Palestine Journal of Mathematics

Vol 14(3)(2025) , 222–231

$\mathcal{L}-$FUZZY WEAKLY $2-$ABSORBING IDEALS IN AN ALMOST DISTRIBUTIVE LATTICE

Natnael Teshale Amare, N. Rafi and S. Nageswara Rao

Communicated by Ayman Badawi

MSC 2010 Classifications: 06D72, 06F15, 08A72.

Keywords and phrases:Almost Distributive Lattice (ADL); $\mathcal{L}-$fuzzy Prime Ideal; $\mathcal{L}-$fuzzy Weakly Prime Ideal; $\mathcal{L}-$fuzzy $2-$absorbing Ideal; $\mathcal{L}-$fuzzy Weakly $2-$absorbing Ideal.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: T.A. Natnael

Abstract *The concept of $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals within an Almost Distributive Lattice (ADL) is introduced, expanding upon the notion of $\mathcal{L}-$fuzzy weakly prime ideals within the same framework. We illustrate the connections between $\mathcal{L}-$fuzzy prime ideals and $\mathcal{L}-$fuzzy weakly prime ideals, $\mathcal{L}-$fuzzy weakly prime ideals and $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals, as well as $\mathcal{L}-$fuzzy $2-$absorbing ideals and $\mathcal{L}-$fuzzy weakly 2-absorbing ideals. Moreover, it is shown that both the image and inverse image of $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals also qualify as $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals.*

1 Introduction

The significance of prime ideals is crucial in exploring the structural theory of distributive lattices in a broad sense, particularly in the context of Boolean algebras. Badawi [3] initially introduced the idea of $2-$absorbing ideals in commutative rings, extending the concept of prime ideals introduced in [7]. Building on this, Chuadhari [5] extended the notion of 2-absorbing ideals to semi-rings. Subsequently, Badawi and Darani [2] introduced the concept of weakly 2- absorbing ideals in commutative rings, a generalization of weakly prime ideals introduced by Anderson and Smith [1]. Extending the concepts of $2-$absorbing and weakly $2-$absorbing ideals from rings to lattices, Wasakidar and Gaikerad [20] introduced the idea of $2-$absorbing ideals and weakly $2-$absorbing ideals in lattices. Later, Natnael [11] introduced the concept of weakly 2- absorbing ideals in an Almost Distributive Lattice. Zadeh [21] initially defined a fuzzy subset of a set X as a function mapping elements of X to real numbers in the interval $[0, 1]$. Goguen [8] extended this by replacing the valuation set $[0, 1]$ with a complete lattice L, aiming to provide a more comprehensive exploration of fuzzy set theory through fuzzy sets. Darani and Ghasemi [6], as well as Mandal [10], introduced the notions of L-fuzzy $2-$absorbing ideals and 2- absorbing fuzzy ideals for commutative rings, respectively, which generalize the concept of fuzzy prime ideals in rings investigated by June [9] and Sharma [17]. Subsequently, Nimbhorkar and Patil [13, 14] introduced the notions of fuzzy weakly $2-$absorbing ideals in lattices. In [18], we presented the ideas of $\mathcal{L}-$fuzzy ideals within an Almost Distributive Lattice (ADL), forming the foundation for our research. Natnael [12] later extended this work by introducing the concept of $\mathcal{L}-$fuzzy $2-$absorbing ideals in an ADL.

In this paper, we have introduced the concept of $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals within an ADL, aiming to extend the idea of $\mathcal{L}-$fuzzy prime ideals in an ADL as presented in [15]. Initially, we define $\mathcal{L}-$fuzzy weakly prime ideals, which are less stringent than $\mathcal{L}-$fuzzy prime ideals. We then proceed to adapt the M.H. Stone theorem [16] concerning prime ideals in distributive lattices to encompass $\mathcal{L}-$fuzzy weakly prime ideals. Our main emphasis is on investigating the connections between $\mathcal{L}-$fuzzy prime ideals and $\mathcal{L}-$fuzzy weakly prime ideals, as well as *the relationships between $\mathcal{L}-$fuzzy weakly prime ideals and $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals, along with $\mathcal{L}-$fuzzy 2-absorbing ideals and $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals. Counter examples are provided to demonstrate that the converses of these relationships do not hold. Furthermore, we demonstrate that the direct product of any two $\mathcal{L}-$fuzzy weakly prime ideals results in an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal. However, it is important to note that the product of $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals may not necessarily yield an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal. Additionally, we establish that both the image and pre-image of any fuzzy weakly $2-$absorbing ideal also qualify as fuzzy weakly 2- absorbing ideals.*

2 Preliminaries

In this portion, we revisit certain definitions and fundamental findings primarily sourced from [18, 15, 19].

Definition 2.1. An algebra $R = (R, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is referred to as an ADL if it meets the subsequent conditions for all r, s and t in R.

(i) $0 \wedge r = 0$

(ii) $r \vee 0 = r$

(iii) $r \wedge (s \vee t) = (r \wedge s) \vee (r \wedge t)$

(iv) $r \vee (s \wedge t) = (r \vee s) \wedge (r \vee t)$

(v) $(r \vee s) \wedge t = (r \wedge t) \vee (s \wedge t)$

(vi) $(r \vee s) \wedge s = s$.

Every distributive lattice with a lower bound is categorized as an ADL.

Example 2.2. For any nonempty set A, it's possible to transform it into an ADL that doesn't constitute a lattice by selecting any element 0 from A and fixing an arbitrary element $u_0 \in R$. For every $u, v \in R$, define $\wedge$ and $\vee$ on R as follows:

$$u \wedge v = \begin{cases} v & \text{if } u \neq u_0 \\ u_0 & \text{if } u = u_0 \end{cases} \quad \text{and} \quad u \vee v = \begin{cases} u & \text{if } u \neq u_0 \\ v & \text{if } u = u_0 \end{cases}$$

Then $(A, \wedge, \vee, u_0)$ is an ADL (called the **discrete ADL**) with u_0 as its zero element.

