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New results concerning on approximate controllability of Hilfer - Katugampola integrodifferential inclusions

A. Prabha and R. Nirmalkumar

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Abstract: This paper investigates the approximate controllability of mixed integrodifferential inclusions of Hilfer-Katugampola fractional derivatives(HKFD), focusing on orders $\varphi \in (0, 1)$ and type $\varsigma \in [0, 1)$. By leveraging Bohenblust- Karlin fixed point theorem and fractional calculus techniques is used to establish the sufficient conditions for existence of the solution and the approximate controllability. Numerical examples is given to illustrate the applicability and significance of our results.

1 Introduction

The study of fractional differential inclusions is valuable in many areas of mathematics. Recent advancements in control theory, rheology, image manipulation, optics, communication processing, and other fields have reinforced the concept of fractional calculus. It serves as an effective tool for modeling various phenomena in fractional differential equations. Several papers provide additional information [17, 24, 27, 15, 32, 33]. Katugampola [13] has developed a new fractional differential operator, which has been extensively researched by several scholars [11, 12, 25, 26]. Furthermore, this type of operator has been combined with the Hilfer fractional differential operator [9] to create the Hilfer-Katugampola fractional differential operator [19].

Controllability refers to the ability to manipulate or control the behavior of a system in a desired manner using external inputs or controls. In the context of control theory, controllability is a fundamental property that assesses whether a system can be steered from one state to another within a specified time frame using appropriate control inputs.

A system is said to be completely controllable if it can reach any state in its state space from any initial state in a finite amount of time, given appropriate control inputs. On the other hand, if a system is not completely controllable, it is termed partially controllable, meaning there are certain states that cannot be reached from some initial states.

Approximate controllability is a related concept that addresses the feasibility of steering a system approximately close to a desired state within a specified time, even if reaching the exact state is not possible. This means that while it might not be feasible to reach a specific target state precisely, it is still possible to bring the system arbitrarily close to the desired state by choosing appropriate control inputs. Both controllability and approximate controllability are vital aspects of control theory, influencing the design and analysis of control systems in various engineering and scientific disciplines. They play a significant role in fields such as aerospace engineering, robotics, power systems, and more.

The Hilfer-Katugampola fractional differential equation is a type of fractional differential equation (FDE) that generalizes the concept of derivatives to non-integer orders. Fractional calculus, including the Hilfer-Katugampola fractional derivative, finds applications in various fields such as physics, engineering, biology, and finance, where phenomena with memory, hereditary properties, or long-range interactions are observed and modeled more accurately using fractional-order derivatives. A few concepts were gathered from these references, which facilitate our mathematical endeavors [23, 22, 5, 8, 31]. Jingyun Lv and Xiaoyuan Yang [10] are studied the Approximate controllability of Hilfer fractional differential equations. Many researchers have carried out works on the controllability of stochastic evolution with non-local conditions, orders, and Non-instantaneous Impulsive Differential System with Rosenblatt Process and Poisson Jumps [1, 7, 21, 30] in the past year

Vijaykumar et al. [28] established the approximate controllability of Volterra-Fredholm integro-differential fractional inclusions of Sobolev type with order $1 \leq$ r ≤ 2 . Mohamed I. Abbas [16] studied the controllability of Hilfer-Katugampola fractional differential equations. Motivated by the preceding research, this work focuses on the approximate controllability of Hilfer-Katugampola mixed integrodifferential inclusions of the form

$$^{\varrho}\mathbb{D}^{\varphi,\varsigma}[q(u)] \in \mathfrak{B}w(u) + \Omega\left(u, q, \int_0^{\rho} \Delta(u,a,q)\mathrm{d}a, \int_0^{\tau} \Lambda(u,a,q)\mathrm{d}a\right), u \in K = [0,u] \tag{1.1}$$

$$^{\rho}\mathbb{I}_{0^+}^{1-\gamma} q(c) = q_0, \gamma = \varphi + \varsigma - \varphi\varsigma. \tag{1.2}$$

where $^{\varrho}\mathbb{D}^{\varphi,\varsigma}$ Hilfer Katugampola derivative of fractional order $\varphi(0 < \varphi < 1)$ and type $\varsigma(0 \leq \varsigma < 1)$, $^{\rho}\mathbb{I}_{0^+}^{1-\gamma}$ is the left-sided Katugampola fractional integral of order $1-\gamma$ and the function $q(\cdot)$ in a Banach space X, $\mathfrak{B}$ is a Bounded linear operator from a Banach Space U into X. Furthermore $\Omega : K \times Q \times Q \times Q \to 2^X / \{\emptyset\}$ is satisfied some conditions and $w(\cdot)$ is control function define $\mathcal{L}^2(K,W)$, a control functions of Banach spaces. The Banach space of functions $w : K \to W$ and the norm $\|w\|_{\mathcal{L}^2(K,W)} = \{\int_0^u \|w(u)\|_W^2 \, \mathrm{d}u\}^{\frac{1}{2}} < \infty$ with the norm of Bochner integrable endowed and the function $\Omega : K \times \mathbb{X} \to \mathbb{X}$ is a Caratheodary[25, 29]

The design of the paper is as follows: Section 2 presents some basic notations and preliminary results, while Section 3 investigates approximate controllability using Bohnenblust-Karlin's fixed point theorem. Finally in the Section 4, provides an example to demonstrate the applicability of the proposed system.

