

INEQUALITIES FOR HIGHER ORDER POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract Let $P(z) = \sum_{\nu=0}^n a_{\nu}z^{\nu}$ be a polynomial of degree n whose all zeros lie in the region $|z| \leq k$, where $k \geq 1$ and $D_{\alpha}P(z)$ represents its polar derivative with respect to the complex number α . For $|\alpha| \geq k$, the following inequality holds for the first-order polar derivative of $P(z)$

$$\max_{|z|=1} |D_{\alpha}P(z)| \geq \frac{(|\alpha| - k)}{1 + k^n} \left\{ n + 1 - \frac{1}{k^{\frac{n}{2}}} \sqrt{\left| \frac{a_0}{a_n} \right|} \right\} \max_{|z|=1} |P(z)|.$$

In this paper, we shall extend the above inequality to the s th-order polar derivative of $P(z)$ without perturbing the condition on zeros of the underlying polynomial. The obtained results besides generalizing the above inequality also unveil a series of interesting inequalities as special cases.

1 Introduction and preliminaries

Let \mathcal{P}_n denote the space of all complex polynomials $P(z) = \sum_{\nu=0}^n a_{\nu}z^{\nu}$ of degree at most n . For any positive real number ρ , let $\mathbb{T}_{\rho}^{-} = \{z \in \mathbb{C} : |z| < \rho\}$, $\mathbb{T}_{\rho}^{+} = \{z \in \mathbb{C} : |z| > \rho\}$, $\partial\mathbb{T}_{\rho} = \{z \in \mathbb{C} : |z| = \rho\}$ and for any analytic function $f : \mathbb{C} \rightarrow \mathbb{C}$, we denote the supnorm of f on $\partial\mathbb{T}_1$ by

$$\max_{z \in \partial\mathbb{T}_1} |f(z)| = \|f(z)\|.$$

For any polynomial $P \in \mathcal{P}_n$, the fundamental inequality relating $\|P(z)\|$ and the norm of its derivative $\|P'(z)\|$, was first established by S. Bernstein [3], who proved that if $P \in \mathcal{P}_n$ has degree n , then

$$\|P'(z)\| \leq n\|P(z)\|. \tag{1.1}$$

Equality in (1.1) holds for the polynomial $P(z) = bz^n$, $0 \neq b \in \mathbb{C}$. However, for $P \in \mathcal{P}_n$ having degree n with $P(z) \neq 0$ for $z \in \mathbb{T}_1^{-}$, Erdős conjectured and later Lax [11] proved that

$$\|P'(z)\| \leq \frac{n}{2}\|P(z)\|. \tag{1.2}$$

Inequality (1.2) is also sharp with equality holding if $P(z) = az^n + b$ with $|a| = |b| \neq 0$. In [12], Malik extended inequality (1.2) and proved that if $P \in \mathcal{P}_n$ is of degree n and $P(z) \neq 0$ for $z \in \mathbb{T}_k^{-}$, where $k \geq 1$, then the following inequality is true

$$\|P'(z)\| \leq \frac{n}{1+k}\|P(z)\|, \tag{1.3}$$

with equality if $P(z) = (z + k)^n$. The above inequality was further extended by Govil and Rahman [10], who under the same hypothesis proved for $1 \leq s < n$, that

$$\|P^{(s)}(z)\| \leq \frac{n(n-1) \cdots (n-s+1)}{1+k^s} \|P(z)\|. \tag{1.4}$$

Inequality (1.4) was refined by Govil [9] who established the following result.

Theorem 1.1. *If $P \in \mathcal{P}_n$ is of degree n and $P(z) \neq 0$ for $z \in \mathbb{T}_k^-$, where $k \geq 1$, then for $1 \leq s < n$, we have*

$$\|P^{(s)}(z)\| \leq \frac{n(n-1) \cdots (n-s+1)}{1+k^s} \left(\|P(z)\| - \min_{z \in \partial \mathbb{T}_k} |P(z)| \right).$$

On the other hand, if we consider the class of polynomials having all their zeros in $\mathbb{T}_1^- \cup \partial \mathbb{T}_1$, then the following relation between $\|P'(z)\|$ and $\|P(z)\|$ is due to Turán [20].

Theorem 1.2. *If $P \in \mathcal{P}_n$ has degree n whose all zeros lie in $\mathbb{T}_1^- \cup \partial \mathbb{T}_1$, then*

$$\|P'(z)\| \geq \frac{n}{2} \|P(z)\|, \tag{1.5}$$

with equality if all the zeros of $P(z)$ lie on $\partial \mathbb{T}_1$.