Definition 2.3. Consider $R = (R, \wedge, \vee, 0)$ be an ADL. For any r and $s \in R$, establish $r \leq s$ if $r = r \wedge s$ (which is equivalent to $r \vee s = s$). Then $\leq$ is a partial order on R with respect to which 0 is the smallest element in R.

Theorem 2.4. *The following conditions are valid for any r, s and t in an ADL R.*

(1) $r \wedge 0 = 0 = 0 \wedge r$ *and* $r \vee 0 = r = 0 \vee r$

(2) $r \wedge r = r = r \vee r$

(3) $r \wedge s \leq s \leq s \vee r$

(4) $r \wedge s = r$ *iff* $r \vee s = s$

(5) $r \wedge s = s$ *iff* $r \vee s = r$

(6) $(r \wedge s) \wedge t = r \wedge (s \wedge t)$ *(in other words, $\wedge$ is associative)*

(7) $r \vee (s \vee r) = r \vee s$

(8) $r \leq s \Rightarrow r \wedge s = r = s \wedge r$ $\big($ *iff* $r \vee s = s = s \vee r\big)$

(9) $(r \wedge s) \wedge t = (s \wedge r) \wedge t$

(10) $(r \vee s) \wedge t = (s \vee r) \wedge t$

(11) $r \wedge s = s \wedge r$ *iff* $r \vee s = s \vee r$

(12) $r \wedge s = \inf\{r, s\}$ *iff* $r \wedge s = s \wedge r$ *iff* $r \vee s = \sup\{r, s\}$.

Definition 2.5. Let R and G be ADLs and form the set $R \times G$ by
$R \times G = \{(r, g) : r \in R \text{ and } g \in G\}$. Define $\wedge$ and $\vee$ in $R \times G$ by,

$$(r_1, g_1) \wedge (r_2, g_2) = (r_1 \wedge r_2, g_1 \wedge g_2)$$
$$\text{and } (r_1, g_1) \vee (r_2, g_2) = (r_1 \vee r_2, g_1 \vee g_2), \text{ for all } (r_1, g_1), (r_2, g_2) \in R \times G.$$

Then $(R \times G, \wedge, \vee, 0)$ is an ADL under the pointwise operations and $0 = (0, 0)$ is the zero element in $R \times G$.

Definition 2.6. A non-empty subset, denoted as I in an ADL R is termed an ideal in R if it satisfies the conditions: if u and v belong to I, then $u \vee v$ is also in F, and for every element r in R, the $u \wedge r$ is in F.

Definition 2.7. A proper ideal I in R is a prime ideal if for any u and v belongs R, $u \wedge v$ belongs F, then either u belongs F or v belongs F.

Theorem 2.8. *Let I be an ideal in R. Let F be a non-empty subset in R such that $r \wedge s \in F$, for all r and $s \in F$. Assume $I \cap F$ is empty set. Then there exists a prime ideal P in R containing I and $P \cap F$ is empty set.*

Theorem 2.9. *Let P be an ideal in R. Then P a weakly prime ideal in R only if P is a prime ideal in R.*

Definition 2.10. Let R and G be ADLs. A mapping $g : R \to G$ is called a homomorphism if the following are satisfied, for any $r, s, t \in R$.
$(1).\ f(r \wedge s \wedge t) = f(r) \wedge f(s) \wedge f(t)$
$(2).\ f(r \vee s \vee t) = f(r) \vee f(s) \vee f(t)$
$(3).\ f(0) = 0.$

Definition 2.11. an $\mathcal{L}-$fuzzy subset η^w is said to be an $\mathcal{L}-$fuzzy ideal in R, if $\eta^w(0) = 1$ and $\eta^w(r \vee s) = \eta^w(r) \wedge \eta^w(s)$, for all r and s in R.

Definition 2.12. A proper $\mathcal{L}-$fuzzy ideal η^w is prime if $\phi \wedge \psi \leq \eta^w$ implies either $\phi \leq \eta^w$ or $\psi \leq \eta^w$, for any $\mathcal{L}-$fuzzy ideals ϕ and ψ in R.

Theorem 2.13. *Let η^w be an $\mathcal{L}-$fuzzy subset. Then η^w is prime iff there exists a prime ideal P in R and a prime element β in L such that η^w equals β_P.*

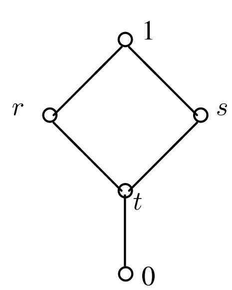
Definition 2.14. A proper $\mathcal{L}-$fuzzy ideal η^w is called an $\mathcal{L}-$fuzzy prime ideal in R if $\eta^w(r \wedge s)$ equals $\eta^w(r)$ or $\eta^w(s)$, for all elements r and s in R.

3 $\mathcal{L}-$fuzzy Weakly Prime Ideals

In the following, we define and characterize the concepts of $\mathcal{L}-$fuzzy weakly prime ideals in terms of weakly prime ideals. A proper ideal P is a weakly prime ideal in R if for all $r, s \in R$, $0 \neq r \wedge s \in P \Rightarrow$ either $r \in P$ or $s \in P$. Here, we extend this result to the case of $\mathcal{L}-$fuzzy ideals in the following.

Definition 3.1. A proper $\mathcal{L}-$fuzzy ideal η^w is an $\mathcal{L}-$fuzzy weakly prime ideal in R if, for any elements r and s in R such that $r \wedge s \neq 0$, the inequality $\eta^w(r \wedge s) \leq \eta^w(r) \vee \eta^w(s)$ holds true.

