean rings and S-Nil_*-coherent rings [9, 3].

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