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Two Efficient Methods for Finding Exact Solutions of Time-Fractional Variant Boussinesq Equations

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Abstract In this paper, we explore novel traveling wave solutions of the time-fractional variant Boussinesq system by employing the Modified Exponential Function Method and the Generalized Kudryashov Method. The analytical exact solutions derived herein are distinguished by their richness, diversity, and generalization to fractional-order derivatives-features that set them apart from previously reported results in the literature. Furthermore, to better illustrate the dynamic behavior of these solutions, we utilize *Mathematica* to generate three-dimensional surface plots, contour plots, and two-dimensional profiles of selected traveling wave solutions.

1 Introduction

NLPDEs constitute a fundamental component in numerous fields of applied science, including quantum mechanics, fluid dynamics, and nonlinear optics, plasma physics, economy, ecology, optics, solid state physics, fluid mechanics, biology, geochemistry, meteorology, electricity, and so forth. Consequently, with various computational programming, new powerful analytical methods have been successfully presented by mathematicians and physicists, some of these are used by authors in treating Variant Boussinesq system.
In this study, we consider the time-fractional coupled Variant Boussinesq equations in the reduced form as follow:

$$\begin{cases} D_t^\alpha u + uu_x + v_x = 0, \\ D_t^\alpha v + (uv)_x + u_{xxx} = 0, \end{cases} \tag{1.1}$$

where $D_t^\alpha u := \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is in the Caputo sense and $0 < \alpha \leq 1$.
We point out that the classical system ($\alpha = 1$) models bidirectional propagation of small-amplitude, long waves in nonlinear dispersive media.
Recently, numerous researchers have investigated this problem in various contexts, including: Yang *et al.* [1] applied An extended Fan's algebraic method for constructing exact traveling wave solution to the classical system of (1.1). While Zhang *et al.* [2] employed an improved algebraic method to investigate the bifurcation structure of the system and derived explicit exact solutions. Using the bifurcation theory of planar dynamical systems, Yuan *et al.* [3] studied the bifurcations of solitary waves and kink waves for the system. Zayed *et al.* [4] utilized the Jacobi elliptic equation expansion method, the general Exp_a-function method, the modified simple equation method , the (G'/G)-expansion method and the Riccati equation expansion method to construct exact solutions of the system. Wenjun *et al.* [5] presented the complex method to obtain all meromorphic solutions of the complex variant Boussinesq equations, and subsequently derived the corresponding traveling wave exact solutions of System (vB). Yan *et al.* [6] fined various types of explicit and exact traveling wave solutions for a system of variant Boussinesq equations by employing an improved sine-cosine method in conjunction with the Wu elimination method. Wen *et al.* [7] proposed an extended Jacobi elliptic function expansion method and successfully applied it to the variant Boussinesq equations, obtaining a rich set of periodic wave solutions expressed in terms of Jacobi elliptic functions. using the homogeneous balance

method, Jiefang [8] obtained multi- solitary wave solutions to the system (VB). Also, By employing the homogeneous balance method, Wang [9] derived solitary wave solutions for two distinct types of variant Boussinesq equations. Hashemi *et al.* [10] employed the invariant subspace method to find the exact solutions for (1.1), where the derivative is in Riemann-Liouville sense. Using the Hirota bilinear method, Guo *et al.* [11] established multiple singular soliton solutions and multiple soliton solutions to the (VB) system. Lu [12] proposed a generalized Jacobi elliptic function expansion method to construct abundant doubly periodic solutions in terms of Jacobi elliptic functions for two forms of the variant Boussinesq equations. Fan *et al.* [13] devised a new algebraic method and applied it to two variat Boussinesq equations to construct a unified series of new traveling wave solutions. Two numerical methods, homotopy analysis and the homotopy Padé methods are investigated by Jabbari *et al.* [14] to obtain approximate solution of the (VB) system.

Khan *et al.* [15] employed the enhanced (G'/G)-expansion method to derive exact solutions of the variant Boussinesq equations. Utilizing the improved $\tanh(\Phi(\zeta)/2)$-expansion method (ITEM) and the improved G'/G-expansion method (IGEM) , Manafian *et al.* [16] constructed hyperbolic function, trigonometric function and rational function solutions for the variant Boussinesq equations. Gao *et al.* [17] derived exact solutions of the variant Boussinesq equations in terms of hyperbolic, trigonometric, and rational functions by using traveling wave hypothesis and auxiliary function method. Jawad *et al.* [18] applied the tanh method to solve the governing equations and further employed the traveling wave hypothesis to address the generalized form of the coupled Boussinesq equations. By employing a new approach based on sine-Gordon equation, Zuntao *et al.* [19] constructed exact periodic solutions to nonlinear (VB) equations.

The goal of this paper is to apply the modified exponential function method [20, 21, 22] and Generalized Kudryashov method [23, 24, 25] to construct exact traveling wave solutions for the nonlinear time-fractional coupled Variant Boussinesq equations. Furthermore, a set of new exact solutions is obtained, among which are solitary wave solutions (hyperbolic function solution), periodic solitary wave (trigonometric function solution) of TFVBEs. To the best of our knowledge, these two methods have not yet been applied to the time-fractional variant Boussinesq system.

The paper consists of five sections. The first section is the introduction, where we reviewed several methods previously employed by researchers to study the considered equation and related ones. The second section serves as a preliminary part, in which we presented the adopted tools and methodologies used in analyzing the problem. Sections three and four contain the main results, through which the aforementioned methods were applied. Finally, the study summaries with a conclusion and remarks on the presented work.

2 Preliminary

2.1 Fractional derivative

Fractional derivation (also known as fractional calculus) is a generalization of ordinary differentiation and integration to non-integer (fractional) orders. In contrast to classical calculus where derivatives and integrals are defined for integer orders (1st derivative, 2nd derivative, etc.), fractional calculus allows operations like:

- $D^{\frac{1}{2}} f(x) \rightarrow$half-derivative of a function
- $D^{-0.3} f(x) \rightarrow$ fractional integral of order 0.3

Fractional derivatives are nonlocal operators, meaning they incorporate memory effects. This makes them ideal for modeling:

Viscoelastic materials, anomalous diffusion, signal processing, Control theory, finance and bioengineering.

There are several definitions, each with its own advantages depending on the context. The most popular are Riemann-Liouville derivative, Grünwald-Letnikov derivative, and Caputo derivative.

Caputo derivative

The Caputo derivative of order α is defined by the following integral formula [26, 27, 28]:

$$D_*^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, & \text{if } m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \text{if } \alpha = m, \end{cases} \tag{2.1}$$

where $m \in \mathbb{N}^*$ and $\Gamma(.)$ denotes the Gamma function, defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \ \ x > 0$. This study makes use of the following essential properties of the Caputo fractional derivative:

$$D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \ \ D^\alpha c = 0 \tag{2.2}$$

$$D_t^\alpha \Big[u(v(t))\Big] = u_v'(v(t)) D_t^\alpha v(t) = D_v^\alpha u(v(t)) \Big[v_t'(t)\Big]^\alpha \tag{2.3}$$

$$d^\alpha w(t) = \Gamma(1+\alpha) d\, w(t) \tag{2.4}$$

2.2 Description of Modified exponential function method

Let us consider a time-fractional nonlinear partial differential equation involving two independent variables, which can be formulated as follows:

$$G(u, D_t^\alpha u, u_x, D_t^{2\alpha} u, u_{xx}, D_t^\alpha u_x, \dots) = 0, \ \ \ 0 < \alpha \le 1, \tag{2.5}$$

In this context, $u = u(x,t)$ is the unknown function, while G denotes a polynomial involving u and its partial fractional derivatives, including the nonlinear and highest-order derivative terms. The exact solution of Eq. (2.5) is constructed by employing the Exp$(-\Phi(\eta))$-expansion method as outlined in the following steps:

- **Step 1:** The exact solution is obtained by applying the following fractional complex transformation: [29, 30]

$$u(x,t) = U(\eta), \ \ \eta = kx - \frac{\omega t^\alpha}{\Gamma(1+\alpha)}, \tag{2.6}$$

  Here, k, ω represent constants to be determined. As a result, Eq (2.5) is transformed into the following NLODE:

$$H(U, -\omega U', kU', \omega^2 U'', k^2 U'', -\omega k U', \dots) = 0, \tag{2.7}$$

  where $U^{(i)} = U_{i\eta}$.