Example 3.2. Let $D = \{0, u, v\}$ be a discrete ADL with 0 as its zero element defined in 2.2 and $L = \{0, r, s, t, 1\}$ be the lattice represented by the Hasse diagram given below:



Consider $D \times L = \{(d,e) \mid d \in D$ and $e \in L\}$. Then $(D \times L, \wedge, \vee, 0)$ is an ADL under the pointwise operations $\wedge$ and $\vee$ on $D \times L$ and $0 = (0,0)$, the zero element in $D \times L$. Now define $\eta^w : D \times L \to [0,1]$ by

$$\eta^w(d,e) = \begin{cases} 1 & \text{if } (d,e) = (0,0) \\ 3/4 & \text{if } d = 0 \text{ and } e \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $(d,e) \in D \times L$. Clearly η^w is an $\mathcal{L}-$fuzzy ideal in $D \times L$. For any $(a,b),(c,d) \in D \times L$ such that $(a,b) \wedge (c,d) \neq (0,0)$, we observe that $\eta^w\big((a,b) \wedge (c,d)\big) \leq \eta^w(a,b) \vee \eta^w(c,d)$. Therefore, η^w is an $\mathcal{L}-$fuzzy weakly prime ideal in $D \times L$.

Recall that for any $\beta \in L$, the set $\eta^w_\beta = \{r \in R : \beta \leq \eta^w(r)\}$ is called the β-cut of η^w.

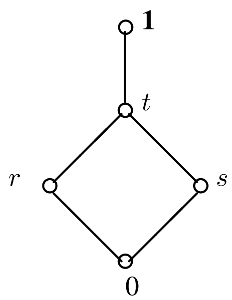
Theorem 3.3. *Let η^w be a proper $\mathcal{L}-$fuzzy ideal in R. Then η^w is an $\mathcal{L}-$fuzzy weakly prime ideal in R iff for all $\beta \in L$, either $\eta^w_\beta = R$ or η^w_β is a weakly prime ideal in R.*

Proof. Assume η^w is an $\mathcal{L}-$fuzzy weakly 2$-$absorbing ideal. Take $r,s \in R$ such that $r \wedge s \neq 0$. Now $r \wedge s \in \eta^w_\beta$ implies $\beta \leq \eta^w(r \wedge s) \leq \eta^w(r) \vee \eta^w(s)$. Consequently, $\beta \leq \eta^w(r)$ or $\beta \leq \eta^w(s)$. It follows that, $r \in \eta^w_\beta$ or $s \in \eta^w_\beta$. Thus η^w_β is a weakly 2$-$absorbing ideal in R. On the other hand, assume that $\beta = \eta^w(r \wedge s)$. Then $r \wedge s \in \eta^w_\beta$. Since η^w_β is a weakly 2$-$absorbing ideal in R, it follows that either $r \in \eta^w_\beta$ or $s \in \eta^w_\beta$. Let $r \in \eta^w_\beta$. Then $\eta^w(r \wedge s) = \beta \leq \eta^w(r) \leq \eta^w(r) \vee \eta^w(s)$. Therefore, η^w qualifies as an $\mathcal{L}-$fuzzy weakly 2$-$absorbing ideal in R. □

Corollary 3.4. *If η^w is an $\mathcal{L}-$fuzzy weakly prime ideal in R, then η^w_1 referred to as a weakly prime ideal.*

The converse of the above corollary is not true; for consider the example below.

Example 3.5. Let $D = \{0,u,v\}$ be a discrete ADL with 0 as its zero element defined in 2.2 and $L = \{0,r,s,t,1\}$ be the lattice represented by the Hasse diagram given below:



Consider $D \times L = \{(d,e) \mid d \in D$ and $e \in L\}$. Then $(D \times L, \wedge, \vee, 0)$ is an ADL under the point-wise operations $\wedge$ and $\vee$ on $D \times L$ and $0 = (0,0)$, the zero element in $D \times L$. Now define $\eta^w : D \times L \to [0,1]$ by

$$\eta^w(d,e) = \begin{cases} 1 & \text{if } (d,e) = (0,0) \\ 2/3 & \text{if } d = 0 \text{ and } e \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $(d,e) \in D \times L$. Clearly η^w is an $\mathcal{L}-$fuzzy ideal in $D \times L$. Then $\eta^w_1 = \{(0,0)\}$. Consequently, η^w_1 is a weakly prime ideal in $D \times L$. While, η^w is not an $\mathcal{L}-$fuzzy weakly prime ideal in $D \times L$, since $\eta^w(0,r) \wedge (0,s) = \eta^w(0,0) = 1 \nleq 2/3 = \eta^w(0,r) \vee \eta^w(0,s)$.

Corollary 3.6. *An ideal P is said to be a weakly prime ideal in R iff χ_P is an $\mathcal{L}-$fuzzy weakly prime ideal in R.*

We have proved that every prime ideal of an ADL is a weakly prime ideal of an ADL (refer Theorem 2.10). Here, we extend this result to the case of $\mathcal{L}-$fuzzy ideals in the following.

Theorem 3.7. *Consider η^w is an $\mathcal{L}-$fuzzy ideal in R. If η^w is an $\mathcal{L}-$fuzzy prime ideal in R, then η^w is an $\mathcal{L}-$fuzzy weakly prime ideal in R. The converse of this result is not true.*

Proof. It is Clear. □

Example 3.8. Let $D \times L = \{(d, e) \mid d \in D \text{ and } e \in L\}$ be an ADL defined in example 3.2. Clearly η^w is an $\mathcal{L}-$fuzzy ideal of $D\times L$. Then $\eta^w_1 = \{(0, 0)\}$ and $\eta^w_{3/4} = \{(0, r), (0, s), (0, t), (0, 1)\}$ are weakly prime ideal of $D \times L$, and $\eta^w_0 = D \times L$. Thus η^w is an $\mathcal{L}-$fuzzy weakly prime ideal of $D \times L$. But η^w is not an $\mathcal{L}-$fuzzy prime ideal of $D \times L$, since η^w_β is not a prime ideal of $D \times L$, for each $\beta \in [0, 1]$, in particular, $\beta = 1$ (η^w_1 is neither an ideal nor a prime ideal).