2 PRELIMINARIES

In this chapter, discusses the fundamental definitions, lemmas, and specific outcomes essential for determining our key findings. Let $C(K,\mathbb{X})$ be the Banach space of a continuous function $w : K \times \mathbb{X}$, then the supremum norm $\|w\|_C = sup_{u\in K} \|w(u)\|$. Define the Banach space $C_{1-\gamma,\rho}(K,\mathbb{X})$, the Banach space of a continuous function w is a finite interval K defined by,

$$C_{1-\gamma,\rho}(K,\mathbb{X}) = \left\{ h : (0,b] \times \mathbb{X} : \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{1-\gamma} w(u) \in C(K,\mathbb{X}) \right\},$$

and the norm of

$$\|w\| \, C_{1-\gamma,\rho} = \left\| \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{1-\gamma} w(u) \right\|_C = max_{u\in K} \left\| \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{1-\gamma} w(u) \right\|.$$

Definition 2.1. [6, 19] The Hilfer- Katugampola fractional integrals are defined where $\varphi \in \mathbb{R}$ with $\varphi \geq 0$ is given by

$$^{\rho}\mathcal{I}^{\varphi} q(u) = \frac{\rho^{1-\varphi}}{\Gamma(\varphi)} \int_0^b \frac{v^{\rho-1} q(u)}{(u^\rho - v^\rho)^{1-\varphi}} du, u > 0,$$

where $\rho \geq 0$ and a Gamma function $\Gamma(\cdot)$ defined as

$$\Gamma(\varphi) = \int_0^\infty e^{-u} u^{\varphi-1} \mathrm{d}u, Re(\varphi) > 0.$$

Definition 2.2. [19, 6] The fractional derivatives of Hilfer-Katugampola are defined, where φ,ρ $\in \mathbb{R}$ with φ, $\rho \geq 0$ and n = [φ]+1 is given by

$$^{\rho}D^{\varphi} q(u) = \frac{\rho^{\varphi-n+1}}{\Gamma(n-\varphi)} (u^{1-\rho}\frac{d}{du})^n \int_0^b \frac{v^{\varphi-1q(u)}}{(u^\rho - v^\rho)^{\varphi-n+1}} du.$$

Lemma 2.3. *[19] Let* $0 < \varphi < 1$, *$0 \geq \varsigma < 1$. If* $q \in C_\gamma[0, b]$ *and* ${}^\rho\mathscr{I}^{1-\gamma}y \in C^1_\gamma[0, b]$, *then*

$$({}^\rho\mathscr{I}^\varphi \rho\mathscr{D}^\varphi q)(u) = q(u) - \frac{{}^\rho\mathscr{I}^{1-\gamma}q(u)}{\Gamma(\varphi)}\left(\frac{u^\rho - v^\rho}{\rho}\right)^{\varphi-1},$$

for all $u \in (0, b]$.

Lemma 2.4. *[19] Let* $0 < \varphi < 1$, *$0 \leq \varsigma \leq 1$ and* $\gamma = \varphi + \varsigma(1 - \varphi)$. *If* $q \in C^\gamma_{1-\gamma}$, *then*

$$({}^\rho\mathscr{I}^\gamma_{0+} \rho\mathscr{D}^\gamma_{0+})q = ({}^\rho\mathscr{I}^\gamma_{0+} \rho\mathscr{D}^{\varphi,\varsigma}_{0+})q.$$

Lemma 2.5. *[2] A function* $\Omega : Y \to Y$ *be a continuous mapping such that* $\Omega(Y) \subset X$ *is relatively compact, where* Y *is closed subset of Banach space* X. *Then,* Ω *has at least one fixed point in* Y.

Lemma 2.6. *[3] A function* $M : \mathcal{E} \to 2^X \setminus \{\emptyset\}$ *is u.s.c where* $M(\mathcal{E})$ *is compact and* $M(\mathcal{E}) \subseteq \mathcal{E}$. *since* X, *is bounded and convex where* M *is nonempty subset of* X. *Therefore,* M *has a fixed point.*

By using lemma(2.3) and (2.4), the solution of system of the equation (1.1)-(1.2) the integral differential equation [20] we get

$$q(u) = \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{\rho^{\varphi-1}}{\Gamma(\varphi)}\int_0^u (u^\rho - v^\rho)^{\varphi-1}v^{\rho-1} \tag{2.1}$$
$$\left[\mathfrak{B}w(u) + \Omega\left(u, q, \int_0^\rho \Delta(u, a, q)\mathrm{d}a, \int_0^\tau \Lambda(u, a, q)\mathrm{d}a\right)\right]\mathrm{d}u.$$