Govil [8] was first to extend inequality (1.5) for the class of polynomials having all zeros in $\mathbb{T}_k^- \cup \partial \mathbb{T}_k$, $k \geq 1$ in the form of following result.

Theorem 1.3. *If all the zeros of $P \in \mathcal{P}_n$ of degree n lie in $\mathbb{T}_k^- \cup \partial \mathbb{T}_k$, $k \geq 1$, then*

$$\|P'(z)\| \geq \frac{n}{1+k^n} \|P(z)\|. \tag{1.6}$$

Inequality (1.6) is sharp and the extremal polynomial is $P(z) = z^n + k^n$.

An inequality for s th-order derivative and analogous to (1.4) for the polynomial $P \in \mathcal{P}_n$ of degree n having all their zeros in $\mathbb{T}_k^- \cup \partial \mathbb{T}_k$, $k \geq 1$ can be easily obtained by repeated applications of Theorem 1.3. In fact, if the polynomial $P(z)$ has all its zeros in $\mathbb{T}_k^- \cup \partial \mathbb{T}_k$, $k \geq 1$, then by Gauss Lucas theorem all the derivatives $P'(z), P''(z), \dots, P^{(s)}(z)$ of degree $n-1, n-2, \dots, n-s$ respectively have their zeros in $\mathbb{T}_k^- \cup \partial \mathbb{T}_k$, $k \geq 1$. Therefore, applying inequality (1.6) to $P'(z), P''(z), \dots, P^{(s)}(z)$, we obtain

$$\|P^{(s)}(z)\| \geq \frac{n(n-1) \cdots (n-s+1)}{(1+k^n)(1+k^{n-1}) \cdots (1+k^{n-s+1})} \|P(z)\|. \tag{1.7}$$

Further note that for $k \geq 1$, we have

$$k^j \geq 1 \implies 1+k^n \geq 1+k^{n-j}, \quad j = 1, 2, \dots, s-1.$$

Using this in (1.7), we get for $1 \leq s < n$

$$\|P^{(s)}(z)\| \geq \frac{n(n-1) \cdots (n-s+1)}{(1+k^n)^s} \|P(z)\|. \tag{1.8}$$

Establishing relationships between the supremum norm of a complex polynomial and that of its derivative is a very engaging topic within the realm of polynomial theory. Consequently, such inequalities have been extended to other types of derivatives with respect to norms different from the supremum norm. For a deeper understanding of this domain, one may refer to ([1], [2], [5], [14], [17], [18], [21]).

Definition 1.4. (Polar derivative of a polynomial) If $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$ is a polynomial of degree n , then for any complex number α , the analytic function $nP(z) + (\alpha - z)P'(z)$ is called

the polar derivative of $P(z)$ and is usually denoted by $D_\alpha P(z)$. It may be noted that the polar derivative

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z) \tag{1.9}$$

generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \rightarrow \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z),$$

uniformly for $|z| \leq R, R > 0$. Further the derivative $D_\alpha P(z)$ is of degree atmost $n - 1$ and has degree exactly $n - 1$ if $\alpha \neq \frac{-a_{n-1}}{na_n}$.

Different results related to the ordinary derivative of a polynomial have been extended to its polar derivative. In this direction, many sharp results concerning the maximum modulus of $D_\alpha P(z)$ on $\partial\mathbb{T}_1$ for the class of polynomials $P \in \mathcal{P}_n$ having zeros in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k$ for different values of $k > 0$ can be seen in ([2], [4], [7], [19]). Based on the focus of our study, we mention the following result (see [2]).

Theorem 1.5. *If all the zeros of $P \in \mathcal{P}_n$ of degree n lie in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k, k \geq 1$, then for any $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,*

$$\|D_\alpha P(z)\| \geq \frac{n(|\alpha| - k)}{1 + k^n} \|P(z)\|. \tag{1.10}$$

Recently, Singh and Chanam [19] have refined inequality (1.10) by introducing the extreme coefficients of $P(z)$ and established the following result.