For any $\beta \in L$, we define the $\mathcal{L}-$fuzzy ideals $\eta^w \vee \beta$ and $\eta^w \wedge \beta$ by $(\eta^w \vee \beta)(r) = \eta^w(r) \vee \beta$ and $(\eta^w \wedge \beta)(r) = \eta^w(r) \wedge \beta$, for all r in R.

Lemma 3.9. *Let η^w be an $\mathcal{L}-$fuzzy weakly prime ideal in R and $\beta \in L$ with $\eta^w(n) \leq \beta < 1$, for any maximal element n in R. Then $\eta^w \vee \beta$ is an $\mathcal{L}-$fuzzy weakly prime ideal in R.*

Theorem 3.10. *Let ψ^w be a proper $\mathcal{L}-$fuzzy ideal such that $\vee\{\psi^w(r) : r \in R, \psi^w(r) < 1\} < 1$. Then there exists an $\mathcal{L}-$fuzzy weakly prime ideal η^w in R such that $\psi^w \leq \eta^w$.*

Proof. Let $\beta = \bigvee \{\psi^w(r) : r \in R \text{ and } \psi^w(r) < 1\}$ and $Q = \{r \in R : \psi^w(r) = 1\}$. Then Q is a proper ideal in R, since ψ^w is proper. Then there exists a weakly prime ideal P in R such that $Q \subseteq P$. Thus χ_P is an $\mathcal{L}-$fuzzy weakly prime ideal in R. Now consider $\chi_P \vee \beta$. For any $r \in R$, $r \in P \Rightarrow (\chi_P \vee \beta)(r) = \chi_P(r) \vee \beta = 1 \vee \beta = 1$ and $r \notin P \Rightarrow (\chi_P \vee \beta)(r) = \chi_P(r) \vee \beta = 0 \vee \beta = \beta$ and hence $\chi_P \vee \beta = \beta_P$. By 3.9, β_P is an $\mathcal{L}-$fuzzy weakly prime ideal in R. Now, if $r \in P$, then $\psi^w(r) \leq 1 = \beta_P(r)$ and if $r \notin P$, then $r \notin Q$ impies that $\psi^w(r) < 1$ and hence $\psi^w(r) \leq \beta = \beta_P(r)$. Therefore, $\psi^w \leq \beta_P$. □

Next, we extend the theorem 2.8 on prime ideals of ADLs (which is analogous to M.H. Stone theorem on prime ideals of distributive lattices) to the case of $\mathcal{L}-$fuzzy weakly prime ideals.

Theorem 3.11. *Let β be a weak prime element in L, ϕ^w be an $\mathcal{L}-$fuzzy ideal in R and ψ^w an $\mathcal{L}-$fuzzy filter in R such that $\phi^w \wedge \psi^w \leq \overline{\beta}$. Then there exists an $\mathcal{L}-$fuzzy weakly prime ideal η^w in R such that $\phi^w \leq \eta^w$ and $\eta^w \wedge \psi^w \leq \overline{\beta}$.*

Proof. Put $P = \{r \in R : \phi^w(r) \nleq \beta\}$ and $Q = \{r \in R : \psi^w(r) \nleq \beta\}$. Clearly, P is an ideal and Q is a filter in R. Since β is weakly prime ideal and for any $r \in R$, $\phi^w(r) \wedge \psi^w(r) \leq \beta$, it follows that $\phi^w(r) \leq \beta$ or $\psi^w(r) \leq \beta$ and hence $r \notin P$ or $r \notin Q$. Therefore, $P \cap Q$ is empty. Then there exists a weakly prime ideal G such that $P \subseteq G$ and $G \cap Q$ is empty. It follows that β_G is a prime $\mathcal{L}-$fuzzy ideal and hence, β_G is an $\mathcal{L}-$fuzzy weakly prime ideal. Also, $r \notin G \Rightarrow r \notin P \Rightarrow \phi^w(r) \leq \beta = \beta_G(r)$ and $r \in G \Rightarrow \phi^w(r) \leq 1 = \beta_G(r)$.
Now, $r \in G \Rightarrow r \notin Q \Rightarrow \beta_G(r) \wedge \psi^w(r) = 1 \wedge \psi^w(r) = \psi^w(r) \leq \beta = \overline{\beta}(r)$
and $r \notin G \Rightarrow \beta_G(r) \wedge \psi^w(r) = \beta \wedge \psi^w(r) \leq \beta = \overline{\beta}(r)$.
Thus $\phi^w \leq \beta_G$ and hence, $\beta_G \wedge \psi^w \leq \overline{\beta}$. □

4 $\mathcal{L}-$fuzzy Weakly $2-$Absorbing Ideals

In the following, we introduce the notions of $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals and their characterizations which is the central part of our investigations. Initially, let us revisit the definition outlined in [11], indicating that a proper ideal P in R is considered a weakly $2-$absorbing ideal in R if, for all elements r, s, and t in R such that $r \wedge s \wedge t \neq 0$, the condition $r \wedge s \wedge t$ belonging to P implies either $r \wedge s$ belonging to P or $r \wedge t$ belonging to P or $s \wedge t$ belonging to P. Now, we aim to extend this outcome to the realm of $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals as elucidated below.