- **Step 2:** Suppose that the solution to Eq. (2.7) admits a finite power series representation of the form:

$$U(\eta) = \frac{\sum_{n=0}^N a_i \Big(\exp(-\Phi(\eta))\Big)^i}{\sum_{n=0}^M b_j \Big(\exp(-\Phi(\eta))\Big)^j} \tag{2.8}$$

  where $a_0, a_1, \dots a_N (a_N \neq 0)$ and $b_0, b_1, \dots b_M (a_M \neq 0)$ are constants .
  Let the function $\Phi = \Phi(\eta)$ be governed by the following ODE:

$$\Phi'(\eta) = \mu e^{\Phi(\eta)} + e^{-\Phi(\eta)} + \lambda \tag{2.9}$$

  where μ, λ denote arbitrary constants, the values of which will be determined later in the analysis.
  The general forms of the solutions to (2.9) are given by:

$$\Phi(\eta) = \begin{cases} \log\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\eta+C)\right)-\lambda}{2\mu} \right), & \lambda^2-4\mu > 0, \ (\mu \neq 0), \\ \log\left(\frac{\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(\eta+C)\right)-\lambda}{2\mu} \right), & \lambda^2-4\mu < 0, \ (\mu \neq 0), \\ \log\left(-\frac{2(C\lambda+\eta\lambda+2)}{\lambda^2(\eta+C)} \right), & \lambda^2-4\mu = 0, \ (\mu \neq 0, \lambda \neq 0), \\ -\log\left(\frac{\lambda}{\sinh(\lambda(\eta+C))+\cosh(\lambda(\eta+C))-1} \right), & \lambda^2-4\mu \neq 0, \ (\mu = 0, \lambda \neq 0), \\ \log(\eta+C), & \lambda^2-4\mu = 0, \ (\mu = 0, \lambda = 0). \end{cases} \tag{2.10}$$

where C is an arbitrary real constant.

- **Step 3:** By applying the homogeneous balance theory between the nonlinear term and the highest-order derivative term in Eq. (2.7) , we determine the positive integers N and M. If the degree of $U(\eta)$ is taken to be $N-M$, then the corresponding degrees of the remaining terms can be computed as follows:

$$D(U^{(\kappa)}) = N - M + \kappa,\ D(U^{\kappa}(U^{(\ell)})^s) = (N-M)\kappa + s(N-M+\ell).$$

- **Step 4:** By substituting Eq.(2.8) into Eq. (2.7) and applying Eq. (2.9), the coefficients of like powers of $e^{n\Phi(\eta)}, (n = 0, 1, 2, \dots,)$ are collected. Equating each coefficient to zero yields a system of nonlinear algebraic equations. This system is then solved using *Mathematica* to determine the values of the unknown constants $a_0, a_1, \dots a_N$, $b_0, b_1, \dots b_N$ as well as k and ω.

- **Step 5:** Substituting the determined values of a_n, k and ω with using (2.10) into (2.8), yields all exact solutions of (2.5).

2.3 Basic ideas of the Generalized Kudryashov method

Consider a time-fractional nonlinear partial differential equation in two independent variables, formulated as follows:

$$G(u, D_t^{\alpha} u, u_x, D_t^{2\alpha} u, u_{xx}, D_t^{\alpha} u_x, \dots) = 0, \quad 0 < \alpha \leq 1. \tag{2.11}$$

Let $u = u(x,t)$ be the unknown function, and let G be a polynomial expression in terms of u and its partial fractional derivatives, encompassing both nonlinearities and the highest derivative orders. The procedure to derive the exact solution of Eq. (2.11) using the generalized Kudryashov method proceeds as follows:

- **Step 1:** To derive the exact solution, the following fractional complex transformation [29, 30] defined in (2.6), is applied: Upon substitution into Eq (2.11), the equation reduces to the NLODE:

$$H(U, -\omega U', kU, \omega^2 U'', k^2 U'', -\omega k U', \dots) = 0, \tag{2.12}$$

  where $U^{(i)} = U_{i\xi}$.

- **Step 2:** Assume that the solution of Eq. (2.12) can be represented as:

$$U(\xi) = \frac{\sum_{n=0}^{N} a_i \Psi(\xi)^i}{\sum_{n=0}^{M} b_j \Psi(\xi)^j} \tag{2.13}$$

  where $a_0, a_1, \dots a_N (a_N \neq 0)$ and $b_0, b_1, \dots b_M (b_M \neq 0)$ are constants.
  Let $\Psi = \Psi(\xi)$ satisfies the NLODE:

$$\Psi' = \Psi^2 - \Psi. \tag{2.14}$$

  The corresponding solutions of Eq.(2.14) take the form:

$$\Psi(\xi) = \frac{1}{Ae^{\xi} + 1} \tag{2.15}$$

  where $A \neq 0$.

- **Step 3:**The values of the positive integers N and M in (2.13) are obtained as described previously.

- **Step 4:** By substituting Eq.(2.13) into Eq. (2.12) and utilizing Eq. (2.14), the resulting expression is expanded in powers of $\Psi(\xi)^n, (n = 0, 1, 2, \dots)$. By collecting the coefficients of like powers and equating each to zero, a system of nonlinear algebraic equations is obtained. This system is subsequently solved using *Mathematica* to determine the unknown constants $a_0, a_1, \dots a_N$, $b_0, b_1, \dots b_M$, k and ω.

- **Step 5:** By substituting the values of a_n, b_m, k, ω and using (2.15) into (2.13), we obtain the complete set of exact solutions to the Eq. (2.11).

3 Application of the Modified exponential function method to variant Boussinesq equations

Let's consider the system of variant Boussinesq equations in the form, equations in the form,

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\dfrac{\partial u(x,t)}{\partial x} + \dfrac{\partial v(x,t)}{\partial x} = 0, \\ \dfrac{\partial^\alpha v(x,t)}{\partial t^\alpha} + \dfrac{\partial^3 u(x,t)}{\partial x^3} + \dfrac{\partial (u(x,t)v(x,t))}{\partial x} = 0, \end{cases} \tag{3.1}$$

The fractional complex traveling wave transformation Eq. (2.1) transforms Eqs. (3.1) into the system of ODEs,

$$\begin{cases} kUU' + kV' - \omega U' = 0, \\ k^3U^{(3)} + k(UV)' - \omega V' = 0 \end{cases} \tag{3.2}$$

Now integrating Eqs. (3.2) with respect to η , we obtain

$$\begin{cases} \frac{kU^2}{2} + kV - \omega U = 0, \\ k^3U'' + kUV - \omega V = 0 \end{cases} \tag{3.3}$$

from the first equation of Eq.(3.3),yields

$$V = \frac{\omega}{k}U - \frac{1}{2}U^2. \tag{3.4}$$

Substituting Eq. (3.4) into the second of Eq.(3.3), we attain

$$2k^4U'' - k^2U^3 - 2\omega^2U + 3k\omega U^2 = 0. \tag{3.5}$$

Now, balancing U'' and U^3 in Eq. (3.5), we get: $N - M = 1$. If we choose $M = 1$ and $N = 2$, then

$$U(\eta) = \frac{a_2 \exp(-2\Phi(\eta)) + a_1 \exp(-\Phi(\eta)) + a_0}{b_1 \exp(-\Phi(\eta)) + b_0}, \tag{3.6}$$