Theorem 1.6. *If $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k, k \geq 1$, then for any complex number α with $|\alpha| \geq k$*

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{(|\alpha| - k)}{1 + k^n} \left\{ n + 1 - \frac{1}{k^{\frac{n}{2}}} \sqrt{\left| \frac{a_0}{a_n} \right|} \right\} \max_{|z|=1} |P(z)|. \tag{1.11}$$

Definition 1.7. (Higher order polar derivative) Given any n th-degree polynomial $P(z)$ and s complex numbers $\alpha_1, \alpha_2, \dots, \alpha_s$ not necessarily distinct, we can construct a sequence of higher order polar derivatives as

$$\begin{aligned} D_{\alpha_1} P(z) &= nP(z) + (\alpha_1 - z)P'(z), \\ D_{\alpha_2} D_{\alpha_1} P(z) &= (n - 1)D_{\alpha_1} P(z) + (\alpha_2 - z)(D_{\alpha_1} P(z))', \\ &\dots \dots \dots \end{aligned}$$

$$\begin{aligned} D_{\alpha_s} D_{\alpha_{s-1}} \dots D_{\alpha_1} P(z) &= (n - s + 1)(D_{\alpha_{s-1}} \dots D_{\alpha_1} P(z)) + \\ &(\alpha_s - z)(D_{\alpha_{s-1}} \dots D_{\alpha_1} P(z))' , \end{aligned}$$

where $1 \leq s \leq n$ and $D_{\alpha_0} P(z) = P(z)$. It may be noted that the s th-order polar derivative of $P(z)$ is a polynomial of degree at most $n - s$.

Following the above definition of the higher-order polar derivative, different researchers have extended the results valid for first-order polar derivative of a polynomial to its higher-order polar derivatives (see [1], [13]). For instance, in [13] Malik and Zargar have extended inequality (1.10) to the s th-order polar derivative of a polynomial in the form of the following result.

Theorem 1.8. *If $P \in \mathcal{P}_n$ is of degree n and has all zeros in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k, k \geq 1$, then for any real or complex numbers $\alpha_1, \alpha_2, \dots, \alpha_s$ with $|\alpha_i| \geq k, i = 1, 2, \dots, s$, we have*

$$\|D_{\alpha_s} D_{\alpha_{s-1}} \dots D_{\alpha_1} P(z)\| \geq \frac{n(n - 1)(n - 2) \dots (n - s + 1)}{(1 + k^n)(1 + k^{n-1}) \dots (1 + k^{n-s+1})} \mathbb{N}_{\alpha_s} \|P(z)\|, \tag{1.12}$$

where $\mathbb{N}_{\alpha_s} = (|\alpha_1| - k)(|\alpha_2| - k) \dots (|\alpha_s| - k)$.

2 Main results

In this paper, we shall present the following refinement of Theorem 1.8 which simultaneously generalizes Theorem 1.6 to the higher-order polar derivative.

Theorem 2.1. *If all the zeros of $P(z) = \sum_{\nu=0}^n a_{\nu}z^{\nu}$ lie in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k$, $k \geq 1$, then for any complex numbers $\alpha_1, \alpha_2, \dots, \alpha_s$, with $|\alpha_i| \geq k$, $i = 1, 2, \dots, s$, $1 \leq s < n$, we have*

$$\|D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_2} D_{\alpha_1} P(z)\| \geq \frac{(|\alpha_1| - k)(|\alpha_2| - k) \cdots (|\alpha_s| - k)}{(1 + k^n)(1 + k^{n-1}) \cdots (1 + k^{n-s+1})} \prod_{\mu=0}^{s-1} \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-\mu}^{\mu}}{\mathbb{A}_{n-\mu}^{\mu}} \right|} \right) \|P(z)\|, \tag{2.1}$$

where for $0 \leq \mu \leq s$

$$\mathbb{A}_{n-\mu}^{\mu} = \sum_{j=0}^{\mu} \frac{\mu! \binom{n}{\mu}}{\binom{n}{j}} \Delta_{\mu-j}^{\mu}(\alpha_1, \alpha_2, \dots, \alpha_{\mu}) a_{n-j},$$

$$\mathbb{B}_{n-\mu}^{\mu} = \sum_{j=0}^{\mu} \frac{\mu! \binom{n}{\mu}}{\binom{n}{j}} \Delta_j^{\mu}(\alpha_1, \alpha_2, \dots, \alpha_{\mu}) a_j,$$

and $\Delta_j^{\mu}(\alpha_1, \alpha_2, \dots, \alpha_{\mu})$ are the symmetric functions consisting of sum of products of $\alpha_1, \alpha_2, \dots, \alpha_{\mu}$ taken j at a time ($0 \leq j \leq \mu$) with $\Delta_0^{\mu}(\alpha_1, \alpha_2, \dots, \alpha_{\mu}) = 1$. Here the terms $\mathbb{A}_{n-\mu}^{\mu}$ and $\mathbb{B}_{n-\mu}^{\mu}$ represent the leading term and the constant term of μ^{th} order polar derivative of $P(z)$ respectively with $\mathbb{A}_{n-\mu}^{\mu} \neq 0$.