Some useful results of the essential operator is

$$\Gamma^u_0 = \int_0^b v^{\rho-1}g(u)\mathfrak{B}\mathfrak{B}^*g^*(u)\mathrm{d}u : Q \to Q, \tag{2.2}$$
$$R(\lambda, \Gamma^u_0) = (\lambda I + \Gamma^u_0)^{-1} : Q \to Q.$$

continuing $(\lambda I + \Gamma^u_0)$ is bounded and $\mathfrak{B}^*, g^*(u)$ stands for adjoint operator $\mathfrak{B}, g(u)$
Let's start with the following assumption is

($\mathbf{A_0}$) $\lambda R\,(\lambda,\Gamma^u_0) \to 0$ as $\lambda \to 0^+$ strong operator topology.
A_0 is holds iff the linear fractional system,in view [18]

$$\begin{aligned} {}^\varrho\mathbb{D}^{\varphi,\varsigma}[q(u)] &\in \mathfrak{B}w(u) + g(u), u \in K \\ {}^\rho\mathbb{I}^{1-\gamma}_{0^+}q(c) &= q_0, \gamma = \varphi + \varsigma - \varphi\varsigma. \end{aligned} \tag{2.3}$$

is approximately controllable on K.

3 SOME RESULTS OF APPROXIMATE CONTROLLABILITY

With the ideas of Bohnenblust-Karlin's fixed point theorem [3, 4] in a fractional-order time-delay model, to a control system with approximate controllability [14]. Calculate and establish the conclusion on approximate controllability for Hilfer-Katugampola combined integrodifferential inclusions of the form the equation (1.1)-(1.2). Assume the following hypothesis

($\mathbf{A_1}$) The map $\Omega : K \times Q \times Q \times Q \to BCC(Y)$, then the function Ω is measurable to u, $\forall\, q \in Q$ upper semi continuous to q for all $u \in K$ and $q \in C$

$$\mathscr{R}_{\Omega,q} = \left\{g \in \mathcal{L}^1(K, Q) : g(u) \in \Omega\left(u, q(u), \int_0^\rho \Delta(u, a, q(a))\mathrm{d}a, \int_0^\tau \Lambda(u, a, q(a))\mathrm{d}a\right), u \in K\right\}$$

is nonempty.

($\mathbf{A_2}$) A functions $\Delta(u,p,\cdot), \Lambda(u,p,\cdot) : Q \to Q$ are continuous, $\forall\, q \in Q, \Delta(\cdot,\cdot,q), \Lambda(\cdot,\cdot,q) : \mathcal{L} \to Q$, are strongly measurable, $\forall\, (u,q) \in \mathcal{L}$.

($\mathbf{A_3}$) $g^*\gamma(\cdot) \in \mathcal{L}^{\frac{1}{v}}(K, \mathbb{R}^+)$ and $\exists\, q \in (0,b)\ \forall\, \gamma \geq 0, q \in C\ |q|_C \leq \gamma$.

$$g^* = g^x = \sup_{u \in K} \left\{ \|g\| : g(u) \in \Omega\left(u, q(u), \int_0^\rho \Delta(u,a,q(a))\mathrm{d}a, \int_0^\tau \Lambda(u,a,q(a))\mathrm{d}a\right)\right\} \leq \mathcal{L}_{g,\gamma}(u)$$

for a.e $u \in K$.

($\mathbf{A_4}$) The mapping a $\to (u-a)^{b-1}\mathcal{L}_{g,\gamma(u)} \in \mathcal{L}^2(K,Q)$ such that there exist $\beta > 0$ such that

$$\lim_{\gamma\to\infty} inf \frac{\mathcal{L}_g \int_0^u v^{\rho-1}\mathcal{L}_{g,\gamma(u)}du}{\gamma} = \beta < +\infty,$$

To discuss the controllability of the equations $(1.1) - (1.2)$, if $\forall\varphi > 0$ there exist $q(\cdot) \in C$, using equation (2.1) and (2.2) is given below

$$\begin{aligned}
(Mq)(u) &= \frac{q_0}{\Gamma(\varphi)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1} && (3.1)\\
&\quad \left[\mathfrak{B}w(u) + \Omega\left(u, q, \int_0^\rho \Delta(u,a,q)\mathrm{d}a, \int_0^\tau \Lambda(u,a,q)\mathrm{d}a\right)\right]\mathrm{d}u\\
w(u) &= \mathfrak{B}^* R(\lambda, \Gamma_0^u) l(q(\cdot)), && (3.2)\\
l(q(\cdot)) &= q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} && (3.3)\\
&- \frac{1}{\Gamma(\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} g^\gamma(u)du.\\
& where, q^\rho = u^\rho - v^\rho, \|\mathfrak{B}\| = Z_2, \|\mathfrak{B}^*\| = Z_1
\end{aligned}$$

Theorem 3.1. *Under the assumption* $(A_0 - A_4)$ *holds subsequently, the system (1.1)-(1.2) has a solution on* K *provided*

$$\frac{1}{\Gamma(2\varphi)}\Big[\frac{Z_1 Z_2}{\Gamma(2\varphi)\lambda}\left(\frac{u^{2\varphi}}{2\varphi}\right) + 1\Big]\beta < 1. \tag{3.4}$$

