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PRODUCT OF PRIME IDEALS AS FACTORIZATION OF SUBMODULES

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Abstract For a proper submodule N of a finitely generated module M over a Noetherian ring, the product of prime ideals which occur in a regular prime extension filtration of M over N is defined as its generalized prime ideal factorization in M. In this article, we find conditions for a product of prime ideals to be the generalized prime ideal factorization of a submodule of some module. We show that a power of a prime ideal occurs in a generalized prime ideal factorization only if it is not equal to its lesser powers. Also, we show that ${\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_n}^{r_n}$ is a generalized prime ideal factorization if and only if for each $1 \leq i \leq n$, ${\mathfrak{p}_i}^{r_i}$ is the generalized prime ideal factorization of some submodule of a module.

1 Introduction

Throughout this article R is a commutative Noetherian ring with identity and all R-modules are assumed to be finitely generated and unitary.

Let N be a proper submodule of an R-module M. If K is a submodule of M such that N is a $\mathfrak{p}$-prime submodule of K, then we say K is a $\mathfrak{p}$-prime extension of N in M and denote it as $N \overset{\mathfrak{p}}{\subset} K$. In this case, $\operatorname{Ass}(K/N) = \{\mathfrak{p}\}$ [4, Theorem 1]. A $\mathfrak{p}$-prime extension K of N is said to be maximal in M if there is no $\mathfrak{p}$-prime extension L of N in M such that $L \supset K$. It is proved that if N is a proper submodule of M and $\mathfrak{p}$ is a maximal element in $\operatorname{Ass}(M/N)$, then $(N : \mathfrak{p})$ is the unique maximal $\mathfrak{p}$-prime extension of N in M [2, Theorem 11] and it is called a regular $\mathfrak{p}$-prime extension of N in M.

A filtration of submodules $\mathcal{F} : N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$ is called a regular prime extension (RPE) filtration of M over N if each M_i is a regular $\mathfrak{p}_i$-prime extension of M_{i-1} in M, $1 \leq i \leq n$. We also have that $M_i \overset{\mathfrak{p}_{i+1}}{\subset} M_{i+1} \subset \cdots \overset{\mathfrak{p}_j}{\subset} M_j$ is an RPE filtration of M_j over M_i for every $0 \leq i < j \leq n$. RPE filtrations are defined and studied in [2] and it is also noted that RPE filtrations are weak prime decompositions defined by Dress in [1]. The set of prime ideals which occur in the RPE filtration $\mathcal{F}$ is precisely $\operatorname{Ass}(M/N)$. This is proved more generally in the following lemma.

Lemma 1.1. *[2, Proposition 14] Let $N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$ be a filtration of submodules such that each $M_{i-1} \overset{\mathfrak{p}_i}{\subset} M_i$ is a maximal $\mathfrak{p}_i$-prime extension. Then $\operatorname{Ass}(M/M_{i-1}) = \{\mathfrak{p}_i, \ldots, \mathfrak{p}_n\}$ for $1 \leq i \leq n$. In particular, $\operatorname{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.*

The submodules occurring in an RPE filtration are characterized as below.

Lemma 1.2. *[3, Lemma 3.1] Let N be a proper submodule of an R-module M. If $N = M_0 \overset{\mathfrak{p}_1}{\subset}$*

$M_1 \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$ *is an RPE filtration of* M *over* N*, then* $M_i = \{x \in M \mid \mathfrak{p}_1 \cdots \mathfrak{p}_i x \subseteq N\} = (N : \mathfrak{p}_1 \cdots \mathfrak{p}_i)$ *for* $1 \leq i \leq n$*.*

It is proved that the number of times a prime ideal occurs in any RPE filtration of M over N is unique [2, Theorem 22]. So if $N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$ is an RPE filtration, then the product $\mathfrak{p}_1 \cdots \mathfrak{p}_n$ is uniquely defined for N in M and it is called the generalized prime ideal factorization of N in M, written as $\mathcal{P}_M(N) = \mathfrak{p}_1 \cdots \mathfrak{p}_n$. Generalized prime ideal factorization of submodules is defined and its various properties are studied in [5].