Replacing (3.6) into (3.5) provides a polynomial of $\exp(-i\Phi(\eta))$, where $(i = 0, 1, 2, ...)$. Then collecting the coefficients of like powers of $\exp(-i\Phi(\eta))$ and equating them to zero, we acquire the following system of algebraic equations:

$$\begin{aligned}
&2a_1b_0^2\lambda k^4\mu - 2a_0b_0b_1\lambda k^4\mu + 4a_2b_0^2k^4\mu^2 + 4a_0b_1^2k^4\mu^2 - 4a_1b_0b_1k^4\mu^2 + 3a_0^2b_0k\omega - 2a_0b_0^2\omega^2 - a_0^3k^2 = 0,\\
&2a_1b_0^2\lambda^2k^4 - 2a_0b_0b_1\lambda^2k^4 + 12a_2b_0^2\lambda k^4\mu + 6a_0b_1^2\lambda k^4\mu - 6a_1b_0b_1\lambda k^4\mu + 4a_1b_0^2k^4\mu - 4a_0b_0b_1k^4\mu\\
&+ 6a_0a_1b_0k\omega + 3a_0^2b_1k\omega - 2a_1b_0^2\omega^2 - 4a_0b_0b_1\omega^2 - 3a_0^2a_1k^2 = 0,\\
&8a_2b_0^2\lambda^2k^4 + 2a_0b_1^2\lambda^2k^4 - 2a_1b_0b_1\lambda^2k^4 + 6a_2b_0b_1\lambda k^4\mu + 6a_1b_0^2\lambda k^4\\
&- 6a_0b_0b_1\lambda k^4 + 16a_2b_0^2k^4\mu + 4a_0b_1^2k^4\mu - 4a_1b_0b_1k^4\mu + 3a_1^2b_0k\omega + 6a_0a_2b_0k\omega\\
&+ 6a_0a_1b_1k\omega - 2a_2b_0^2\omega^2 - 2a_0b_1^2\omega^2 - 4a_1b_0b_1\omega^2 - 3a_0a_1^2k^2 - 3a_0^2a_2k^2 = 0,\\
&6a_2b_0b_1\lambda^2k^4 + 2a_2b_1^2\lambda k^4\mu + 20a_2b_0^2\lambda k^4 + 2a_0b_1^2\lambda k^4 - 2a_1b_0b_1\lambda k^4 + 12a_2b_0b_1k^4\mu + 4a_1b_0^2k^4\\
&- 4a_0b_0b_1k^4 + 6a_1a_2b_0k\omega + 3a_1^2b_1k\omega + 6a_0a_2b_1k\omega - 2a_1b_1^2\omega^2 - 4a_2b_0b_1\omega^2 - 6a_0a_1a_2k^2 - a_1^3k^2 = 0,\\
&2a_2b_1^2\lambda^2k^4 + 18a_2b_0b_1\lambda k^4 + 4a_2b_1^2k^4\mu + 12a_2b_0^2k^4 + 3a_2^2b_0k\omega\\
&+ 6a_1a_2b_1k\omega - 2a_2b_1^2\omega^2 - 3a_0a_2^2k^2 - 3a_1^2a_2k^2 = 0,\\
&6a_2b_1^2\lambda k^4 + 12a_2b_0b_1k^4 + 3a_2^2b_1k\omega - 3a_1a_2^2k^2 = 0,\\
&4a_2b_1^2k^4 - a_2^3k^2 = 0,
\end{aligned} \tag{3.7}$$

The system (3.7) is solved using *Mathematica*, and three separate cases are analyzed to obtain the values of the unknown constants a_0, a_1, a_2 and ω.

- if $\lambda^2-4\mu>0, \mu\neq 0$:

$$\left\{a_0\to -b_0k\lambda-b_0k\sqrt{\lambda^2-4\mu}, a_1\to -b_1k\sqrt{\lambda^2-4\mu}-b_1k\lambda-2b_0k, a_2\to -2b_1k, \omega\to -k^2\sqrt{\lambda^2-4\mu}\right\},$$
$$\left\{a_0\to b_0k\lambda-b_0k\sqrt{\lambda^2-4\mu}, a_1\to -b_1k\sqrt{\lambda^2-4\mu}+b_1k\lambda+2b_0k, a_2\to 2b_1k, \omega\to -k^2\sqrt{\lambda^2-4\mu}\right\},$$
$$\left\{a_0\to b_0k\sqrt{\lambda^2-4\mu}-b_0k\lambda, a_1\to b_1k\sqrt{\lambda^2-4\mu}-b_1k\lambda-2b_0k, a_2\to -2b_1k, \omega\to k^2\sqrt{\lambda^2-4\mu}\right\},$$
$$\left\{a_0\to b_0k\sqrt{\lambda^2-4\mu}+b_0k\lambda, a_1\to b_1k\sqrt{\lambda^2-4\mu}+b_1k\lambda+2b_0k, a_2\to 2b_1k, \omega\to k^2\sqrt{\lambda^2-4\mu}\right\}, \tag{3.8}$$

- if $\lambda^2-4\mu<0, \mu\neq 0$:

$$\left\{a_0\to -b_0k\lambda-ib_0k\sqrt{4\mu-\lambda^2}, a_1\to -ib_1k\sqrt{4\mu-\lambda^2}-b_1k\lambda-2b_0k, a_2\to -2b_1k, \omega\to -ik^2\sqrt{4\mu-\lambda^2}\right\},$$
$$\left\{a_0\to b_0k\lambda-ib_0k\sqrt{4\mu-\lambda^2}, a_1\to -ib_1k\sqrt{4\mu-\lambda^2}+b_1k\lambda+2b_0k, a_2\to 2b_1k, \omega\to -ik^2\sqrt{4\mu-\lambda^2}\right\},$$
$$\left\{a_0\to -b_0k\lambda+ib_0k\sqrt{4\mu-\lambda^2}, a_1\to ib_1k\sqrt{4\mu-\lambda^2}-b_1k\lambda-2b_0k, a_2\to -2b_1k, \omega\to ik^2\sqrt{4\mu-\lambda^2}\right\},$$
$$\left\{a_0\to b_0k\lambda+ib_0k\sqrt{4\mu-\lambda^2}, a_1\to ib_1k\sqrt{4\mu-\lambda^2}+b_1k\lambda+2b_0k, a_2\to 2b_1k, \omega\to ik^2\sqrt{4\mu-\lambda^2}\right\}. \tag{3.9}$$

- if $\mu=0, \lambda\neq 0$:

$$\begin{aligned}&\left\{a_0\to -2b_0k\lambda, a_1\to -2b_1k\lambda-2b_0k, a_2\to -2b_1k, \omega\to -k^2\lambda\right\},\\ &\left\{a_0\to 0, a_1\to 2b_0k, a_2\to 2b_1k, \omega\to -k^2\lambda\right\},\\ &\left\{a_0\to 0, a_1\to -2b_0k, a_2\to -2b_1k, \omega\to k^2\lambda\right\},\\ &\left\{a_0\to 2b_0k\lambda, a_1\to 2b_1k\lambda+2b_0k, a_2\to 2b_1k, \omega\to k^2\lambda\right\}\end{aligned} \tag{3.10}$$

Substituting these values in (3.6) with using (2.10) then substituting (3.6) into (3.4), we required the following solutions of the system (3.1):