Remark 2.2. Since all the zeros of $P(z)$ lie in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k$, $k \geq 1$, we have by repeated applications of Laguerre’s theorem that all the zeros of $D_{\alpha_{\mu}} D_{\alpha_{\mu-1}} \cdots D_{\alpha_2} D_{\alpha_1} P(z)$ lie in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k$, $k \geq 1$. Therefore if $\omega_1, \omega_2, \dots, \omega_{n-\mu}$, $1 \leq \mu < n$ are the zeros of $D_{\alpha_{\mu}} D_{\alpha_{\mu-1}} \cdots D_{\alpha_2} D_{\alpha_1} P(z)$, then

$$\omega_1 \omega_2 \cdots \omega_{n-\mu} = (-1)^{n-\mu} \frac{\mathbb{B}_{n-\mu}^{\mu}}{\mathbb{A}_{n-\mu}^{\mu}},$$

which implies

$$\frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-\mu}^{\mu}}{\mathbb{A}_{n-\mu}^{\mu}} \right|} \leq 1. \tag{2.2}$$

Using inequality (2.2) in (2.1), we see that Theorem 2.1 is a refinement of Theorem 1.8.

Remark 2.3. Theorem 2.1 is a generalization of inequality (1.11) due to Singh and Chanam [19] which can be seen by taking $s = 1$ in (2.1).

Taking $s = 2$ in Theorem 1.6, we get the following result.

Corollary 2.4. *If all the zeros of $P(z) = \sum_{\nu=0}^n a_{\nu}z^{\nu}$ lie in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k$, $k \geq 1$, then for any complex numbers α_1, α_2 , with $|\alpha_1| \geq k$ and $|\alpha_2| \geq k$, we have*

$$\|D_{\alpha_2} D_{\alpha_1} P(z)\| \geq \frac{(|\alpha_1| - k)(|\alpha_2| - k)}{(1 + k^n)(1 + k^{n-1})} \left(n + 1 - \frac{1}{k^{\frac{n}{2}}} \sqrt{\left| \frac{a_0}{a_n} \right|} \right) \left(n - \frac{1}{k^{\frac{n-1}{2}}} \sqrt{\left| \frac{a_1 \alpha_1 + n a_0}{n \alpha_1 a_n + a_{n-1}} \right|} \right) \|P(z)\|.$$

If we divide $|\prod_{i=1}^s \alpha_i|$ to both sides of inequality (2.1) and let $|\alpha_i| \rightarrow \infty, \forall i = 1, 2, \dots, s$, we obtain the following result for s^{th} order derivative.

Corollary 2.5. *If all the zeros of $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$ lie in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k$, $k \geq 1$, then for $1 \leq s < n$,*

$$\|P^{(s)}(z)\| \geq \frac{1}{(1+k^n)(1+k^{n-1}) \cdots (1+k^{n-s+1})} \prod_{\mu=0}^{s-1} \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{a_\mu}{\binom{n}{\mu} a_n} \right|} \right) \|P(z)\|. \tag{2.3}$$

In particular taking $s = k = 1$, inequality (2.3) reduces to the following inequality due to Dubinin (see [6] Theorem 4)

$$\|P'\| \geq \frac{1}{2} \left(n + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}} \right) \|P\|. \tag{2.4}$$

The above inequality is best possible and equality holds for the polynomial $P(z) = 1 + z^n$.

Instead of proving Theorem 2.1, we prove the following more general result which contains Theorem 2.1 as a special case and provides refinement of inequality (1.8).