In general, a product of prime ideals need not be a generalized prime ideal factorization of some submodule of a module.

Example 1.3. Let $R = \frac{k[x,y,z]}{(xy-z^2, x^2-yz)}$ and $\mathfrak{p}$ be the prime ideal $(\overline{x}, \overline{z})$. Suppose there exists an R-module M and a submodule N with $\mathcal{P}_M(N) = \mathfrak{p}^2$. Then there is an RPE filtration $N \overset{\mathfrak{p}}{\subset} N_1 \overset{\mathfrak{p}}{\subset} M$. By Lemma 1.1, $\mathrm{Ass}(M/N_1) = \{\mathfrak{p}\}$. So $\mathfrak{p} = (N_1 : m)$ for some $m \in M$. We have $\mathfrak{p}(\overline{x}, \overline{y}, \overline{z}) = \mathfrak{p}^2$, which implies $\mathfrak{p}(\overline{x}, \overline{y}, \overline{z})M = \mathfrak{p}^2 M \subseteq N$. Therefore, $(\overline{x}, \overline{y}, \overline{z})m \subseteq (N : \mathfrak{p}) = N_1$ [Lemma 1.2] which implies $(\overline{x}, \overline{y}, \overline{z}) \subseteq (N_1 : m) = \mathfrak{p}$, a contradiction. Therefore, we cannot have an R-module M and a submodule N with $\mathcal{P}_M(N) = \mathfrak{p}^2$.

In this article, for a product of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, we find conditions for the existence of an R-module M and a submodule N of M with $\mathcal{P}_M(N) = \mathfrak{p}_1 \cdots \mathfrak{p}_n$. We show that a power of a prime ideal occurs in a generalized prime ideal factorization only if it is not equal to its lesser powers. We show that ${\mathfrak{p}_i}^{r_i} \neq {\mathfrak{p}_i}^{r_i - 1}$ and $\mathrm{Ass}(R/{\mathfrak{p}_i}^{r_i}) = \{\mathfrak{p}_i\}$ for every $1 \leq i \leq n$ is a sufficient condition for the existence of an R-module M and a submodule N of M with $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$. Also, we show that ${\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$ is a generalized prime ideal factorization if and only if for each $1 \leq i \leq n$, ${\mathfrak{p}_i}^{r_i}$ is the generalized prime ideal factorization of some submodule of a module.

2 Conditions for ideals to be generalized prime ideal factorization of submodules

We need the following lemmas.

Lemma 2.1. *Let* N *be a proper submodule of an* R*-module* M*. If a submodule* K *of* M *lies in an RPE filtration of* M *over* N*, then* $\mathcal{P}_M(N) = \mathcal{P}_M(K)\mathcal{P}_K(N)$*.*

Proof. If a submodule K of M lies in an RPE filtration of M over N, then we have an RPE filtration

$$N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \overset{\mathfrak{p}_r}{\subset} N_r \overset{\mathfrak{p}_{r+1}}{\subset} \cdots \overset{\mathfrak{p}_n}{\subset} N_n = M,$$

where $N_r = K$ for some r. So $\mathcal{P}_M(N) = \mathfrak{p}_1 \cdots \mathfrak{p}_n$. Then

$$N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \overset{\mathfrak{p}_r}{\subset} N_r = K \text{ and } K = N_r \overset{\mathfrak{p}_{r+1}}{\subset} N_{r+1} \subset \cdots \overset{\mathfrak{p}_n}{\subset} N_n = M$$

are RPE filtrations, and therefore, $\mathcal{P}_K(N) = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ and $\mathcal{P}_M(K) = \mathfrak{p}_{r+1} \cdots \mathfrak{p}_n$. This proves the lemma. □

Lemma 2.2. *[2, Lemma 20] Let* N *be a proper submodule of* M *and* $N = M_0 \subset \cdots \subset M_{i-1} \overset{\mathfrak{p}_i}{\subset} M_i \overset{\mathfrak{p}_{i+1}}{\subset} M_{i+1} \subset \cdots \subset M_n = M$ *be an RPE filtration of* M *over* N*. If* $\mathfrak{p}_{i+1} \not\subseteq \mathfrak{p}_i$ *for some* i*, then there exists a submodule* K_i *of* M *such that* $N = M_0 \subset \cdots \subset M_{i-1} \overset{\mathfrak{p}_{i+1}}{\subset} K_i \overset{\mathfrak{p}_i}{\subset} M_{i+1} \subset \cdots \subset M_n = M$ *is an RPE filtration of* M *over* N*.*