Definition 4.1. A proper $\mathcal{L}-$fuzzy ideal η^w in R is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in R if for any elements r, s and t in R with $r \wedge s \wedge t \neq 0$, the inequality $\eta^w(r \wedge s \wedge t) \leq \eta^w(r \wedge s) \vee \eta^w(r \wedge t) \vee \eta^w(s \wedge t)$ holds true.

Example 4.2. Let $R = \{0, r, s, t\}$ and L be 4 elements chain $\{0, \gamma, \beta, 1\}$, where $0 < \gamma < \beta < 1$ and let $\vee$ and $\wedge$ be binary operations on A defined by:

$\vee$	0	r	s	t
0	0	r	s	t
r	r	r	r	r
s	s	s	s	s
t	t	r	s	t

$\wedge$	0	r	s	t
0	0	0	0	0
r	0	r	s	t
s	0	r	s	t
t	0	t	t	t

Define an $\mathcal{L}-$fuzzy subset η^w in R as follows: $\eta^w(0) = 1$, $\eta^w(r) = \gamma$, $\eta^w(s) = \gamma$ and $\eta^w(t) = \beta$. It is evident that η^w is an $\mathcal{L}-$fuzzy ideal in R. Next we observe that, for any r, s and $t \in R$, $\eta^w(r \wedge s \wedge t) = \eta^w(s \wedge t)$ or $\eta^w(r \wedge t)$. Therefore, η^w is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in R.

Following that, we define the concept of an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal with respect to β-cut.

Theorem 4.3. *Let η^w be an $\mathcal{L}-$fuzzy ideal in R. An ideal η^w_β is a weakly $2-$absorbing ideal in R or $\eta^w_\beta = R$, for all $\beta \in L$ iff η^w is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in R.*

Corollary 4.4. *An ideal P in R is a weakly $2-$absorbing ideal in R iff χ_P is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in R.*

In the following, we facilitate the inter-relationship between $\mathcal{L}-$fuzzy weakly prime ideals and $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals in an ADL.

Theorem 4.5. *Every $\mathcal{L}-$fuzzy weakly prime ideal in R is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in R.*

The converse of the above result is not true; consider the following example.

Example 4.6. Let $D \times L = \{(d, e) \mid d \in D \text{ and } e \in L\}$ be an ADL defined in example 3.2. Now define $\eta^w : D \times L \to [0, 1]$ by

$$\eta^w(d,e) = \begin{cases} 1 & \text{if } (d,e) = (0,0) \\ 3/4 & \text{if } d \neq 0 \textit{ and } e \in \{0,t\} \\ 0 & \text{otherwise} \end{cases}$$

for all $(d, e) \in D \times L$. Clearly η^w is an $\mathcal{L}-$fuzzy ideal in $D \times L$. Then $\eta^w_1 = \{(0, 0)\}$ and $\eta^w_{3/4} = \{(u, 0), (v, 0), (u, t), (v, t)\}$. Thus, η^w_β is a weakly $2-$absorbing ideal in $D \times L$, for all $\beta \in (0, 1]$ or $\eta^w_0 = D \times L$. Consequently, η^w emerges as an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal. However, η^w does not qualify as $\mathcal{L}-$fuzzy weakly prime ideal, as $\eta^w_{3/4}$ is not weakly prime ideal. This is demonstrated by considering, $(0, 0) \neq (u, r) \wedge (v, s) \in \eta^w_{3/4}$ implies that $(u, r) \notin \eta^w_{3/4}$ and $(v, s) \notin \eta^w_{3/4}$, for all $(u, r), (v, s) \in D \times L$.

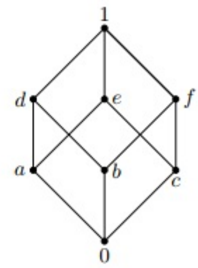
Theorem 4.7. *Assume η^w is an $\mathcal{L}-$fuzzy ideal in R. If η^w is an $\mathcal{L}-$fuzzy $2-$absorbing ideal in R, then η^w is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in R.*

The converse of the above result is not true; consider the following example.

Example 4.8. Let $D = \{0, u, v\}$ be a discrete ADL with 0 as its zero element defined in 2.2 and $L = \{0, a, b, c, d, e, f, 1\}$ be a lattice whose Hasse diagram is given below. Define $\mathcal{L}-$fuzzy ideal $\eta^w : R \to [0, 1]$ by

$$\eta^w(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0) \\ 1/2 & \text{if } x = 0 \textit{ and } y \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

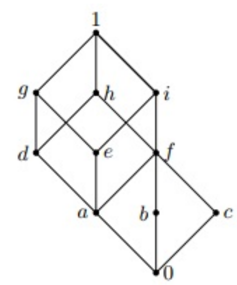
for all $(x, y) \in D \times L$. Clearly η^w is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal of $D \times L$ but not $\mathcal{L}-$fuzzy $2-$absorbing ideal, since $\eta^w_1 = \{(0, 0)\}$ is a weakly $2-$absorbing ideal but not $2-$absorbing ideal. This is demonstrated by considering, $(0, d) \wedge (u, e) \wedge (v, f) = (0, 0) \in \eta^w_1$ implies $(0, d) \wedge (u, e) = (0, a) \notin \eta^w_1$, $(0, d) \wedge (v, f) = (0, b) \notin \eta^w_1$ and $(u, e) \wedge (v, f) = (v, c) \notin \eta^w_1$.



Theorem 4.9. *Let η^w and ψ^w be $\mathcal{L}-$fuzzy weakly prime ideals in R. Then $\eta^w \cap \psi^w$ is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in R.*

The intersection of any two $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals may not be $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal; consider the following example.