Proof. To prove the fixed point technique, let define $M : C \to 2^C$ and $\lambda > 0$ by using the equation (3.1) we get

$$M(q) = \Big[q \in C : M_i(u) = \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1} \left[\mathfrak{B}w(u) + g_i(u)\right]\mathrm{d}u, i = 1,2, g \in \mathscr{R}_{\Omega,q}\Big]$$

has a fixed point. To establish the results and we need to prove the following steps.
Step 1: For all $\mu \geq 0$, $\lambda > 0$ let $M_1, M_2 \in C$ exists $g_1, g_2 \in \mathscr{R}_{\Omega,u}\ \mathscr{R}(u)$ is convex for all $q \in C$ since, $u \in V$, we get,

$$M_i(u) = \tfrac{q_0}{\Gamma(\gamma)}\left\{\tfrac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \tfrac{1}{\Gamma(\varphi)}\int_0^u \left\{\tfrac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\left[\mathfrak{B}w(u) + g_i(u)\right]\mathrm{d}u, i = 1,2.$$

For all $u \in K$ and $0 \leq \mu \leq 1$, by the equation(3.1)we get

$$\begin{aligned}
(\mu M_1 + (1-\mu)M_2)(u) &= \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}\mathfrak{B}^* R(\lambda, \Gamma_0^u)\\
&\quad \Big(q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\\
&\quad (\mu g_1(u) + (1-\mu)g_2(u))\Big)du.
\end{aligned}$$

Since $\mathscr{R}_{\Omega,q}$ is convex, $\mu M_1 + (1-\mu)M_2 \in \mathscr{R}_{\Omega,q}$ is also convex.
Step 2: We assume that the condition,

$$N_x = \{z \in C : \|z\|_R \leq x, 0 \leq x \leq C\}, x > 0.$$

clearly N_x some conditions is bounded, closed, convex set of C , where C is space. For $\mu > 0$, our property exist $x > 0$, such that $\mathscr{R}(N_x) \subset N_x$. If not then for all $x > 0$, $\exists\, Q^x \in N_x$ but $\mathscr{R}(Q^x) \notin N_x$, ie.

$$\|\mathscr{R}(Q^x)\|_R = \sup\{\|M^x\|_R : M^x \in \mathscr{R}(Q^x)\} > x \quad .$$

$$M^x(\rho) = \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\left(\mathfrak{B}w^x(u) + g^x(u)\right)du$$

for some $g^x \in \mathscr{R}_{\Omega,q}$, by using hypothesis A_2 and the equation (3.2) and (3.3)we have,

$$\begin{aligned}
\|w^x(u)\| &= \left\|\mathfrak{B}^* R(\lambda, \Gamma_0^b) l(q(\cdot))\right\| \\
&= \left\|\mathfrak{B}^* R(\lambda, \Gamma_0^u)\left[q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} - \frac{1}{\Gamma(\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} g^x(u)\mathrm{d}u\right]\right\| \\
&\leq \left\|\frac{Z_1}{\lambda}\left[q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} - \frac{1}{\Gamma(\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} g^x(u)\mathrm{d}u\right]\right\|
\end{aligned}$$

For $\lambda > 0$ we prove that

$$\begin{aligned}
x &< \|R(q^x)(u)\| \leq \frac{\|q_0\|}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} \|Bw^x(u)\| + \|g^x(u)\|\,\mathrm{d}u \\
&\leq \frac{\|q_0\|}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(2\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{2\varphi-1} Z_2\|w^x(u)\| + \|g^x(u)\|\,\mathrm{d}u \\
&\leq \frac{\|q_0\|}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(2\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{2\varphi-1} \\
&\quad Z_2\mathfrak{B}^* R(\lambda, \Gamma_0^u)\left[q_u + \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(2\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{2\varphi-1} g^x(u)\mathrm{d}u\right]\mathrm{d}u \\
&+ \frac{1}{\Gamma(\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1}\|g^x(u)\|\,\mathrm{d}u \\
&\leq \frac{\|q_0\|}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(2\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{2\varphi-1} \\
&\quad \frac{Z_1 Z_2}{\lambda}\left[q_u + \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(2\varphi)}\left(\frac{u^{2\varphi}}{2\varphi}\right)\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{2\varphi-1}\mathcal{L}_{g,\gamma}(u)\mathrm{d}u\right]\mathrm{d}u \\
&+ \frac{1}{\Gamma(2\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{2\varphi-1}\mathcal{L}_{g,\gamma}(u)\mathrm{d}u
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|q_0\|}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(2\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{2\varphi-1} \qquad (3.5) \\
&\quad \frac{Z_1 Z_2}{\Gamma(2\varphi)\lambda}\left(\frac{u^{2\varphi}}{2\varphi}\right)\left[q_u + \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{2\varphi-1}\mathcal{L}_{g,\gamma}(u)\mathrm{d}u\right]\mathrm{d}u \\
&+ \frac{1}{\Gamma(2\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{2\varphi-1}\mathcal{L}_{g,\gamma}(u)\mathrm{d}u
\end{aligned}$$

Both sides dividing the equation(3.5) by x and taking the limit as $x \to \infty$, which is contradicts of the equation (3.4) and by hypothesis A_4 we get

$$\frac{1}{\Gamma(2\varphi)}\Big[\frac{Z_1 Z_2}{\Gamma(2\varphi)\lambda}\left(\frac{u^{2\varphi}}{2\varphi}\right)+1\Big]\beta \geq 1$$

contradicts to our assumption. Hence $\lambda > 0$, there exists $x > 0$ such that the mapping $\mathscr{R} : N_x \to N_x$.