Remark 2.3. Suppose a product of prime ideals $\mathfrak{p}_1 \cdots \mathfrak{p}_r$ is $\mathcal{P}_M(N)$ for some module M and a submodule N of M. For every reordering of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ with $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i < j$, by applying Lemma 2.2 for a sufficient number of times, we can get an RPE filtration $N \overset{\mathfrak{p}_1}{\subset} N_1 \overset{\mathfrak{p}_2}{\subset} N_2 \subset \cdots \subset N_{r-1} \overset{\mathfrak{p}_r}{\subset} N_r = M$.

Proposition 2.4. *For a prime ideal $\mathfrak{p}$ and a positive integer r, if $\mathfrak{p}^r$ is the generalized prime ideal factorization of a submodule of an R-module, then $\mathfrak{p}^r \neq \mathfrak{p}^{r-1}$.*

Proof. Suppose M is an R-module and N is a submodule of M with $\mathcal{P}_M(N) = \mathfrak{p}^r$. Then we have an RPE filtration

$$N \overset{\mathfrak{p}}{\subset} N_1 \overset{\mathfrak{p}}{\subset} N_2 \subset \cdots \subset N_{r-1} \overset{\mathfrak{p}}{\subset} N_r = M.$$

If $\mathfrak{p}^r = \mathfrak{p}^{r-1}$, then by Lemma 1.2, $M = N_r = \{x \in M \mid \mathfrak{p}^r x \subseteq N\} = \{x \in M \mid \mathfrak{p}^{r-1} x \subseteq N\} = N_{r-1}$, a contradiction. □

Example 1.3 shows that the converse of the above result is not true.

Corollary 2.5. *Let $\mathfrak{p}$ be a prime ideal in R and r be a positive integer. Then $\mathcal{P}_R(\mathfrak{p}^r) = \mathfrak{p}^r$ if and only if $\mathfrak{p}^r \neq \mathfrak{p}^{r-1}$ and $\operatorname{Ass}(R/\mathfrak{p}^r) = \{\mathfrak{p}\}$.*

Proof. If $\mathfrak{p}^r \neq \mathfrak{p}^{r-1}$ and $\operatorname{Ass}(R/\mathfrak{p}^r) = \{\mathfrak{p}\}$, then by [5, Corollary 2.16], $\mathcal{P}_R(\mathfrak{p}^r) = \mathfrak{p}^r$. Conversely if $\mathcal{P}_R(\mathfrak{p}^r) = \mathfrak{p}^r$, then we have an RPE filtration $\mathfrak{p}^r \overset{\mathfrak{p}}{\subset} \mathfrak{a}_1 \overset{\mathfrak{p}}{\subset} \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_{r-1} \overset{\mathfrak{p}}{\subset} \mathfrak{a}_r = R$. By Lemma 1.1, $\operatorname{Ass}(R/\mathfrak{p}^r) = \{\mathfrak{p}\}$ and by Proposition 2.4, $\mathfrak{p}^r \neq \mathfrak{p}^{r-1}$. □

Corollary 2.6. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct prime ideals in R and $r_1, \ldots, r_n$ be positive integers. If there exists an R-module M and a submodule N of M with $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$, then ${\mathfrak{p}_i}^{r_i} \neq {\mathfrak{p}_i}^{r_i - 1}$ for every $1 \leq i \leq n$.*