Example 4.10. Let $R = \{0, a, b, c, d, e, f, g, h, i, 1\}$ be the lattice represented by the Hasse diagram given below:



Define an $\mathcal{L}-$fuzzy ideals η^w and ϕ^w from R to $[0, 1]$ as follows: $\eta^w(0) = 1, \eta^w(a) = 3/4, \eta^w(b) = 3/4, \eta^w(c) = 1, \eta^w(d) = 0, \eta^w(e) = 3/4$ and $\eta^w(f) = \eta^w(g) = \eta^w(h) = \eta^w(i) = \eta^w(1) = 0$; $\phi^w(0) = 1, \phi^w(a) = \phi^w(b) = \phi^w(c) = \phi^w(d) = 7/8$ and $\phi^w(e) = \phi^w(f) = \phi^w(g) = \phi^w(h) = \phi^w(i) = \phi^w(1) = 0$. Clearly η^w and ϕ^w are $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals in R but $\eta^w \cap \phi^w$ is not an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in R, since $(\eta^w \cap \phi^w)(g \wedge h \wedge i) = (\eta^w \cap \phi^w)(a) = 3/4 \nleq 0 = (\eta^w \cap \phi^w)(g \wedge h) = (\eta^w \cap \phi^w)(g \wedge i) = (\eta^w \cap \phi^w)(h \wedge i)$.

Let us recall that $\mathcal{L}-$fuzzy subsets η^w and ϕ^w of ADLs R and G respectively. Then the product of η^w and ϕ^w is denoted by $\eta^w \times \phi^w$ and defined by, $(\eta^w \times \phi^w)(a, b) = \eta^w(a) \wedge \phi^w(b)$, for all $(a, b) \in R \times G$.

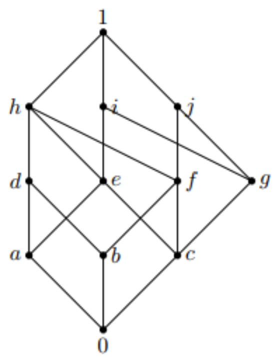
Theorem 4.11. *Let η^w and ϕ^w be $\mathcal{L}-$fuzzy ideals in R and G respectively. If $\eta^w \times \phi^w$ is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in $R \times G$, then η^w and ϕ^w are $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals in R and G, respectively.*

Proof. Suppose that $\eta^w \times \phi^w$ is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in $R \times G$. Consider,
$\eta^w(r \wedge s \wedge t) \wedge \phi^w(x \wedge y \wedge z) = (\eta^w \times \phi^w)(r \wedge s \wedge t, x \wedge y \wedge z)$
$= (\eta^w \times \phi^w)\big((r, x) \wedge (s, y) \wedge (t, z)\big)$
$\leq (\eta^w \times \phi^w)\big((r, x) \wedge (s, y)\big) \vee (\eta^w \times \phi^w)\big((r, x) \wedge (t, z)\big) \vee (\eta^w \times \phi^w)\big((s, y) \wedge (t, z)\big)$
$= \Big(\eta^w(r \wedge s) \wedge \phi^w(x \wedge y)\Big) \vee \Big(\eta^w(r \wedge t) \wedge \phi^w(x \wedge z)\Big) \vee \Big(\eta^w(s \wedge t) \wedge \phi^w(y \wedge z)\Big)$
$= \Big(\eta^w(r \wedge s) \wedge \phi^w(x \wedge y)\Big) \vee \Big(\big(\eta^w(r \wedge t) \vee \eta^w(s \wedge t)\big) \wedge \big(\eta^w(s \wedge t) \vee \phi^w(y \wedge z)\big) \wedge \big(\phi^w(x \wedge z) \vee \eta^w(s \wedge t)\big) \wedge \big(\eta^w(x \wedge z) \vee \phi^w(y \wedge z)\big)\Big)$
$= \Big(\eta^w(r\wedge s)\vee\eta^w(r\wedge t)\vee\eta^w(s\wedge t)\Big)\wedge\Big(\eta^w(r\wedge s)\vee\eta^w(r\wedge t)\vee\phi^w(y\wedge z)\Big)\wedge\Big(\eta^w(r\wedge s)\vee\phi^w(x\wedge z)\vee \eta^w(s\wedge t)\Big)\wedge\Big(\eta^w(r\wedge s)\vee\phi^w(x\wedge z)\vee\phi^w(y\wedge z)\Big)\wedge\Big(\phi^w(x\wedge y)\vee\eta^w(r\wedge t)\vee\eta^w(s\wedge t)\Big)\wedge\Big(\phi^w(x\wedge y)\vee$

$\eta^w(r\wedge t)\vee\phi^w(y\wedge z)\Big)\wedge\Big(\phi^w(x\wedge y)\vee\phi^w(x\wedge z)\vee\eta^w(s\wedge t)\Big)\wedge\Big(\phi^w(x\wedge y)\vee\phi^w(x\wedge z)\vee\phi^w(y\wedge z)\Big)$
$\leq \Big(\eta^w(r\wedge s)\vee\eta^w(r\wedge t)\vee\eta^w(s\wedge t)\Big)\wedge\Big(\phi^w(x\wedge y)\vee\phi^w(x\wedge z)\vee\phi^w(y\wedge z)\Big).$
Hence the result. □

If there are $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals, then their direct product may not $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal; consider the following example.