Theorem 2.6. *If $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$ has all its zeros in the region $\mathbb{T}_k^- \cup \partial\mathbb{T}_k$, $k \geq 1$, then for any complex numbers $\alpha_1, \alpha_2, \dots, \alpha_s$ with $|\alpha_i| \geq k, \forall i = 1, 2, \dots, s, 1 \leq s < n$ and $|\lambda| < 1$, we have*

$$\begin{aligned} \|D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_2} D_{\alpha_1} P(z)\| \geq & \left\{ \mathbb{M}_{\alpha_s}^k \prod_{\mu=0}^{s-1} \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-\mu}^\mu + \mu! \binom{n}{\mu} \lambda m}{\mathbb{A}_{n-\mu}^\mu} \right|} \right) \right\} \|P(z)\| + \\ & \left\{ \mathbb{M}_{\alpha_s}^k \prod_{\mu=0}^{s-1} \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-\mu}^\mu + \mu! \binom{n}{\mu} \lambda m}{\mathbb{A}_{n-\mu}^\mu} \right|} \right) - \prod_{\mu=0}^{s-1} (n - \mu) \right\} |\lambda| \min_{z \in \partial\mathbb{T}_k} |P(z)|, \end{aligned} \tag{2.5}$$

where

$$\mathbb{M}_{\alpha_s}^k = \frac{(|\alpha_1| - k)(|\alpha_2| - k) \cdots (|\alpha_s| - k)}{(1+k^n)(1+k^{n-1}) \cdots (1+k^{n-s+1})},$$

and $\mathbb{A}_{n-\mu}^\mu$ and $\mathbb{B}_{n-\mu}^\mu$ are defined as in Theorem 2.1.

Remark 2.7. Theorem 2.1 follows by choosing $\lambda = 0$ in Theorem 2.6.

If we divide both sides of inequality 2.5 by $|\prod_{i=1}^s \alpha_i|$ and let $|\alpha_i| \rightarrow \infty, \forall i = 1, 2, \dots, s$, we get the following refinement of Corollary 2.5.

Corollary 2.8. *If all the zeros of $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$ lie in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k$, $k \geq 1$, then*

$$\|P^{(s)}(z)\| \geq \frac{1}{K_s} \prod_{\mu=0}^{s-1} \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{a_\mu}{\binom{n}{\mu} a_n} \right|} \right) \left\{ \|P(z)\| + |\lambda| \min_{z \in \partial\mathbb{T}_k} |p(z)| \right\}, \tag{2.6}$$

where $K_s = (1+k^n)(1+k^{n-1}) \cdots (1+k^{n-s+1})$.

In particular, taking $s = k = 1$ and making $|\lambda| \rightarrow 1$ in inequality (2.6), we obtain

$$\|P'\| \geq \frac{1}{2} \left(n + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}} \right) \left\{ \|P\| + \min_{z \in \partial\mathbb{T}_1} |P| \right\}. \tag{2.7}$$

Inequality (2.7) is sharp and equality holds for the polynomial $P(z) = 1 + z^n$. Moreover inequality (2.7) is clearly the refinement of Dubinin’s inequality (2.4).

Using inequality (2.2) in corollary 2.8 and letting $|\lambda| \rightarrow 1$, we obtain the following refinement of inequality (1.8).

Corollary 2.9. *If $P \in \mathcal{P}_n$ is of degree n with all the zeros in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k$, $k \geq 1$, then for $1 \leq s < n$*

$$\|P^{(s)}(z)\| \geq \frac{n(n-1)\cdots(n-s+1)}{(1+k^n)^s} \left(\|P(z)\| + \min_{z \in \partial\mathbb{T}_k} |P(z)| \right).$$

Computation analysis: Taking $s = 2$ and $P(z) = z^2 + 3z - 1$. Then all the zeros of $P(z)$ lie in $|z| \leq k$ with $k = 3.3028$ and $\|P(z)\| = \sqrt{13}$. Choose $\alpha_1 = 5$ and $\alpha_2 = 6$, then based on the known result (1.12), the lower bound of $\|D_6D_5P(z)\|$ is approximately 0.6442. However, using our refined result (2.1), the lower bound of $\|D_6D_5P(z)\|$ for the same polynomial comes out to be 1.1909 approximately. This example demonstrates a significant improvement in the accuracy of the lower bound achieved by our result.

3 Auxiliary results

For the proofs of main results, we need the following Lemmas. The first lemma can be verified by the repeated application of Laguerre’s theorem [14].

Lemma 3.1. *If $P(z)$ is a polynomial of degree n whose all the zeros lie in a circular region \mathcal{C} and if none of the points $\alpha_1, \alpha_2, \dots, \alpha_s$ lie in the region \mathcal{C} , then each of the polar derivatives*

$$D_{\alpha_s}D_{\alpha_{s-1}} \cdots D_{\alpha_1}P(z), \quad s = 1, 2, \dots, n - 1$$

has all of its zeros in \mathcal{C} .