Proof. Reorder $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ so that $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i < j$. By Remark 2.3, we have an RPE filtration

$$N = L^{(0)} \overset{\mathfrak{p}_1}{\subset} L_1^{(1)} \overset{\mathfrak{p}_1}{\subset} L_2^{(1)} \overset{\mathfrak{p}_1}{\subset} \cdots \overset{\mathfrak{p}_1}{\subset} L_{r_1}^{(1)} \overset{\mathfrak{p}_2}{\subset} L_1^{(2)} \subset \cdots$$
$$\overset{\mathfrak{p}_{n-1}}{\subset} L_{r_{n-1}}^{(n-1)} \overset{\mathfrak{p}_n}{\subset} L_1^{(n)} \overset{\mathfrak{p}_n}{\subset} L_2^{(n)} \subset \cdots \overset{\mathfrak{p}_n}{\subset} L_{r_n}^{(n)} = M$$

of M over N. Then for every $1 \leq i \leq n$,

$$L_{r_{i-1}}^{(i-1)} \overset{\mathfrak{p}_i}{\subset} L_1^{(i)} \overset{\mathfrak{p}_i}{\subset} L_2^{(i)} \subset \cdots \overset{\mathfrak{p}_i}{\subset} L_{r_i - 1}^{(i)} \overset{\mathfrak{p}_i}{\subset} L_{r_i}^{(i)}$$

is an RPE filtration of $L_{r_i}^{(i)}$ over $L_{r_{i-1}}^{(i-1)}$, and therefore, ${\mathfrak{p}_i}^{r_i}$ is the generalized prime ideal factorization of $L_{r_{i-1}}^{(i-1)}$ in $L_{r_i}^{(i)}$. By Proposition 2.4, ${\mathfrak{p}_i}^{r_i} \neq {\mathfrak{p}_i}^{r_i - 1}$. □

Theorem 2.7. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct prime ideals in R and $r_1, \ldots, r_n$ be positive integers. If there exists an R-module M and a submodule N of M with $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$, then ${\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_i}^{r_i - 1} \cdots {\mathfrak{p}_n}^{r_n} \neq {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$ whenever $\mathfrak{p}_i$ is minimal among $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.*

Proof. Let $\mathfrak{p}_i$ be minimal among $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. We can reorder $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that $\mathfrak{p}_n = \mathfrak{p}_i$ and $\mathfrak{p}_j \not\subset \mathfrak{p}_k$ for $j < k$. So without loss of generality, we assume that $i = n$. Then by Remark 2.3, we have the RPE filtration

$$N = L^{(0)} \overset{\mathfrak{p}_1}{\subset} L_1^{(1)} \overset{\mathfrak{p}_1}{\subset} L_2^{(1)} \overset{\mathfrak{p}_1}{\subset} \cdots \overset{\mathfrak{p}_1}{\subset} L_{r_1}^{(1)} \overset{\mathfrak{p}_2}{\subset} L_1^{(2)} \subset \cdots$$
$$\overset{\mathfrak{p}_{n-1}}{\subset} L_{r_{n-1}}^{(n-1)} \overset{\mathfrak{p}_n}{\subset} L_1^{(n)} \overset{\mathfrak{p}_n}{\subset} L_2^{(n)} \subset \cdots \overset{\mathfrak{p}_n}{\subset} L_{r_n - 1}^{(n)} \overset{\mathfrak{p}_n}{\subset} L_{r_n}^{(n)} = M$$

of M over N. Then for every $x \in M$, ${\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n} x \subseteq N$ [Lemma 1.2]. Suppose ${\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_{n-1}}^{r_{n-1}} {\mathfrak{p}_n}^{r_n - 1} = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$. Then ${\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_{n-1}}^{r_{n-1}} {\mathfrak{p}_n}^{r_n - 1} x \subseteq N$. By Lemma 1.2, $x \in L_{r_n - 1}^{(n)}$. This implies $L_{r_n - 1}^{(n)} = M$, which is not true. Therefore, ${\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_i}^{r_i - 1} \cdots {\mathfrak{p}_n}^{r_n} \neq {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$. □