Example 4.12. Let $R=\{0,a,b,c,d,e,f,g,h,i,1\}$ be the lattice and η^w an $\mathcal{L}-$fuzzy ideal in R define in Example 4.10. Also let $G=\{0,a,b,c,d,e,f,g,h,i,j,1\}$ be a lattice whose Hasse diagram is given below:



Define $\phi^w: G\to[0,1]$ by $\phi^w(0)=1,\ \phi^w(b)=1,\ \phi^w(a)=\phi^w(c)=\phi^w(e)=7/8$ and $\phi^w(d)=\phi^w(f)=\phi^w(g)=\phi^w(h)=\phi^w(i)=\phi^w(j)=\phi^w(1)=0$. It is evidence that both η^w and ϕ^w emerges as $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals in R and G respectively. However, $\eta^w\times\phi^w$ is not an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in $R\times G$, since
$(\eta^w\times\phi^w)(g\wedge h\wedge i,h\wedge i\wedge j)=(\eta^w\times\phi^w)(a,c)=\eta^w(a)\wedge\phi^w(c)=3/4$
$\nleq 0=(\eta^w\times\phi^w)(g\wedge h,h\wedge i)\vee(\eta^w\times\phi^w)(g\wedge i,h\wedge j)\vee(\eta^w\times\phi^w)(h\wedge i,i\wedge j).$

The following is a consequences of the above results 4.3,4.5,4.7 and 4.11. First let us recall from [18] that if $\chi_{\{0\}}$ is the characteristics map of $\{0\}$ in an ADL R (i.e., $\chi_{\{0\}}(x)=1$ or 0 depending the values of x), then $\chi_{\{0\}}$ is the smallest $\mathcal{L}-$fuzzy ideal in R.

Theorem 4.13. *Let $\eta^w(\neq\chi_{\{0\}})$ and $\phi^w(\neq\chi_{\{0\}})$ be proper $\mathcal{L}-$fuzzy ideals in R and G respectively. Then the following are equivalent to each other.*
(1). $\eta^w\times\phi^w$ is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in $R\times G$
(2). $\phi^w_\beta=G$, for all $\beta\in L$ and η^w is an $\mathcal{L}-$fuzzy $2-$absorbing ideal, or η^w is an $\mathcal{L}-$fuzzy prime ideal in R and ϕ^w is an $\mathcal{L}-$fuzzy prime ideal in G
(3). $\eta^w\times\phi^w$ is an $\mathcal{L}-$fuzzy $2-$absorbing ideal in $R\times G$.

Corollary 4.14. *Let η^w and ϕ^w be $\mathcal{L}-$fuzzy ideals in R and G with 0 respectively. Then the following are equivalent.*
(1). $\eta^w\times\chi_G$ is an $\mathcal{L}-$fuzzy $2-$absorbing ideal in $R\times G$
(2). $\eta^w\times\chi_G$ is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in $R\times G$
(3). $\eta^w\times\phi^w$ is an $\mathcal{L}-$fuzzy $2-$absorbing ideal in $R\times G$
(4). $\eta^w\times\phi^w$ is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in $R\times G$.

Finally, we discuss the homomorphism of $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals in ADLs.

Theorem 4.15. *Let R and G be ADLs, and $g: R\to G$ a lattice homomorphism. Then the following holds:*
(1). If ψ^w is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in G, then $g^{-1}(\psi^w)$ is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in R
(2). If g is an epimorphism and η^w is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in R, then $g(\eta^w)$ is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal of G.

Proof. (1). Let $g : R \to G$ be a lattice homomorphism. Assume that ψ^w is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in G. For all elements $r, s, t \in G$ such that $r \wedge s \wedge t \neq 0$. Then
$g^{-1}(\psi^w)(r \wedge s \wedge t) = \phi^w\big(g(r \wedge s \wedge t)\big)$
$= \phi^w\big(g(r) \wedge g(s) \wedge g(t)\big)$
$\leq \phi^w\big(g(r) \wedge g(s)\big) \vee \phi^w\big(g(r) \wedge g(t)\big) \vee \phi^w\big(g(s) \wedge g(t)\big)$
$= \phi^w\big(g(r \wedge s)\big) \vee \phi^w\big(g(r \wedge t)\big) \vee \phi^w\big(g(s \wedge t)\big)$
$= g^{-1}(\psi^w)(r \wedge s) \vee g^{-1}(\psi^w)(r \wedge t) \vee g^{-1}(\psi^w)(s \wedge t)$.
Thus $g^{-1}(\psi^w)$ is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in R.
(2). Let g be an isomorphism. Assume that η^w is an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in R. For all $a, b, c \in R$ such that $a \wedge b \wedge c \neq 0$. Now, consider,
$g(\eta^w)(a \wedge b) \vee g(\eta^w)(a \wedge c) \vee g(\eta^w)(b \wedge c) = \Big(Sup\{\eta^w(r \wedge s) : r \wedge s \in g^{-1}(a \wedge b)\}\Big) \vee \Big(Sup\{\eta^w(r \wedge t) : r \wedge t \in g^{-1}(a \wedge c)\}\Big) \vee \Big(Sup\{\eta^w(s \wedge t) : s \wedge t \in g^{-1}(b \wedge c)\}\Big)$
$\geq \big\{\eta^w(r \wedge s \wedge t) : r \wedge s \wedge t \in g^{-1}(a \wedge b \wedge c)\big\}$
$= g(\eta^w)(a \wedge b \wedge c)$.
Thus, $g(\eta^w)$ emerges as an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideal in G. □

5 Conclusion

In this paper, we study an $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals in an almost distributive lattice (ADL) which is weaker than that of an $\mathcal{L}-$fuzzy weakly prime ideals and discussed properties of these. Furthermore, the relationship between $\mathcal{L}-$fuzzy weakly prime ideal and $\mathcal{L}-$fuzzy weakly $2-$absorbing ideals in ADLs are introduced and there are examples that shown that the converse of these is not true.