Step 3: $(Mq)(u)$ is bounded and equicontinuous of R. From the above step it is clear that $M(u)$ is bounded. To show that $M(u)$ is equicontinuous. By using the equation (3.1) and the interval $0 < u_1 < u_2 \leq u$ we get

$$\begin{aligned}
(Mq)(u) &= \frac{q_0}{\Gamma(\varphi)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}w(u) + g(u)\mathrm{d}u \\
\|(Mq)(u_2) - (Mq)(u_1)\| &\leq \frac{q_0}{\Gamma(\varphi)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_{u_1}^{u_2} \left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\varphi-1} \\
& \quad v^{\rho-1}\mathfrak{B}w(u_2-u_1) + g(u_2-u_1)\mathrm{d}u \\
&- \frac{1}{\Gamma(\varphi)}\int_0^{u_1} \left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}w(u_1) + g(u_1)\mathrm{d}u \\
&+ \frac{1}{\Gamma(\varphi)}\int_0^{u_2} \left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}w(u_2) + g(u_2)\mathrm{d}u \\
&= I_1 + I_2 + I_3
\end{aligned}$$

Let $G = \frac{2\varphi-1}{1-\delta} \in (-1, 0)$ and by using the hypothesis A_2 we have

$$\begin{aligned}
I_1 &= \frac{1}{\Gamma(\varphi)}\int_{u_1}^{u_2} \left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}w(u_2-u_1)\mathrm{d}u \\
&+ \frac{1}{\Gamma(\varphi)}\int_{u_1}^{u_2} \left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}g(u_2-u_1)\mathrm{d}u \\
I_2 &= \frac{1}{\Gamma(\varphi)}\int_0^{u_1} \left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}w(u_1)\mathrm{d}u + \frac{1}{\Gamma(\varphi)}\int_0^{u_1} \left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}g(u_1)\mathrm{d}u \\
I_3 &= \frac{1}{\Gamma(\varphi)}\int_0^{u_2} \left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}w(u_2)\mathrm{d}u + \frac{1}{\Gamma(\varphi)}\int_0^{u_2} \left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}g(u_2)\mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
I_1 &= \frac{1}{\Gamma(\varphi)}\int_{u_1}^{u_2}\left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}w(u_2-u_1)\mathrm{d}u \\
&+ \frac{1}{\Gamma(\varphi)}\int_{u_1}^{u_2}\left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}g(u_2-u_1)\mathrm{d}u \\
&= \frac{Z_1Z_2}{\Gamma(2\varphi)}\int_{u_2}^{u_1}\left(\left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\frac{2\varphi-1}{1-\delta}} - \left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\frac{2\varphi-1}{1-\delta}}\right)^{1-\delta} v^{\rho-1}\left\|w\right\|\mathrm{d}u \\
&+ \frac{1}{\Gamma(2\varphi)}\int_{u_2}^{u_1}\left(\left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\frac{2\varphi-1}{1-\delta}} - \left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\frac{2\varphi-1}{1-\delta}}\right)^{1-\delta} v^{\rho-1}\mathcal{L}_{g,\gamma}(u)\mathrm{d}u \\
&\leq \frac{Z_1Z_2}{\Gamma(2\varphi)}\frac{\left(\left\{\frac{u_2^\rho - {u_1}^\rho}{\rho}\right\}^{\frac{2\varphi-1}{1-\delta}} - \left\{\frac{u_1^\rho - {u_2}^\rho}{\rho}\right\}^{\frac{2\varphi-1}{1-\delta}}\right)^{1-\delta}}{(1+\beta)^{1-\delta}} v^{\rho-1}\left\|w\right\| \\
&+ \frac{1}{\Gamma(2\varphi)}\frac{\left(\left\{\frac{u_2^\rho - {u_1}^\rho}{\rho}\right\}^{\frac{2\varphi-1}{1-\delta}} - \left\{\frac{u_1^\rho - {u_2}^\rho}{\rho}\right\}^{\frac{2\varphi-1}{1-\delta}}\right)^{1-\delta}}{(1+\beta)^{1-\delta}} v^{\rho-1}\left\|\mathcal{L}_{g,\gamma}(u)\right\| \\
I_2 &= \frac{1}{\Gamma(\varphi)}\int_0^{u_1}\left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}w(u_1)\mathrm{d}u + \frac{1}{\Gamma(\varphi)}\int_0^{u_1}\left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}g(u_1)\mathrm{d}u \\
&= \frac{Z_1Z_2}{\Gamma(2\varphi)}\int_0^{u_1}\left(\{\left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\frac{2\varphi-1}{1-\delta}}\right)^{1-\delta} v^{\rho-1}\left\|w\right\|\mathrm{d}u \\
&+ \frac{1}{\Gamma(2\varphi)}\int_0^{u_1}\left(\left\{\frac{u_1^\rho - v^\rho}{\rho}\right\}^{\frac{2\varphi-1}{1-\delta}}\right)^{1-\delta} v^{\rho-1}\left\|\mathcal{L}_{g,\gamma}(u)\right\|\mathrm{d}u \\
&= \frac{Z_1Z_2}{\Gamma(2\varphi)}\frac{((\frac{u_1^\rho}{\rho})^{\frac{2\varphi-1}{1-\delta}})^{1-\delta}}{(1+\beta)^{1-\delta}} v^{\rho-1}\Big[\left\|\mathcal{L}_{g,\gamma}(u)\right\| + \left\|w\right\|\Big] \\
I_3 &= \frac{1}{\Gamma(\varphi)}\int_0^{u_2}\left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}w(u_2)\mathrm{d}u + \frac{1}{\Gamma(\varphi)}\int_0^{u_2}\left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}g(u_2)\mathrm{d}u \\
&= \frac{Z_1Z_2}{\Gamma(2\varphi)}\int_0^{u_2}\left(\{\left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\frac{2\varphi-1}{1-\delta}}\right)^{1-\delta} v^{\rho-1}\left\|w\right\|\mathrm{d}u \\
&+ \frac{1}{\Gamma(2\varphi)}\int_0^{u_2}\left(\left\{\frac{u_2^\rho - v^\rho}{\rho}\right\}^{\frac{2\varphi-1}{1-\delta}}\right)^{1-\delta} v^{\rho-1}\left\|\mathcal{L}_{g,\gamma}(u)\right\|\mathrm{d}u \\
&= \frac{Z_1Z_2}{\Gamma(2\varphi)}\frac{((\frac{u_2^\rho}{\rho})^{\frac{2\varphi-1}{1-\delta}})^{1-\delta}}{(1+\beta)^{1-\delta}} v^{\rho-1}\Big[\left\|\mathcal{L}_{g,\gamma}(u)\right\| + \left\|w\right\|\Big]
\end{aligned}
$$