Lemma 3.2. *If $P(z) = \sum_{\nu=0}^n a_\nu z^\nu$ is a polynomial of degree n and $\alpha_1, \alpha_2, \dots, \alpha_s$ are any real or complex numbers. Further, if \mathbb{A}_{n-s}^s and \mathbb{B}_{n-s}^s represent the leading coefficient and constant term of s th-order polar derivative of $P(z)$ of degree $n - s$ respectively, then we can write*

$$\mathbb{A}_{n-s}^s = \sum_{j=0}^s \frac{s! \binom{n}{s}}{\binom{n}{j}} \Delta_{s-j}^s(\alpha_1, \alpha_2, \dots, \alpha_s) a_{n-j}, \tag{3.1}$$

and

$$\mathbb{B}_{n-s}^s = \sum_{j=0}^s \frac{s! \binom{n}{s}}{\binom{n}{j}} \Delta_j^s(\alpha_1, \alpha_2, \dots, \alpha_s) a_j, \tag{3.2}$$

where $\Delta_j^s(\alpha_1, \alpha_2, \dots, \alpha_s)$ are symmetric functions consisting of sum of products of $\alpha_1, \alpha_2, \dots, \alpha_s$ taken j at a time ($0 \leq j \leq s$), $\Delta_0^s(\alpha_1, \alpha_2, \dots, \alpha_s) = 1$ and $\binom{n}{r}$, $0 \leq r \leq n$ represents the standard binomial coefficients.

The above Lemma follows by the simple computation of sequence of polar derivatives of $P(z)$.

4 Proof of the main result

Proof of Theorem 2.6. To prove the theorem, we shall use the principle of mathematical induction, and we induct on the order of polar derivative. First we assume $P(z)$ has all zeros in \mathbb{T}_k^- , $k \geq 1$, so that $0 < m = \min_{z \in \partial\mathbb{T}_k} |P(z)|$ and $|P(z)| \geq m$ on $\partial\mathbb{T}_k$. Thus for every complex number λ with $|\lambda| < 1$, we have $|\lambda|m < |P(z)|$ on $\partial\mathbb{T}_k$. Therefore, by Rouché’s theorem, it follows that the polynomial $F(z) = P(z) + \lambda m$ has all zeros in \mathbb{T}_k^- , $k \geq 1$. Thus applying Theorem 1.6 with $\alpha = \alpha_1$ to the n th degree polynomial $F(z)$, and noting that $|\alpha_1| \geq k$, we get

$$\|D_{\alpha_1}F(z)\| \geq \frac{(|\alpha_1| - k)}{1 + k^n} \left\{ n + 1 - \frac{1}{k^{\frac{n}{2}}} \sqrt{\frac{|a_0 + \lambda m|}{|a_n|}} \right\} \|F(z)\|, \tag{4.1}$$

that is

$$\|D_{\alpha_1}P(z) + n\lambda m\| \geq \frac{(|\alpha_1| - k)}{1 + k^n} \left\{ n + 1 - \frac{1}{k^{\frac{n}{2}}} \sqrt{\frac{|a_0 + \lambda m|}{|a_n|}} \right\} \|P(z) + \lambda m\|. \tag{4.2}$$

Now let z_0 be a point on $\partial\mathbb{T}_1$ such that $|P(z_0)| = \max_{z \in \partial\mathbb{T}_1} |P(z)|$, then we have

$$\begin{aligned} \|P(z) + \lambda m\| &\geq |P(z_0) + \lambda m| \\ &= \left| |P(z_0)|e^{i\theta_0} + |\lambda|e^{i\beta}m \right| \\ &= \left| |P(z_0)| + |\lambda|e^{i(\beta-\theta_0)}m \right| \end{aligned}$$

Choosing λ such that $\arg(\lambda) = \theta_0$ that is $\beta = \theta_0$ and noting that $m > 0$, it follows that $|P(z_0) + \lambda m| = |P(z_0)| + |\lambda|m$. Using this and triangle inequality in (4.2), we get

$$\|D_{\alpha_1}P(z)\| + n|\lambda|m \geq \frac{(|\alpha_1| - k)}{1 + k^n} \left\{ n + 1 - \frac{1}{k^{\frac{n}{2}}} \sqrt{\frac{|a_0 + \lambda m|}{|a_n|}} \right\} (\|P(z)\| + |\lambda|m),$$

or equivalently

$$\begin{aligned} \|D_{\alpha_1}P(z)\| &\geq \left\{ \frac{(|\alpha_1| - k)}{1 + k^n} \left(n + 1 - \frac{1}{k^{\frac{n}{2}}} \sqrt{\frac{|a_0 + \lambda m|}{|a_n|}} \right) \right\} \|P(z)\| \\ &\quad + \left\{ \frac{(|\alpha_1| - k)}{1 + k^n} \left(n + 1 - \frac{1}{k^{\frac{n}{2}}} \sqrt{\frac{|a_0 + \lambda m|}{|a_n|}} \right) - n \right\} |\lambda| \min_{z \in \partial\mathbb{T}_k} |P(z)|. \end{aligned}$$