Remark 2.8. The above result is not true if $\mathfrak{p}_i$ is not minimal among $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. For example, in the ring $R = \frac{k[x,y,z]}{(xy-z, yz-x)}$, let $\mathfrak{p}$ and $\mathfrak{q}$ be the prime ideals $(\overline{x}, \overline{y}, \overline{z})$ and $(\overline{x}, \overline{z})$ respectively. In this case, $\mathfrak{p}\mathfrak{q} = \mathfrak{q}$. For the R-module $M = \frac{R}{\mathfrak{q}} \oplus \frac{R}{\mathfrak{q}}$ and its submodule $N = \frac{\mathfrak{p}}{\mathfrak{q}} \oplus 0$, we have the RPE filtration

$$N = \frac{\mathfrak{p}}{\mathfrak{q}} \oplus 0 \quad \overset{\mathfrak{p}}{\subset} \quad \frac{R}{\mathfrak{q}} \oplus 0 \quad \overset{\mathfrak{q}}{\subset} \quad \frac{R}{\mathfrak{q}} \oplus \frac{R}{\mathfrak{q}} = M$$

of M over N. So we have $\mathcal{P}_M(N) = \mathfrak{p}\mathfrak{q}$ and $\mathfrak{p}\mathfrak{q} = \mathfrak{q}$.

The next lemma is about the regular prime extensions on direct sums.

Lemma 2.9. *Let N, N' be submodules of modules M, M' respectively and $N \overset{\mathfrak{p}}{\subset} K$ be a regular $\mathfrak{p}$-prime extension in M.*

(i) If $N' \overset{\mathfrak{p}}{\subset} K'$ is a regular $\mathfrak{p}$-prime extension in M', then $N \oplus N' \overset{\mathfrak{p}}{\subset} K \oplus K'$ is a regular $\mathfrak{p}$-prime extension in $M \oplus M'$.

(ii) If $\mathfrak{p} \notin \operatorname{Ass}(M'/N')$ and $\mathfrak{p}$ is a maximal element in $\operatorname{Ass}(M/N) \cup \operatorname{Ass}(M'/N')$, then $N \oplus N' \overset{\mathfrak{p}}{\subset} K \oplus N'$ is a regular $\mathfrak{p}$-prime extension in $M \oplus M'$.

Proof. *(i)* We have $\mathfrak{p}K \subseteq N$ and $\mathfrak{p}K' \subseteq N'$. Therefore, $\mathfrak{p} \subseteq (N \oplus N' : K \oplus K')$. Let $a \in (N \oplus N' : K \oplus K')$ and $x \in K \setminus N$. Then $(x, 0) \in K \oplus K' \setminus N \oplus N'$ and $a(x, 0) \in N \oplus N'$. This implies $ax \in N$. Since $x \notin N$ and $N \overset{\mathfrak{p}}{\subset} K$ is a $\mathfrak{p}$-prime extension in M, we get $a \in \mathfrak{p}$. Hence, $(N \oplus N' : K \oplus K') = \mathfrak{p}$.

Let $a \in R$, $(x, y) \in K \oplus K'$ such that $a(x, y) \in N \oplus N'$. Suppose $(x, y) \notin N \oplus N'$. Then either $x \notin N$ or $y \notin N'$. If $x \notin N$, since $ax \in N$ and $N \overset{\mathfrak{p}}{\subset} K$ is a $\mathfrak{p}$-prime extension, we get $a \in \mathfrak{p}$. If $x \in N$, then $y \notin N'$ and since $N' \overset{\mathfrak{p}}{\subset} K'$ is a $\mathfrak{p}$-prime extension in M', $ay \in N'$ implies $a \in \mathfrak{p}$. Hence, $N \oplus N' \overset{\mathfrak{p}}{\subset} K \oplus K'$ is a $\mathfrak{p}$-prime extension in $M \oplus M'$.

Suppose it is not maximal. Let $L \oplus L'$ be a $\mathfrak{p}$-prime extension of $N \oplus N'$ in $M \oplus M'$ with $K \oplus K' \subsetneq L \oplus L'$. Then $\mathfrak{p}(L \oplus L') \subseteq N \oplus N'$. This implies $\mathfrak{p}L \subseteq N$. Since $N \subsetneq K \subset L$, we have $N \subsetneq L$. Let $x \in L \setminus N$. Then for every $a \in (N : L)$, $ax \in N$, and therefore, $a(x, 0) \in N \oplus N'$. Since $(x, 0) \notin N \oplus N'$, we get $a \in \mathfrak{p}$. Therefore, L is a $\mathfrak{p}$-prime extension of N in M and $K \subset L$. Since $N \overset{\mathfrak{p}}{\subset} K$ is a maximal $\mathfrak{p}$-prime extension in M, we get $K = L$. Similarly for $N' \subsetneq K' \subset L'$, we get $K' = L'$. This implies $K \oplus K' = L \oplus L'$, a contradiction. Hence, $N \oplus N' \overset{\mathfrak{p}}{\subset} K \oplus K'$ is a maximal $\mathfrak{p}$-prime extension in $M \oplus M'$.