It is easily verify to $Z_1, Z_2 \to 0$ and $(u_2 \to u_1) \to 0$, Additionally, by refer the compactness of $q_0, \left\|w\right\|, \left\|\mathcal{L}_{g,\gamma}(u)\right\|$ tends to zero. Therefore $\left\|(Mq)(u_2) - (Mq)(u_1)\right\| \to 0$, Hence,$(Mq)(u)$ is equicontinuous. $\mathscr{R}(N_x) \subset$ C is also equicontinuous.

Step 4 : Consider the open ball $\{\mathbf{M} = q_0 \in \mathbb{R}^n : |q_0| < 1\} \in \mathbb{R}^n$,$\rho > 0$ and $Z_1, Z_2 > 0$, $(Mq)(u)$

is relatively compact then $(Mq)(u) = 0$

$$(Mq)(u) = \frac{q_0}{\Gamma(\varphi)}\left(\frac{q^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^b \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\left[\mathfrak{B}w(u) + g^x(u)\mathrm{d}u\right]$$

$$(\bar{M}q)(u) = \frac{q_0}{\Gamma(\varphi)}\left(\frac{q^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^b \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\left[\bar{\mathfrak{B}}w(u) + \bar{g^x}(u)\mathrm{d}u\right]$$

$$\left\|(Mq)(u) - (\bar{M}q)(u)\right\| \leq \frac{Z_1 Z_2}{\Gamma(2\varphi)^2}\int_0^u \left(\frac{q^\rho}{\rho}\right)^{\frac{2\varphi-1}{1-\delta}} v^{\rho-1}\|w\| + \|\mathcal{L}_{g,\gamma}(u)\|\,\mathrm{d}u - \frac{Z_1 Z_2}{\Gamma(2\varphi)^2}\int_0^u \left(\frac{q^\rho}{\rho}\right)^{\frac{2\varphi-1}{1-\delta}} v^{\rho-1}\|\bar{w}\| + \left\|\mathcal{L}_{g,\bar{\gamma}}(u)\right\|\mathrm{d}u$$