This shows that the result holds for $s = 1$. Now if $\alpha_1 \neq \frac{-a_{n-1}}{na_n}$, then $D_{\alpha_1}F(z)$ is a polynomial of degree $n - 1$. Since $|\alpha_1| \geq k$, therefore by Lemma 3.1 all the zeros of $D_{\alpha_1}P(z)$ lie in \mathbb{T}_k^- , $k \geq 1$. Applying Theorem 1.6 to the $(n - 1)$ th degree polynomial $D_{\alpha_1}F(z)$ with $\alpha = \alpha_2$, we get for $\alpha_2 \geq k$

$$\|D_{\alpha_2}D_{\alpha_1}F(z)\| \geq \frac{(|\alpha_2| - k)}{1 + k^{n-1}} \left\{ n - \frac{1}{k^{\frac{n-1}{2}}} \sqrt{\frac{|\alpha_1 a_1 + na_0 + n\lambda m|}{|n\alpha_1 a_n + a_{n-1}|}} \right\} \|D_{\alpha_1}F(z)\|,$$

which by using (4.1) gives

$$\begin{aligned} \|D_{\alpha_2}D_{\alpha_1}F(z)\| &\geq \frac{(|\alpha_1| - k)(|\alpha_2| - k)}{(1 + k^n)(1 + k^{n-1})} \left\{ n + 1 - \frac{1}{k^{\frac{n}{2}}} \sqrt{\frac{|a_0 + \lambda m|}{|a_n|}} \right\} \\ &\quad \left\{ n - \frac{1}{k^{\frac{n-1}{2}}} \sqrt{\frac{|\alpha_1 a_1 + na_0 + n\lambda m|}{|n\alpha_1 a_n + a_{n-1}|}} \right\} \|F(z)\|. \end{aligned}$$

or equivalently

$$\begin{aligned} \|D_{\alpha_2}D_{\alpha_1}P(z) + n(n - 1)\lambda m\| &\geq \\ &\frac{(|\alpha_1| - k)(|\alpha_2| - k)}{(1 + k^n)(1 + k^{n-1})} \prod_{\mu=0}^1 \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-\mu}^\mu + \mu! \binom{n}{\mu} \lambda m}{\mathbb{A}_{n-\mu}^\mu} \right|} \right) \|P(z) + \lambda m\|. \end{aligned}$$

Again choosing argument of λ as above in the first induction step, we get

$$\begin{aligned} \|D_{\alpha_2}D_{\alpha_1}P(z)\| &\geq \left\{ \mathbb{M}_{\alpha_2}^k \prod_{\mu=0}^1 \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-\mu}^\mu + \mu! \binom{n}{\mu} \lambda m}{\mathbb{A}_{n-\mu}^\mu} \right|} \right) \right\} \|P(z)\| + \\ &\left\{ \mathbb{M}_{\alpha_2}^k \prod_{\mu=0}^1 \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-\mu}^\mu + \mu! \binom{n}{\mu} \lambda m}{\mathbb{A}_{n-\mu}^\mu} \right|} \right) - \prod_{\mu=0}^1 (n - \mu) \right\} |\lambda| \min_{z \in \partial\mathbb{T}_k} |P(z)|. \end{aligned}$$

This shows the result is true for $s = 2$. Suppose the result is true for the polar derivative of order less or equal to s . That is, for the suitable choices of $\alpha_1, \alpha_2, \dots, \alpha_{s-1}$, if all the zeros of $(s - 1)$ th-order polar derivative of degree $(n - s + 1)$ lie in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k$, $k \geq 1$ and $|\alpha_i| > k$, $i = 1, 2, \dots, (s - 1)$, then we have