Since $\mathfrak{p}$ is a maximal element in both $\operatorname{Ass}(M/N)$ and $\operatorname{Ass}(M'/N')$, $\mathfrak{p}$ is maximal in $\operatorname{Ass}(M/N) \cup \operatorname{Ass}(M'/N')$. Therefore, $\mathfrak{p}$ is maximal in $\operatorname{Ass}(\frac{M \oplus M'}{N \oplus N'})$. Hence, $N \oplus N' \overset{\mathfrak{p}}{\subset} K \oplus K'$ is a regular $\mathfrak{p}$-prime extension in $M \oplus M'$.

(ii) Taking $K' = N'$ in the proof of (i), we get $N \oplus N' \overset{\mathfrak{p}}{\subset} K \oplus N'$ is a $\mathfrak{p}$-prime extension in $M \oplus M'$. Suppose this $\mathfrak{p}$-prime extension is not maximal. Let $L \oplus L'$ be a $\mathfrak{p}$-prime extension of $N \oplus N'$ in $M \oplus M'$ with $K \oplus N' \subsetneq L \oplus L'$. Then $\mathfrak{p}(L \oplus L') \subseteq N \oplus N'$. This implies $\mathfrak{p}L \subseteq N$. Since $N \subsetneq K \subset L$, we have $N \subsetneq L$. As in the proof of (i), we get $K = L$. We have $N' \subseteq L'$. If $N' \subsetneq L'$, then using the same argument we get $N' \overset{\mathfrak{p}}{\subset} L'$ is a $\mathfrak{p}$-prime extension in M' and therefore, $\mathfrak{p} \in \operatorname{Ass}(M'/N')$ [Lemma 1.1], which is not the case. Therefore, $N' = L'$. Then $K \oplus N' = L \oplus L'$, which is a contradiction. Hence, $N \oplus N' \overset{\mathfrak{p}}{\subset} K \oplus N'$ is a maximal $\mathfrak{p}$-prime extension in $M \oplus M'$.

Since $\mathfrak{p}$ is a maximal element in $\operatorname{Ass}(M/N) \cup \operatorname{Ass}(M'/N')$, $\mathfrak{p}$ is maximal in $\operatorname{Ass}(\frac{M \oplus M'}{N \oplus N'})$. Hence, $N \oplus N' \overset{\mathfrak{p}}{\subset} K \oplus N'$ is a regular $\mathfrak{p}$-prime extension in $M \oplus M'$. □

Lemma 2.10. *Let $N_1, \ldots, N_n$ be submodules of R-modules $M_1, \ldots, M_n$ respectively and let $\mathfrak{p}$ be a maximal element in $\cup_{i=1}^{n} \operatorname{Ass}(M_i/N_i)$. For $1 \leq i \leq n$, let K_i be the regular $\mathfrak{p}$-prime extension of N_i in M_i if $\mathfrak{p} \in \operatorname{Ass}(M_i/N_i)$ and $K_i = N_i$ otherwise. Then $\oplus_{i=1}^{n} N_i \overset{\mathfrak{p}}{\subset} \oplus_{i=1}^{n} K_i$ is a regular $\mathfrak{p}$-prime extension in $\oplus_{i=1}^{n} M_i$.*

Proof. Since $\operatorname{Ass}\Big(\dfrac{\oplus_{i=1}^{n} M_i}{\oplus_{i=1}^{n} N_i}\Big) = \bigcup_{i=1}^{n} \operatorname{Ass}\Big(\dfrac{M_i}{N_i}\Big)$ the proof follows by induction on n using Lemma 2.9. □