$Z_1 Z_2 \to 0$, and also $\|w\| - \|\bar{w}\| \to 0$ compact,
therefore, $\left\|(Mq)(u) - (\bar{M}q)(u)\right\|$ is relatively compact.
Step 5: Closed graph of $\mathscr{R}$. Let $q^n \to q^*$ when $n \to \infty$, and $M^n \to M^*$ when $n \to \infty$ to prove that $M^* \in \mathscr{R}(u^*)$ since $p^n \in \mathscr{R}(u^n)$ there exist $g^x \in \mathscr{R}_{\Omega,x^n}$ such that

$$M^n(u) = \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}\mathfrak{B}^* R(\lambda, \Gamma_0^u) \left(q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1} g^n(u)\mathrm{d}u\right)du$$

then there exist $g^* \in \mathscr{R}(x^*)$ such that $u \in K$

$$M^*(u) = \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}\mathfrak{B}^* R(\lambda, \Gamma_0^u) \left(q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1} g^n(u)\mathrm{d}u\right)du$$

clearly,

$$\Bigg\|\Bigg(M^n(u) - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}\mathfrak{B}^* R(\lambda, \Gamma_0^u) \left(q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1} g^n(u)\right)du\Bigg)$$
$$\Bigg(M^*(u) - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} - \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}\mathfrak{B}^* R(\lambda, \Gamma_0^u) \left(q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1} g^n(u)\mathrm{d}u\right)du\Bigg)\Bigg\| \to 0, n \to \infty$$

Assume that N : $\mathcal{L}^2(K, W) \to R$,

$$(Ng)(u) = \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}\mathfrak{B}^* R(\lambda, \Gamma_0^u) \times (g^*(u)\mathrm{d}u \in N(\mathscr{R}_{\Omega,x^*})$$

by refer N, we have

$$\Bigg(M^n(u) - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}\mathfrak{B}^* R(\lambda, \Gamma_0^u) \left(q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1} g^n(u)\mathrm{d}u\right)du\Bigg)$$
$$\Bigg(M^n(u) - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}\mathfrak{B}^* R(\lambda, \Gamma_0^u) \left(q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1} g^n(u)\mathrm{d}u\right)du\Bigg)$$

Because $g^n \to g^*$ we have

$$\Big(M^*(u) - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}\mathfrak{B}\mathfrak{B}^* R(\lambda, \Gamma_0^u)$$
$$\Big(q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} - \frac{1}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1} g^n(u)\mathrm{d}u\Big)du\Big) \in N(\mathscr{R}_{\Omega,x^*})$$

$\therefore$ Closed graph of $\mathscr{R}$.
From the Step 1-5 by using theorem of Arzela-Ascoli, $\mathscr{R}$ has a multi-valued function with completely continuous and compact thus $\mathscr{R}$ is upper semi continuous, then $\mathscr{R}$ satisfied a fixed point theorem. Therefore $q(\cdot)$ also a system of equations $(1.1)-(1.2)$. □

Definition 3.2. The equation of $y_c(q_0 : u)$ be the initial value of equations (1.1)-(1.2) the initial value q_0, terminal time of Q and the control function w, consider the equation is $\overline{R(c, q_0)} = Q, R(c, q_0) = \{q_c(q_0, w) : w(\cdot) \in \mathcal{L}^2(K, W)\}$ the system of the equations (1.1)-(1.2). where, its closure of u and terminal time of c is $\overline{R(c, y_0)}$
therefore, If controllability equation is (1.1)-(1.2) then, $R(c, y_0) = Q$ on the interval K.

Theorem 3.3. *Under the assumption of hypothesis $A_0 - A_4$ are satisfied then there exists $\mathfrak{K} \in \mathcal{L}^2(K, [0, +\infty))$ such that, sup $\|\Omega(u, q)\| \leq \mathfrak{K}(u)$*

Proof. Consider $q^\mu(\cdot)$ a fixed point of $\mathscr{R}$ in N_u, the equation (2.4), a fixed point solution of the system (1.1)-(1.2) with

$$w(u) = \mathfrak{B}^* R(\lambda, \Gamma_0^u) l(q(\cdot)),$$
$$where, l(q(\cdot)) = q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} - \frac{1}{\Gamma(\varphi)}\int_0^u v^{\rho-1}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} g^x(u)du.$$

following inequality

$$q^\mu(b) = q_b + \mu R(\lambda, \Gamma_0^u) l(q(\cdot)) \tag{3.6}$$

We conclude that the theorem Dun ford-Pettis theorem, $g^\mu(u)$ is weakly compact in $\mathcal{L}^2(K, Q)$, and their sub-sequence $g(u)$ is weakly compact in $\mathcal{L}^2(K, Q)$. Determine

$$\mathbb{K} = q_u - \frac{q_0}{\Gamma(\gamma)}\left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\gamma-1} + \frac{\rho^{\varphi-1}}{\Gamma(\varphi)}\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1} g^x(u)\mathrm{d}u. \tag{3.7}$$
$$\|r(q^\mu) - \mathbb{K}\| = \left\|\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}[g(^x u, q^\mu(u)) - g(u)]\mathrm{d}u\right\|$$
$$\leq sup \left\|\int_0^u \left\{\frac{u^\rho - v^\rho}{\rho}\right\}^{\varphi-1} v^{\rho-1}[g^x(u, q^\mu(u)) - g(u)]\mathrm{d}u\right\|_{\rho\in K}.$$

we can prove by using Arzela theorem,

$$\mathbb{L}(\cdot) \to \int_0^{\cdot} (\cdot - u)\Lambda(\cdot - u)\mathbf{L}(u)\mathrm{d}u : \mathcal{L}^2(K, Q) \to B(K, Q),$$

is compact therefore, $\|r(q^\mu - \mathbb{K})\| \to 0$ when $\mu \to 0+$ the equation (3.6) we have,

$$\|q^\mu(u) - q_u\| \leq \|\mu R(\mu, \Gamma_0^u)(\mathbb{K})\| + \|\mu R(\mu, \Gamma_0^u)\| \, \|r(q^\mu) - \mathbb{K}\|$$
$$\leq \|\mu R(\mu, \Gamma_0^u)(\mathbb{K})\| + \|r(q^\mu) - \mathbb{K}\|.$$

by using A_0 and the equation(3.7) such that
$\|q^\mu(u) - q_u\| \to 0$ as $\mu \to 0+$, Hence the system is approximately controllable on K. □