(i) If $\lambda^2-4\mu>0, \mu\neq 0$: Then we have the hyperbolic solutions

$$u_1(x,t)=-\frac{k\left(\lambda\sqrt{\lambda^2-4\mu}+\lambda^2-4\mu\right)\left(\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)+\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)\right)}{\sqrt{\lambda^2-4\mu}\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)+\lambda\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)} \tag{3.11}$$

$$v_1(x,t)=\frac{2k^2\mu\left(\lambda^2-4\mu\right)}{\left(\sqrt{\lambda^2-4\mu}\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)+\lambda\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)\right)^2}, \tag{3.12}$$

$$u_2(x,t)=\frac{k\left(\lambda\sqrt{\lambda^2-4\mu}-\lambda^2+4\mu\right)\left(\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)-\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)\right)}{\sqrt{\lambda^2-4\mu}\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)+\lambda\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)}, \tag{3.13}$$

$$v_2(x,t)=\frac{2k^2\mu\left(\lambda^2-4\mu\right)}{\left(\sqrt{\lambda^2-4\mu}\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)+\lambda\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)\right)^2}, \tag{3.14}$$

where $\eta=kx+k^2\sqrt{\lambda^2-4\mu}\frac{t^\alpha}{\Gamma(1+\alpha)}$.

$$u_3(x,t)=\frac{k\left(\lambda\sqrt{\lambda^2-4\mu}-\lambda^2+4\mu\right)\left(\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)-\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)\right)}{\sqrt{\lambda^2-4\mu}\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)+\lambda\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)}, \tag{3.15}$$

$$v_3(x,t)=\frac{2k^2\mu\left(\lambda^2-4\mu\right)}{\left(\sqrt{\lambda^2-4\mu}\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)+\lambda\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)\right)^2}, \tag{3.16}$$

$$u_4(x,t)=\frac{k\left(\lambda\sqrt{\lambda^2-4\mu}+\lambda^2-4\mu\right)\left(\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)+\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)\right)}{\sqrt{\lambda^2-4\mu}\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)+\lambda\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)}, \tag{3.17}$$

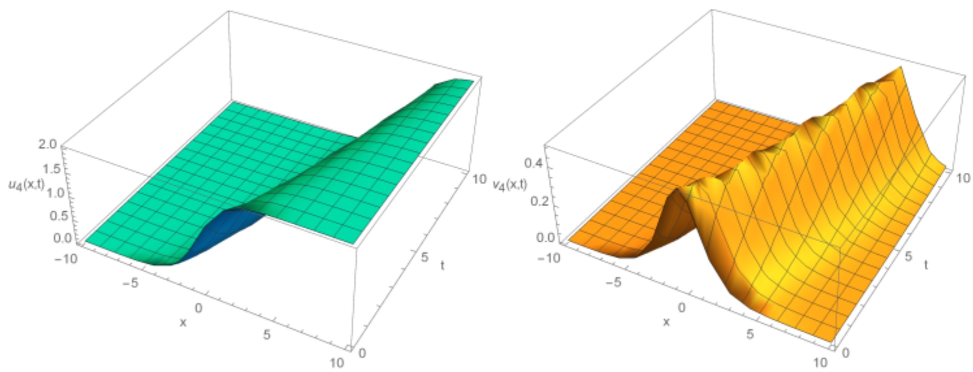
$$v_4(x,t)=\frac{2k^2\mu\left(\lambda^2-4\mu\right)}{\left(\sqrt{\lambda^2-4\mu}\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)+\lambda\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,(C_1+\eta)\right)\right)^2}, \tag{3.18}$$

where $\eta = kx - k^2\sqrt{\lambda^2-4\mu}\frac{t^\alpha}{\Gamma(1+\alpha)}$.

in particular case $(k=1,\lambda=3,\mu=2,C_1=0,\alpha=0.75)$:

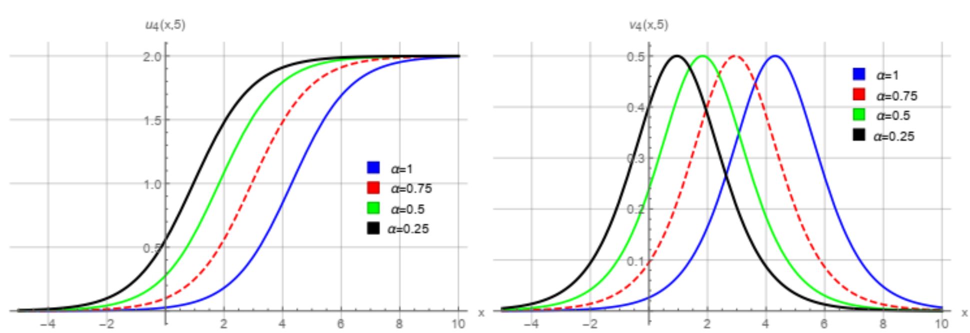
$$u_4(x,t)=\frac{4\left(\cosh\left(\frac{1}{2}\left(x-\frac{t^{0.75}}{\Gamma(1.75)}\right)\right)+\sinh\left(\frac{1}{2}\left(x-\frac{t^{0.75}}{\Gamma(1.75)}\right)\right)\right)}{3\cosh\left(\frac{1}{2}\left(x-\frac{t^{0.75}}{\Gamma(1.75)}\right)\right)+\sinh\left(\frac{1}{2}\left(x-\frac{t^{0.75}}{\Gamma(1.75)}\right)\right)}, \tag{3.19}$$

$$v_4(x,t)=\frac{4}{\left(3\cosh\left(\frac{1}{2}\left(x-\frac{t^{0.75}}{\Gamma(1.75)}\right)\right)+\sinh\left(\frac{1}{2}\left(x-\frac{t^{0.75}}{\Gamma(1.75)}\right)\right)\right)^2}. \tag{3.20}$$



(a) $u_4(x,t)$ $(k=1,\lambda=3,\mu=2,C_1=0,\alpha=0.75$ **(b)** $v_4(x,t)$ $(k=1,\lambda=3,\mu=2,C_1=0,\alpha=0.75$

Figure 1. The solution Profiles of Eq. (3.1) expressed in Eqs.(3.17) and (3.18) are illustrated over the domain $(x,t)\in[-10,10]\times[0,10]$



(a) $u_4(x,t)$ $(k=1,\lambda=3,\mu=2,C_1=0,\alpha=1,0.75,0.5,0.25.$ **(b)** $v_4(x,t)$ $(k=1,\lambda=3,\mu=2,C_1=0,\alpha=1,0.75,0.5,0.25.$

Figure 2. Plot2D of the solution to Eq. (3.1) corresponding to Eqs. (3.17) and (3.18) are presented for $x\in[-5,10]$ and $t=5$

(ii) If $\lambda^2-4\mu<0, \mu\neq 0$: Then we have the trigonometric solutions

$$u_5(x,t)=\frac{k\left(i\lambda\sqrt{4\mu-\lambda^2}+\lambda^2-4\mu\right)\left(\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)+i\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)\right)}{\sqrt{4\mu-\lambda^2}\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)-\lambda\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)}, \tag{3.21}$$

$$v_5(x,t)=\frac{2k^2\mu\left(\lambda^2-4\mu\right)}{\left(\lambda\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)-\sqrt{4\mu-\lambda^2}\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)\right)^2}, \tag{3.22}$$

$$u_6(x,t)=\frac{k\left(\lambda\sqrt{4\mu-\lambda^2}+i\lambda^2-4i\mu\right)\left(\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)+i\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)\right)}{\sqrt{4\mu-\lambda^2}\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)-\lambda\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)}, \tag{3.23}$$

$$v_6(x,t)=\frac{2k^2\mu\left(\lambda^2-4\mu\right)}{\left(\lambda\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)-\sqrt{4\mu-\lambda^2}\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)\right)^2}, \tag{3.24}$$

where $\eta=kx+ik^2\sqrt{\lambda^2-4\mu}\frac{t^\alpha}{\Gamma(1+\alpha)}$.