Proposition 2.11. *Let N, N' be proper submodules of modules M, M' respectively. If $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$ and $\mathcal{P}_{M'}(N') = {\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k}$ for some non-negative integers r_i and s_i with either $r_i > 0$ or $s_i > 0$, then $\mathcal{P}_{M \oplus M'}(N \oplus N') = {\mathfrak{p}_1}^{t_1} \cdots {\mathfrak{p}_k}^{t_k}$, where $t_i = \max\{r_i, s_i\}$ for $i = 1, \ldots, k$.*

Proof. We prove by induction on k. If $\mathcal{P}_M(N) = \mathfrak{p}^r$ and $\mathcal{P}_{M'}(N') = \mathfrak{p}^s$ for some prime ideal $\mathfrak{p}$. Without loss of generality, we assume $r \geq s$. Then we have RPE filtrations

$$N \overset{\mathfrak{p}}{\subset} N_1 \overset{\mathfrak{p}}{\subset} N_2 \subset \cdots \subset N_s \overset{\mathfrak{p}}{\subset} N_{s+1} \subset \cdots \subset N_r = M$$

$$N' \overset{\mathfrak{p}}{\subset} N'_1 \overset{\mathfrak{p}}{\subset} N'_2 \subset \cdots \subset N'_s = M'.$$

Then by Lemma 2.9,

$$N \oplus N' \overset{\mathfrak{p}}{\subset} N_1 \oplus N'_1 \overset{\mathfrak{p}}{\subset} N_2 \oplus N'_2 \overset{\mathfrak{p}}{\subset} \cdots \overset{\mathfrak{p}}{\subset} N_s \oplus N'_s = N_s \oplus M' \overset{\mathfrak{p}}{\subset} N_{s+1} \oplus M' \overset{\mathfrak{p}}{\subset} \cdots \overset{\mathfrak{p}}{\subset} N_r \oplus M' = M \oplus M'$$

is an RPE filtration. Therefore, $\mathcal{P}_{M\oplus M'}(N \oplus N') = \mathfrak{p}^r$. Hence, the result is true for $k = 1$.

Now let $k > 1$. By reordering $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ we assume that $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ whenever $i < j$. Then $\mathfrak{p}_1$ is maximal in $\mathrm{Ass}(M/N) \cup \mathrm{Ass}(M'/N')$. By Remark 2.3, there exists RPE filtrations

$$N \overset{\mathfrak{p}_1}{\subset} N_1 \overset{\mathfrak{p}_1}{\subset} N_2 \subset \cdots \overset{\mathfrak{p}_1}{\subset} N_{r_1} \overset{\mathfrak{p}_2}{\subset} K_1 \subset \cdots \subset M$$

$$N' \overset{\mathfrak{p}_1}{\subset} N'_1 \overset{\mathfrak{p}_1}{\subset} N'_2 \subset \cdots \overset{\mathfrak{p}_1}{\subset} N'_{s_1} \overset{\mathfrak{p}_2}{\subset} K'_1 \subset \cdots \subset M'.$$

So we have $\mathcal{P}_M(N_{r_1}) = {\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_k}^{r_k}$ and $\mathcal{P}_{M'}(N'_{s_1}) = {\mathfrak{p}_2}^{s_2} \cdots {\mathfrak{p}_k}^{s_k}$. By induction assumption, $\mathcal{P}_{M\oplus M'}(N_{r_1} \oplus N'_{s_1}) = {\mathfrak{p}_2}^{t_2} \cdots {\mathfrak{p}_k}^{t_k}$, where $t_i = \max\{r_i, s_i\}$ for $i = 2, \ldots, k$. By the previous paragraph, we have $\mathcal{P}_{N_{r_1}\oplus N'_{s_1}}(N \oplus N') = {\mathfrak{p}_1}^{t_1}$, where $t_1 = \max\{r_1, s_1\}$. Then by Lemma 2.10, $N_{r_1} \oplus N'_{s_1}$ lies in an RPE filtration of $M \oplus M'$ over $N \oplus N'$. So by Lemma 2.1, we get $\mathcal{P}_{M\oplus M'}(N \oplus N') = \mathcal{P}_{M\oplus M'}(N_{r_1} \oplus N'_{s_1})\, \mathcal{P}_{N_{r_1}\oplus N'_{s_1}}(N \oplus N') = {\mathfrak{p}_1}^{t_1} \cdots {\mathfrak{p}_k}^{t_k}$. □

Using induction, we have the following theorem.