$$\begin{aligned} \|D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_2} D_{\alpha_1} P(z)\| \geq & \left\{ \mathbb{M}_{\alpha_s}^k \prod_{\mu=0}^{s-1} \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-\mu}^\mu + \mu! \binom{n}{\mu} \lambda m}{\mathbb{A}_{n-\mu}^\mu} \right|} \right) \right\} \|P(z)\| + \\ & \left\{ \mathbb{M}_{\alpha_s}^k \prod_{\mu=0}^{s-1} \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-\mu}^\mu + \mu! \binom{n}{\mu} \lambda m}{\mathbb{A}_{n-\mu}^\mu} \right|} \right) - \prod_{\mu=0}^{s-1} (n - \mu) \right\} |\lambda| \min_{z \in \partial\mathbb{T}_k} |P(z)|, \end{aligned}$$

which by our technique actually follows by replacing $F(z)$ by $P(z) + \lambda m$ in the inequality

$$\|D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_1} F(z)\| \geq \left\{ \mathbb{M}_{\alpha_s}^k \prod_{\mu=0}^{s-1} \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-\mu}^\mu + \mu! \binom{n}{\mu} \lambda m}{\mathbb{A}_{n-\mu}^\mu} \right|} \right) \right\} \|F(z)\|. \tag{4.3}$$

We now show that the result is true for $(s + 1)$ th-order polar derivative ($s < n$) as well. For the complex numbers $\alpha_1, \alpha_2, \dots, \alpha_s$ choosen suitably as above, the s th order polar derivative $D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_2} D_{\alpha_1} F(z)$ is a polynomial of degree $n - s$. Also we have $|\alpha_i| > k$ for $i = 1, 2, \dots, s$, therefore by Lemma 3.1 $D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_1} P(z)$ has all its zeros in $\mathbb{T}_k^- \cup \partial\mathbb{T}_k$, $k \geq 1$. Now applying Theorem 1.6 to the $(n - s)$ th degree polynomial $D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_1} F(z)$ with $\alpha = \alpha_{s+1}$ and noting that $|\alpha_i| \geq k, \forall i = 1, 2, \dots, s + 1$, we get

$$\|D_{\alpha_{s+1}} \{D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_1} F(z)\}\| \geq \frac{(|\alpha_{s+1}| - k)}{1 + k^{n-s}} \left\{ n - s + 1 - \frac{1}{k^{\frac{n-s}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-s}^s + (s - 1)! \binom{n}{s-1} \lambda m}{\mathbb{A}_{n-s}^s} \right|} \right\} \|D_{\alpha_s} D_{\alpha_{s-1}} \cdots D_{\alpha_1} F(z)\|.$$

This by using induction hypothesis (4.3) gives

$$\begin{aligned} \|D_{\alpha_{s+1}} D_{\alpha_s} \cdots D_{\alpha_1} F(z)\| \geq & \frac{(|\alpha_{s+1}| - k)}{1 + k^{n-s}} \left\{ n - s + 1 - \frac{1}{k^{\frac{n-s}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-s}^s + (s - 1)! \binom{n}{s-1} \lambda m}{\mathbb{A}_{n-s}^s} \right|} \right\} \\ & \times \left\{ \mathbb{M}_{\alpha_s}^k \prod_{\mu=0}^{s-1} \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-\mu}^\mu + \mu! \binom{n}{\mu} \lambda m}{\mathbb{A}_{n-\mu}^\mu} \right|} \right) \right\} \|F(z)\|, \end{aligned}$$

which is equivalent to

$$\|D_{\alpha_{s+1}} D_{\alpha_s} \cdots D_{\alpha_1} F(z)\| \geq \left\{ \mathbb{M}_{\alpha_{s+1}}^k \prod_{\mu=0}^s \left(n - \mu + 1 - \frac{1}{k^{\frac{n-\mu}{2}}} \sqrt{\left| \frac{\mathbb{B}_{n-\mu}^\mu + \mu! \binom{n}{\mu} \lambda m}{\mathbb{A}_{n-\mu}^\mu} \right|} \right) \right\} \|F(z)\|.$$

Again, replacing $F(z)$ by $P(z) + \lambda m$, choosing the argument of λ suitably, and using triangle inequality as before, we see that the result holds for the polar derivative of order $s + 1$ as well. Hence, by the principle of mathematical induction, it follows that inequality (2.5) holds for every $s = 1, 2, \dots, n$.

Now, if the polynomial $P(z)$ has a zero on $\partial\mathbb{T}_k$, then $m = \min_{z \in \partial\mathbb{T}_k} |P(z)| = 0$, and the result follows on the similar lines as above by considering polynomial $P(z)$ instead of $P(z) + \lambda m$. This completes the proof of Theorem 2.6. \square

Conflict of Interest. The authors declare that they have no conflicts of interest.

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