$$u_7(x,t)=\frac{k\left(-i\lambda\sqrt{4\mu-\lambda^2}+\lambda^2-4\mu\right)\left(\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)-i\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)\right)}{\sqrt{4\mu-\lambda^2}\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)-\lambda\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)} \tag{3.25}$$

$$v_7(x,t)=\frac{2k^2\mu\left(\lambda^2-4\mu\right)}{\left(\lambda\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)-\sqrt{4\mu-\lambda^2}\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)\right)^2}, \tag{3.26}$$

$$u_8(x,t)=\frac{k\left(-i\lambda\sqrt{4\mu-\lambda^2}-\lambda^2+4\mu\right)\left(\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)+i\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)\right)}{\sqrt{4\mu-\lambda^2}\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)-\lambda\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)} \tag{3.27}$$

$$v_8(x,t)=\frac{2k^2\mu\left(\lambda^2-4\mu\right)}{\left(\lambda\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)-\sqrt{4\mu-\lambda^2}\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(C_1+\eta\right)\right)\right)^2}, \tag{3.28}$$

where $\eta=kx-ik^2\sqrt{\lambda^2-4\mu}\frac{t^\alpha}{\Gamma(1+\alpha)}$.

In particular case:

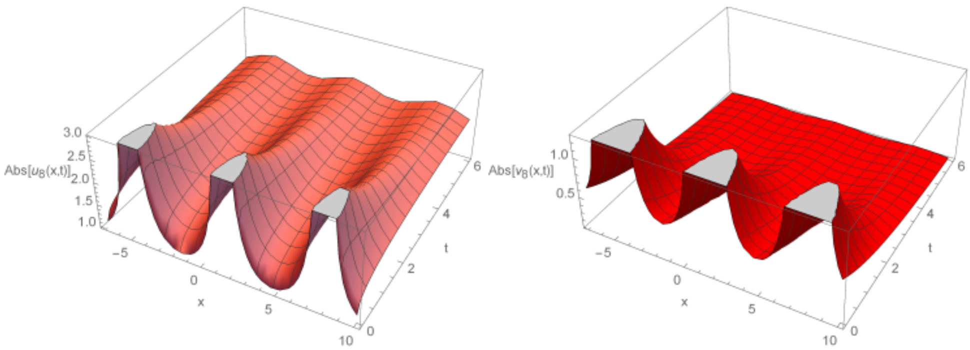
$$u_8(x,t)=\frac{(1-i)\left(\cos\left(\frac{1}{2}\left(x-\frac{it^{0.5}}{\Gamma(1.5)}\right)\right)+i\sin\left(\frac{1}{2}\left(x-\frac{it^{0.5}}{\Gamma(1.5)}\right)\right)\right)}{\sin\left(\frac{1}{2}\left(x-\frac{it^{0.5}}{\Gamma(1.5)}\right)\right)-\cos\left(\frac{1}{2}\left(x-\frac{it^{0.5}}{\Gamma(1.5)}\right)\right)}, \tag{3.29}$$

$$v_8(x,t)=-\frac{1}{\left(\cos\left(\frac{1}{2}\left(x-\frac{it^{0.5}}{\Gamma(1.5)}\right)\right)-\sin\left(\frac{1}{2}\left(x-\frac{it^{0.5}}{\Gamma(1.5)}\right)\right)\right)^2}. \tag{3.30}$$

(iii) If $\mu=0, \lambda\neq 0$: Then we have the hyperbolic solutions

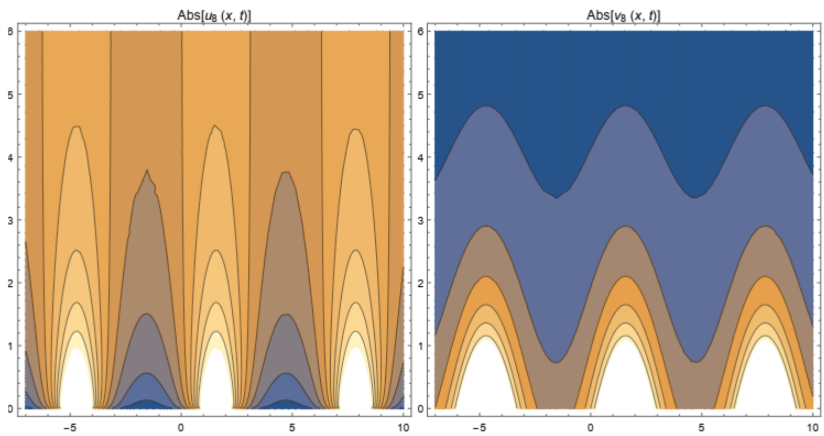
$$u_9(x,t)=\frac{2k\lambda}{\sinh\left(\lambda\left(C_1+\eta\right)\right)+\cosh\left(\lambda\left(C_1+\eta\right)\right)-1}, v_9(x,t)=-\frac{1}{2}k^2\lambda^2\operatorname{csch}^2\left(\frac{1}{2}\lambda\left(C_1+\eta\right)\right), \tag{3.31}$$

$$u_{10}(x,t)=-\frac{2k\lambda}{\sinh\left(\lambda\left(C_1+\eta\right)\right)+\cosh\left(\lambda\left(C_1+\eta\right)\right)-1}, v_{10}(x,t)=-\frac{1}{2}k^2\lambda^2\operatorname{csch}^2\left(\frac{1}{2}\lambda\left(C_1+\eta\right)\right), \tag{3.32}$$



(a) $u_8(x,t)$ $(\lambda = 1, \mu = \frac{1}{2}, C_1 = 0, k = 1, \alpha = 0.5)$ **(b)** $v_8(x,t)$ $(\lambda = 1, \mu = \frac{1}{2}, C_1 = 0, k = 1, \alpha = 0.5)$

Figure 3. The solution Profiles of Eq. (3.1) expressed in Eqs. (3.29) and (3.30) are illustrated over the domain $(x,t) \in [-7, 10] \times [0, 6]$



(a) $u_8(x,t)$ $(\lambda = 1, \mu = \frac{1}{2}, C_1 = 0, k = 1, \alpha = 0.5)$ **(b)** $v_8(x,t)$ $(\lambda = 1, \mu = \frac{1}{2}, C_1 = 0, k = 1, \alpha = 0.5)$

Figure 4. ContourPlot of the solution to Eq. (3.1) corresponding to Eqs. (3.29) and (3.30) are illustrated over the domain $(x,t) \in [-7, 10] \times [0, 6]$

where $\eta = kx + k^2\lambda\frac{t^\alpha}{\Gamma(1+\alpha)}$.

$$u_{11}(x,t) = -\frac{2k\lambda\left(\sinh\left(\lambda\left(C_1+\eta\right)\right)+\cosh\left(\lambda\left(C_1+\eta\right)\right)\right)}{\sinh\left(\lambda\left(C_1+\eta\right)\right)+\cosh\left(\lambda\left(C_1+\eta\right)\right)-1}, v_{11}(x,t) = -\frac{1}{2}k^2\lambda^2\text{csch}^2\left(\frac{1}{2}\lambda\left(C_1+\eta\right)\right), \tag{3.33}$$

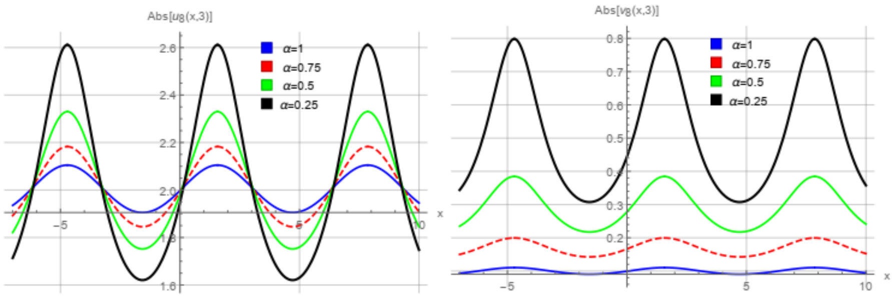
$$u_{12}(x,t) = k\lambda\left(\coth\left(\frac{1}{2}\lambda\left(C_1+\eta\right)\right)+1\right), v_{12}(x,t) = -\frac{1}{2}k^2\lambda^2\text{csch}^2\left(\frac{1}{2}\lambda\left(C_1+\eta\right)\right), \tag{3.34}$$

where $\eta = kx - k^2\lambda\frac{t^\alpha}{\Gamma(1+\alpha)}$.