Author contribution statement: *I hereby declare that I am the sole author of this work and that I have not used any sources other than those listed in the references. I further declare that I didn't submit this manuscript to any other journal.*

Data Availability: *No data were used to support this study.*

Conflicts of Interest: *The author(s) declare(s) that there are no conflicts of interest regarding the publication of this paper.*

Funding Statement: *no grants or support received.*

Acknowledgment: *The author wishes to express their sincere appreciation to the reviewers for their insightful comments and suggestions, which have greatly enhanced the paper's presentation.*

References

[1] D.D. Anderson and E. Smith, Weakly Prime Ideals, Houston Journal of Math, 29(4), (2003) 831–840. https://iro.uiowa.edu/esploro/outputs/9984230419402771.

[2] A. Badawi and A.Y. Darani, "On weakly 2-absorbing ideals of commutative rings," Houston Journal of Mathematics, Vol.39, No.2, (2013) pp. 441-452. https://www.math.uh.edu/ hjm/Vol39-2.html.

[3] A. Badawi, "On 2-absorbing ideals of commutative rings," Bull. Austral. Math.Soc., Vol. 75, (2007) pp. 417-429. DOI: https://doi.org/10.1017/S0004972700039344.

[4] C. beddani and W. Messirdi, "2-prime ideals and their applications," Journal of Algebra and Its Application, Vol. 15, No. 3, (2016) (11 pages). DOI: 10.1142/S0219498816500511.

[5] J.N. Chuadhari, "2-absorbing ideals in semirings," International Journal of Algebra, Vol. 6, No. 6, (2012) pp. 265-270. https://www.m-hikari.com/ija/ija-2012/ija-5-8-2012/chaudhariija5-8-2012-1.

[6] A.Y. Darani and G.Ghasemi, On L-fuzzy 2-absorbing ideal, Italian Journal of Pure and Applied Mathematics, NO36, (2016) pp. 147–154.

[7] M.K. Dubey, "prime and weakly prime ideals," Quasi-groups and Related Systems, Vol. 20, (2012) pp. 197-202. http://www.quasigroups.eu/contents/20.php.

[8] *J. A. Goguen, "L-fuzzy sets," Journal of Mathematical Analysis and Applications, vol. 18, no. 1, (1967) pp. 145–174. DOI: 10.1016/0022-247X(67)90189-8.*

[9] *Y.B. Jun, On fuzzy prime ideals of* Γ*-rings, Soochow J. Math.21, (1995) pp. 41–48.*

[10] *D. Mandal, On 2-absorbing fuzzy ideals of Commutative semi rings, JWMS J.App and Eng. Math, Vol. 11, No. 2 (2021) 368–373. https://jaem.isikun.edu.tr/web/index.php/archive/719-jaem-vol-11-no-2-2021.*

[11] *Natnael Teshale A., Weakly 2-absorbing Ideals in Almost Distributive Lattices, Journal of Mathematics (Hindawi), Volume (2022), Article ID 9252860, 1-9. DOI: 10.1155/2022/9252860.*

[12] *Natnael Teshale A., L-fuzzy 2-absorbing Ideals in Almost Distributive Lattices, (Communicated Paper).*

[13] *S.K. Nimbhorkar and Y.S. Patil, Fuzzy Weakly 2-absorbing ideals of a lattice, Discussions Mathematical, (42), (2022) pp. 255–277. https://doi.org/10.7151/dmgaa.1389.*

[14] *S.K. Nimbhorkar and Y.S. Patil, Generalizations of prime fuzzy ideals of a lattices, Journal of Hyperstructures, Vol. 9,No. 2 (2020) 1–33. DOI: 10.22098/JHS.2020.2629.*

[15] *Ch. Santhi Sundar Raj, A. Natnael Teshale and U.M. Swamy,* Fuzzy Prime Ideals of ADLs, *International Journal of Computing Science and Applied Mathematics, Vol. 4, No. 2, (2018) pp. 32–36. DOI: 10.12962/j24775401.v4i2.3187.*

[16] *M.H. Stone, Topological representation of distributive lattices and Brouwerian logic, Casopis. Mat. Fyz., 67, (1937), 1 - 25. http://eudml.org/doc/27235.*

[17] *T.R. Sharma and R. Sharma, Completely Prime Fuzzy Ideals of* Γ*-Semi-rings, International Journal of Pure and Applied Mathematical Sciences, Vol(16), NO1, (2023) pp. 31–41. https://dx.doi.org/10.37622/IJPAMS/16.1.2023.31-41.*

[18] *U.M. Swamy, Ch. Santhi Sundar Raj and Natnael Teshale A.,* Fuzzy Ideals of Almost Distributive Lattices, *Annals of Fuzzy Mathematics and Informatics, Vol. 14, No. 4, (2017) pp. 371-379. http://www.afmi.or.kr(AFMI-14-4-H-170201R1).*

[19] *U.M. Swamy and G.C. Rao, "Almost Distributive Lattices," J. Australian Math. Soc., (Series A), Vol. 31, (1981) pp. 77–91. DOI: 10.1017/S1446788700018498.*

[20] *M.P. Wasadikar and K.T. Gaikevad, "On 2-absorbing ideals and weakly 2-absorbing ideals of lattice," Mathematical Sciences International Research Journal, Vol. 4, No. 2, (2015) pp. 82-85.*

[21] *L.A. Zadeh, Fuzzy set, Inform and control, 8, (1965) pp. 338–353.DOI: 10.1016/S0019-9958(65)90241-X.*

Author information

Natnael Teshale Amare, Department of Mathematics, College of NCS, University of Gondar, Ethiopia.
E-mail: yenatnaelteshale@gmail.com

N. Rafi, Department of Mathematics, Bapatla Engineering College, Bapatla, Andhra Pradesh, India.
E-mail: rafimaths@gmail.com

S. Nageswara Rao, Department of Mathematics, Mallareddy College of Engineering, Hyderabad, India.
E-mail: snraomaths@gmail.com

Received: 2024-03-25
Accepted: 2025-07-13