Theorem 2.12. *Let $N_1, \ldots, N_n$ be proper submodules of R-modules $M_1, \ldots, M_n$ respectively. If $\mathcal{P}_{M_i}(N_i) = {\mathfrak{p}_1}^{r_{i_1}} \cdots {\mathfrak{p}_k}^{r_{i_k}}$ for $i = 1, \ldots, n$ and $r_{i_j} \geq 0$, then $\mathcal{P}_{\oplus_{i=1}^n M_i}(\bigoplus_{i=1}^{n} N_i) = {\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k}$, where $s_j = \max\{r_{1_j}, \ldots, r_{n_j}\}$ for $j = 1, \ldots, k$.*

Now we get the following sufficient condition for the existence of an R-module M and a submodule N of M with $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$.

Corollary 2.13. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct prime ideals in R and $r_1, \ldots, r_n$ be positive integers. If ${\mathfrak{p}_i}^{r_i} \neq {\mathfrak{p}_i}^{r_i - 1}$ and $\mathrm{Ass}(R/{\mathfrak{p}_i}^{r_i}) = \{\mathfrak{p}_i\}$ for every $1 \leq i \leq n$, then $\mathcal{P}_{R^n}({\mathfrak{p}_1}^{r_1} \oplus \cdots \oplus {\mathfrak{p}_n}^{r_n}) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$.*

Proof. By Corollary 2.5, $\mathcal{P}_R({\mathfrak{p}_i}^{r_i}) = {\mathfrak{p}_i}^{r_i}$ for every $1 \leq i \leq n$. Then by Theorem 2.12, $\mathcal{P}_{R^n}({\mathfrak{p}_1}^{r_1} \oplus \cdots \oplus {\mathfrak{p}_n}^{r_n}) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$. □

The next theorem gives a necessary and sufficient condition for the existence of an R-module M with a submodule N such that $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$.

Theorem 2.14. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct prime ideals in R and $r_1, \ldots, r_n$ be positive integers. There exists an R-module M and a submodule N of M with $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$ if and only if there exist R-modules M_i and submodules N_i of M_i such that $\mathcal{P}_{M_i}(N_i) = {\mathfrak{p}_i}^{r_i}$, $1 \leq i \leq n$.*

Proof. If there exist R-modules M_i and submodules N_i of M_i such that $\mathcal{P}_{M_i}(N_i) = {\mathfrak{p}_i}^{r_i}$, $1 \leq i \leq n$, then by Theorem 2.12, for $N = \oplus_{i=1}^n N_i$ and $M = \oplus_{i=1}^n M_i$ we have $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$.

Conversely if there exists an R-module M and a submodule N of M with $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$, then taking $M_i = L^{(i)}_{r_i}$ and $N_i = L^{(i-1)}_{r_{i-1}}$ in the RPE filtration

$$N = L^{(0)} \overset{\mathfrak{p}_1}{\subset} L^{(1)}_1 \overset{\mathfrak{p}_1}{\subset} L^{(1)}_2 \overset{\mathfrak{p}_1}{\subset} \cdots \overset{\mathfrak{p}_1}{\subset} L^{(1)}_{r_1} \overset{\mathfrak{p}_2}{\subset} L^{(2)}_1 \subset \cdots \overset{\mathfrak{p}_{n-1}}{\subset} L^{(n-1)}_{r_{n-1}} \overset{\mathfrak{p}_n}{\subset} L^{(n)}_1 \overset{\mathfrak{p}_n}{\subset} L^{(n)}_2 \subset \cdots \overset{\mathfrak{p}_n}{\subset} L^{(n)}_{r_n} = M,$$

we get $\mathcal{P}_{M_i}(N_i) = {\mathfrak{p}_i}^{r_i}$, $1 \leq i \leq n$. □

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