In particular case: If $\lambda = 1, k = 1, C_1 = 0, \alpha = 0.75$, yields

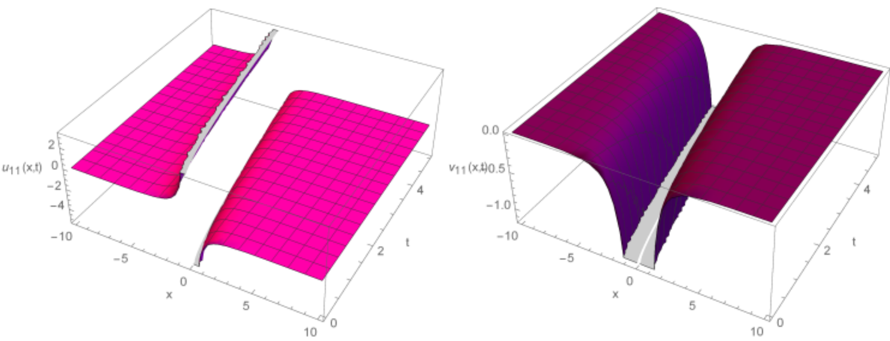
$$u_{11}(x,t) = -\frac{2\left(\cosh\left(\frac{t^{0.75}}{\Gamma(1.75)}+x\right)+\sinh\left(\frac{t^{0.75}}{\Gamma(1.75)}+x\right)\right)}{\cosh\left(\frac{t^{0.75}}{\Gamma(1.75)}+x\right)+\sinh\left(\frac{t^{0.75}}{\Gamma(1.75)}+x\right)-1}, \tag{3.35}$$

$$v_{11}(x,t) = -\frac{1}{2}\text{csch}^2\left(\frac{1}{2}\left(\frac{t^{0.75}}{\Gamma(1.75)}+x\right)\right). \tag{3.36}$$



(a) $u_8(x,t)$ ($\lambda = 1, \mu = \frac{1}{2}, C_1 = 0, k = 1, \alpha = 1, 0.75, 0.5, 0.25.$) **(b)** $v_8(x,t)$ ($\lambda = 1, \mu = \frac{1}{2}, C_1 = 0, k = 1, \alpha = 1, 0.75, 0.5, 0.25.$)

Figure 5. Plot2D of the solution to Eq. (3.1) corresponding to Eqs. (3.29) and (3.30) are illustrated over the domain $x \in [-7, 10]$ and $t = 3$



(a) $u_{11}(x,t)$ ($\lambda = 1, k = 1, C_1 = 0, \alpha = 0.75$) **(b)** $v_{11}(x,t)$ ($\lambda = 1, k = 1, C_1 = 0, \alpha = 0.75$)

Figure 6. The solution Profiles of Eq. (3.1) expressed in Eqs. (3.35) and (3.36) are illustrated over the domain $(x, t) \in [-10, 10] \times [0, 5]$

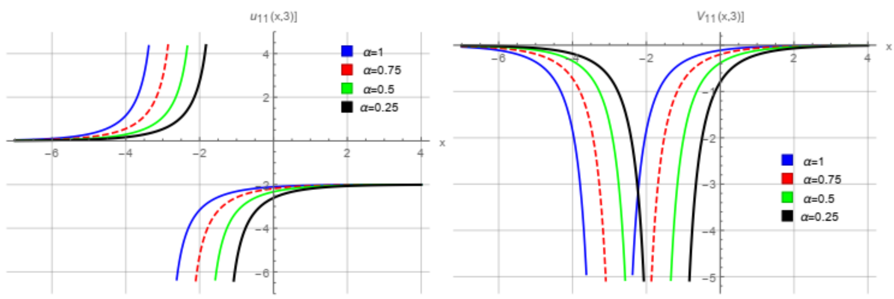
4 Application of the Generalized Kudryashov method to variant Boussinesq equations

Proceeding as above from Eq.(3.1) to (3.5), using the equation (2.13) with the fact that $M = 1, N = 2$, therefore

$$U(\xi) = \frac{a_0 + a_1\Psi(\xi) + a_2\Psi(\xi)^2}{b_0 + b_1\Psi(\xi)} \tag{4.1}$$

By applying **step 4** of the method, we obtain the following algebraic system

$$\begin{aligned}
&-3a_0^2b_0k\omega + 2a_0b_0^2\omega^2 + a_0^3k^2 = 0,\\
&-2a_1b_0^2k^4 + 2a_0b_0b_1k^4 - 6a_0a_1b_0k\omega - 3a_0^2b_1k\omega + 2a_1b_0^2\omega^2 + 4a_0b_0b_1\omega^2 + 3a_0^2a_1k^2 = 0,\\
&6a_1b_0^2k^4 - 8a_2b_0^2k^4 - 2a_0b_1^2k^4 - 6a_0b_0b_1k^4 + 2a_1b_0b_1k^4 - 3a_1^2b_0k\omega - 6a_0a_2b_0k\omega - 6a_0a_1b_1k\omega\\
&+ 2a_2b_0^2\omega^2 + 2a_0b_1^2\omega^2 + 4a_1b_0b_1\omega^2 + 3a_0a_1^2k^2 + 3a_0^2a_2k^2 = 0,\\
&-4a_1b_0^2k^4 + 20a_2b_0^2k^4 + 2a_0b_1^2k^4 + 4a_0b_0b_1k^4 - 2a_1b_0b_1k^4 - 6a_2b_0b_1k^4 - 6a_1a_2b_0k\omega\\
&-3a_1^2b_1k\omega - 6a_0a_2b_1k\omega + 2a_1b_1^2\omega^2 + 4a_2b_0b_1\omega^2 + a_1^3k^2 + 6a_0a_1a_2k^2 = 0,\\
&-12a_2b_0^2k^4 - 2a_2b_1^2k^4 + 18a_2b_0b_1k^4 - 3a_2^2b_0k\omega - 6a_1a_2b_1k\omega + 2a_2b_1^2\omega^2 + 3a_0a_2^2k^2 + 3a_1^2a_2k^2 = 0,\\
&6a_2b_1^2k^4 - 12a_2b_0b_1k^4 - 3a_2^2b_1k\omega + 3a_1a_2^2k^2 = 0,\\
&a_2^3k^2 - 4a_2b_1^2k^4 = 0.
\end{aligned} \tag{4.2}$$



(a) $u_{11}(x,t)$ $(\lambda = 1, k = 1, C_1 = 0, \alpha = 1, 0.75, 0.5, 0.25.)$

(b) $v_{11}(x,t)$ $(\lambda = 1, k = 1, C_1 = 0, \alpha = 1, 0.75, 0.5, 0.25.)$

Figure 7. Plot2D of the solution to Eq. (3.1) corresponding to Eqs. (3.35) and (3.36) examined within the domain $x \in [-7, 4]$ and $t = 3$

The values of the unknown constants a_0, a_1, a_2 and ω are determined by solving the system with the assistance of *Mathematica*.

$$
\begin{aligned}
&\left\{a_0 \to 0, a_1 \to 0, a_2 \to 2b_1 k, b_0 \to 0, \omega \to k^2\right\},\\
&\left\{a_0 \to 0, a_1 \to 0, a_2 \to -2b_1 k, b_0 \to 0, \omega \to -k^2\right\},\\
&\left\{a_0 \to 0, a_1 \to 0, a_2 \to 4b_0 k, b_1 \to -2b_0, \omega \to -2k^2\right\},\\
&\left\{a_0 \to 0, a_1 \to 0, a_2 \to -4b_0 k, b_1 \to -2b_0, \omega \to 2k^2\right\},\\
&\left\{a_0 \to 0, a_1 \to -2b_1 k, a_2 \to 2b_1 k, b_0 \to 0, \omega \to -k^2\right\},\\
&\left\{a_0 \to 0, a_1 \to 2b_1 k, a_2 \to -2b_1 k, b_0 \to 0, \omega \to k^2\right\},\\
&\left\{a_0 \to 0, a_1 \to -2b_0 k, a_2 \to 4b_0 k, b_1 \to -2b_0, \omega \to -k^2\right\},\\
&\left\{a_0 \to 0, a_1 \to 2b_0 k, a_2 \to -4b_0 k, b_1 \to -2b_0, \omega \to k^2\right\},\\
&\left\{a_0 \to -4b_0 k, a_1 \to 8b_0 k, a_2 \to -4b_0 k, b_1 \to -2b_0, \omega \to -2k^2\right\},\\
&\left\{a_0 \to 4b_0 k, a_1 \to -8b_0 k, a_2 \to 4b_0 k, b_1 \to -2b_0, \omega \to 2k^2\right\},\\
&\left\{a_0 \to -2b_0 k, a_1 \to 6b_0 k, a_2 \to -4b_0 k, b_1 \to -2b_0, \omega \to -k^2\right\},\\
&\left\{a_0 \to 2b_0 k, a_1 \to -6b_0 k, a_2 \to 4b_0 k, b_1 \to -2b_0, \omega \to k^2\right\},\\
&\left\{a_0 \to -2b_0 k, a_1 \to -2b_0 k, a_2 \to 4b_0 k, b_1 \to 2b_0, \omega \to -k^2\right\}.
\end{aligned} \tag{4.3}
$$

Substituting these values in (4.1) with using (2.15) then substituting (4.1) into (3.4), In this way, all exact solutions of the system (3.1) are derived

$$u_1(x,t) = \frac{2k}{Ae^{kx - \frac{k^2 t^\alpha}{\Gamma(\alpha+1)}} + 1}, v_1(x,t) = \frac{2Ak^2 e^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)} + x\right)}}{\left(Ae^{kx} + e^{\frac{k^2 t^\alpha}{\Gamma(\alpha+1)}}\right)^2}. \tag{4.4}$$

$$u_2(x,t) = -\frac{2k}{Ae^{\frac{k^2 t^\alpha}{\Gamma(\alpha+1)} + kx} + 1}, v_2(x,t) = \frac{2Ak^2 e^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)} + x\right)}}{\left(Ae^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)} + x\right)} + 1\right)^2}. \tag{4.5}$$

$$u_3(x,t) = \frac{4k}{A^2 e^{2k\left(\frac{2kt^\alpha}{\Gamma(\alpha+1)} + x\right)} - 1}, v_3(x,t) = -\frac{8A^2 k^2 e^{2k\left(\frac{2kt^\alpha}{\Gamma(\alpha+1)} + x\right)}}{\left(A^2 e^{2k\left(\frac{2kt^\alpha}{\Gamma(\alpha+1)} + x\right)} - 1\right)^2}. \tag{4.6}$$

$$u_4(x,t) = \frac{4ke^{\frac{4k^2 t^\alpha}{\Gamma(\alpha+1)}}}{e^{\frac{4k^2 t^\alpha}{\Gamma(\alpha+1)}} - A^2 e^{2kx}}, v_4(x,t) = -\frac{8A^2 k^2 e^{2k\left(\frac{2kt^\alpha}{\Gamma(\alpha+1)} + x\right)}}{\left(e^{\frac{4k^2 t^\alpha}{\Gamma(\alpha+1)}} - A^2 e^{2kx}\right)^2}. \tag{4.7}$$

$$u_5(x,t)=\frac{2k}{Ae^{\frac{k^2t^\alpha}{\Gamma(\alpha+1)}+kx}+1}-2k, v_5(x,t)=\frac{2Ak^2e^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}}{\left(Ae^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}+1\right)^2}. \tag{4.8}$$

$$u_6(x,t)=2k-\frac{2k}{Ae^{kx-\frac{k^2t^\alpha}{\Gamma(\alpha+1)}}+1}, v_6(x,t)=\frac{2Ak^2e^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}}{\left(Ae^{kx}+e^{\frac{k^2t^\alpha}{\Gamma(\alpha+1)}}\right)^2}. \tag{4.9}$$

$$u_7(x,t)=-\frac{2k}{Ae^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}+1}, v_7(x,t)=\frac{2Ak^2e^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}}{\left(Ae^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}+1\right)^2}. \tag{4.10}$$

$$u_8(x,t)=\frac{2ke^{\frac{k^2t^\alpha}{\Gamma(\alpha+1)}}}{Ae^{kx}+e^{\frac{k^2t^\alpha}{\Gamma(\alpha+1)}}}, v_8(x,t)=\frac{2Ak^2e^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}}{\left(Ae^{kx}+e^{\frac{k^2t^\alpha}{\Gamma(\alpha+1)}}\right)^2}. \tag{4.11}$$

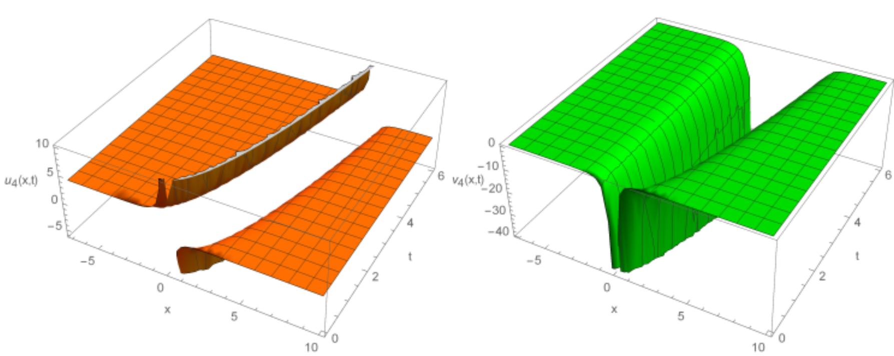
$$u_9(x,t)=\frac{4A^2ke^{2k\left(\frac{2kt^\alpha}{\Gamma(\alpha+1)}+x\right)}}{A^2e^{2k\left(\frac{2kt^\alpha}{\Gamma(\alpha+1)}+x\right)}-1}, v_9(x,t)=-\frac{8A^2k^2e^{2k\left(\frac{2kt^\alpha}{\Gamma(\alpha+1)}+x\right)}}{\left(A^2e^{2k\left(\frac{2kt^\alpha}{\Gamma(\alpha+1)}+x\right)}-1\right)^2}. \tag{4.12}$$

$$u_{10}(x,t)=\frac{4A^2ke^{2kx}}{A^2e^{2kx}-e^{\frac{4k^2t^\alpha}{\Gamma(\alpha+1)}}}, v_{10}(x,t)=-\frac{8A^2k^2e^{2k\left(\frac{2kt^\alpha}{\Gamma(\alpha+1)}+x\right)}}{\left(e^{\frac{4k^2t^\alpha}{\Gamma(\alpha+1)}}-A^2e^{2kx}\right)^2}. \tag{4.13}$$

$$u_{11}(x,t)=-\frac{2Ake^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}}{Ae^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}+1}, v_{11}(x,t)=\frac{2Ak^2e^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}}{\left(Ae^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}+1\right)^2}. \tag{4.14}$$

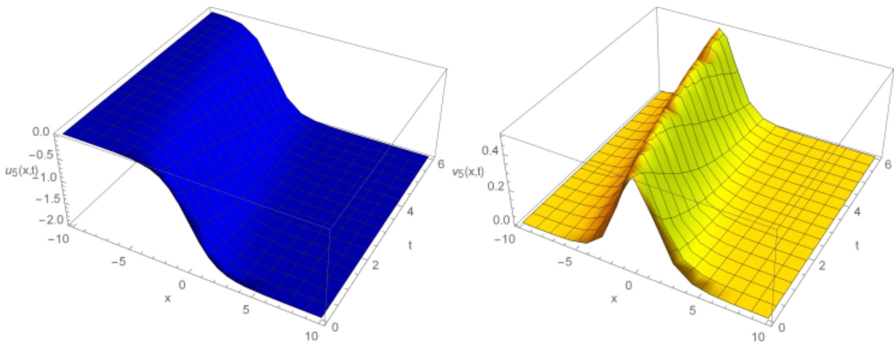
$$u_{12}(x,t)=\frac{2Ake^{kx}}{Ae^{kx}+e^{\frac{k^2t^\alpha}{\Gamma(\alpha+1)}}}, v_{12}(x,t)=\frac{2Ak^2e^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}}{\left(Ae^{kx}+e^{\frac{k^2t^\alpha}{\Gamma(\alpha+1)}}\right)^2}. \tag{4.15}$$

$$u_{13}(x,t)=-\frac{2Ake^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}}{Ae^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}+1}, v_{13}(x,t)=\frac{2Ak^2e^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}}{\left(Ae^{k\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}+x\right)}+1\right)^2}. \tag{4.16}$$



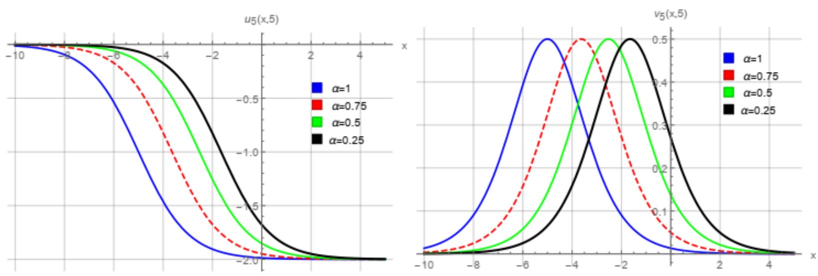
(a) $u_4(x,t)$ $(A=1, k=1, \alpha=0.5)$ **(b)** $v_4(x,t)$ $(A=1, k=1, \alpha=0.5)$

Figure 8. The solution Profiles of Eq. (3.1) expressed in Eqs. (4.7) are illustrated over the domain $(x,t)\in[-7,10]\times[0,6]$



(a) $u_5(x,t)$ $(A=1, k=1, \alpha=0.75)$ (b) $v_5(x,t)$ $(A=1, k=1, \alpha=0.75)$

Figure 9. The solution Profiles of Eq. (3.1) expressed in (4.8) are illustrated over the domain $(x,t) \in [-10,10] \times [0,6]$



(a) $u_5(x,t)$ $(A=1, k=1, \alpha=1, 0.75, 0.5, 0.25)$ (b) $v_5(x,t)$ $(A=1, k=1, \alpha=1, 0.75, 0.5, 0.25)$

Figure 10. Plot2D of the solution to Eq. (3.1) corresponding to Eq. (4.8), presented for $x \in [-10,5]$ and $t=5$

5 Conclusion and remarks

In this study, a variety of traveling wave solutions to the variant Boussinesq equations were obtained using the Modified Exponential Function Method and the Generalized Kudryashov Method. The results demonstrate that both methods are powerful and effective analytical techniques, suitable for solving a broad class of NLPDEs and systems. The 2D, 3D, and contour plots of the solutions-particularly the hyperbolic, exponential, and periodic ones-generated using the *Mathematica* software package, reveal significant geometrical structures inherent in these solutions. These findings confirm the effectiveness of the applied methods in deriving traveling wave solutions of the considered fractional nonlinear partial differential equation (NLPDE). In addition, the presented approaches can be effectively generalized to address other nonlinear equations arising in a range of fields such as physics, finance, biology, and applied sciences.

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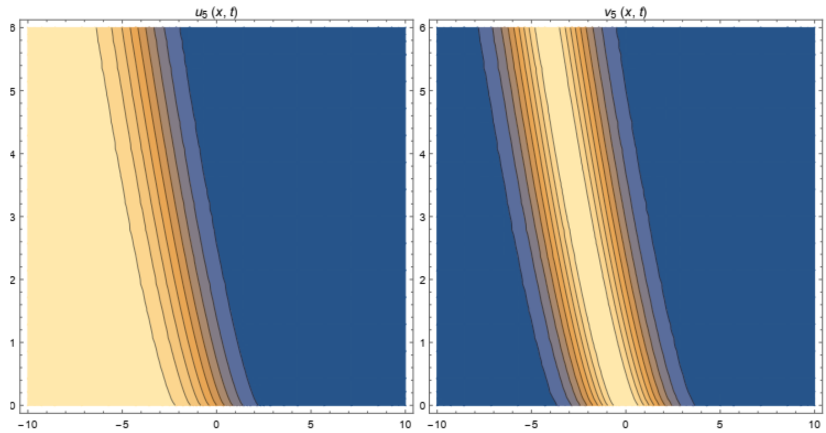
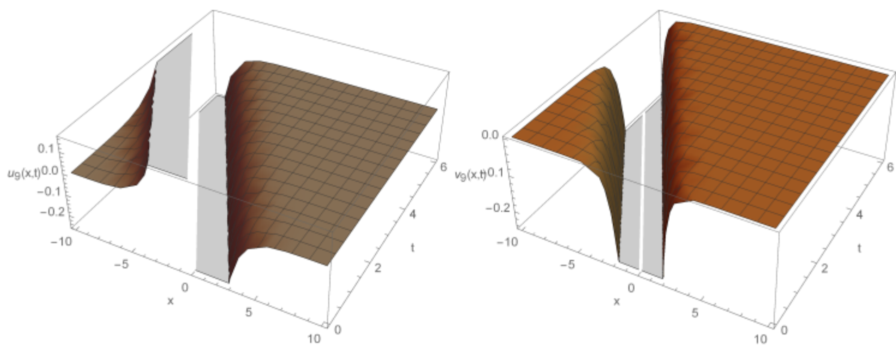


(a) $u_5(x,t)$ $(A=1, k=1, \alpha=0.75)$ **(b)** $v_5(x,t)$ $(A=1, k=1, \alpha=0.75)$

Figure 11. ContourPlot of the solution to Eq. (3.1) given by (4.8), for $(x,t) \in [-10,10] \times [0,6]$



(a) $u_9(x,t)$ $(A=1, k=1, \alpha=1)$ **(b)** $v_9(x,t)$ $(A=1, k=1, \alpha=1)$

Figure 12. The solution Profiles of Eq. (3.1) expressed in (4.12) are illustrated over the domain $(x,t) \in [-10,10] \times [0,6]$

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