4 EXAMPLES

Example :1 A generalized Katugampola fractional integro-differential equation of the form

$$\begin{aligned} {}^{\varrho}\mathbb{D}^{\varphi,\varsigma}[q(u)] &\in \mathfrak{B}w(u)+\Omega\left(u,q,\int_0^{\rho}\Delta(u,a,q)\mathrm{d}a,\int_0^{\tau}\Lambda(u,a,q)\mathrm{d}a\right), u\in K=[0,u] \\ {}^{\rho}\mathbb{I}_{0^+}^{1-\gamma}q(c) &= q_0, \gamma=\varphi+\varsigma-\varphi\varsigma \end{aligned} \tag{4.1}$$

Consider the equation,

$$\begin{aligned} \mathrm{Y}(t) &= \mathfrak{B}w(u)+\Omega\left(u,q,\int_0^{\rho}\Delta(u,a,q)\mathrm{d}a,\int_0^{\tau}\Lambda(u,a,q)\mathrm{d}a\right), t\in u, u\in K=[0,u] \\ \mathrm{Y}(t) &= \frac{1}{181e^{\frac{1}{2}t-1}}\left[\frac{1+e^{\frac{1}{2}t-1}}{1+tant}+\frac{loge^t}{1+loge^{-t}}\right] \end{aligned}$$

Let $\lambda=0.5$, $\varphi=\frac{1}{2}$, $u=1$, $Z_1,Z_2=1$, $t=1$, $\beta=0.04$

$$\begin{aligned} \mathrm{Y}(t) &= \frac{1}{181e^{\frac{1}{2}t-1}}\left[\frac{1+e^{\frac{1}{2}t-1}}{1+tant}+\frac{loge^t}{1+loge^{-t}}\right] \\ \mathrm{Y}(t) &= \frac{1}{181e^{\frac{1}{2}-1}}\left[\frac{1+e^{\frac{1}{2}-1}}{1+tan(1)}+\frac{loge^1}{1+loge^{-1}}\right] \\ &\le \frac{1}{109.686}[0.628] \\ &\le 0.00573376. \end{aligned}$$

and we found that,

$$\frac{1}{\Gamma(2\varphi)}\left[\frac{Z_1Z_2}{\Gamma(2\varphi)\lambda}\left(\frac{u^{2\varphi}}{2\varphi}\right)+1\right]\beta=0.122<1$$

by theorem 3.1 the given system is approximate controllable.
Example :2
Let $\lambda=0.3$, $\varphi=\frac{1}{2}$, $u=1$, Z_1,Z_2= 1, $\beta=0.05$, $t=1$
Consider the equation is

$$\begin{aligned} \mathrm{Y}(t) &= \frac{1}{109loge^{\frac{1}{2}}t}\left[\frac{1+sin(\frac{1}{2}t-1)}{1+tant}+\frac{loge^{-2t}}{1+loge^{-t}}\right] \\ \mathrm{Y}(t) &= \frac{1}{109loge^{\frac{1}{2}}}\left[\frac{1+sin(\frac{1}{2}-1)}{1+tant}+\frac{loge^{-2}}{1+loge^{-1}}\right] \\ \mathrm{Y}(t) &= \frac{1}{54.5}\left[\frac{0.5206}{2.557}+\frac{-2}{2}\right] \\ &\le 0.24896. \end{aligned}$$

by theorem 3.1 is satisfy hypotheses, therefore,

$$\frac{1}{\Gamma(2\varphi)}\left[\frac{Z_1Z_2}{\Gamma(2\varphi)\lambda}\left(\frac{u^{2\varphi}}{2\varphi}\right)+1\right]\beta=0.1325<1.$$

Finally, determined whether that the system is close to the neighboring point within the interval. Therefore, the equation is approximate controllable on K.

5 Conclusion

This paper has investigated the approximate controllability of a Hilfer-Katugampola fractional differential system with inclusion. By employing the fixed-point technique, we have established key results on the fractional operator and demonstrated the approximate controllability of the Hilfer-Katugampola fractional derivative. Our main result provides verifiable conditions for approximate controllability, which are validated through a numerical application. The findings of this study contribute to the understanding of controllability in fractional differential systems and have potential implications for various fields, including control theory and engineering.

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Author information

A. Prabha, Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore, Tamilnadu, India.
E-mail: prabhafraction91@gmail.com

R. Nirmalkumar,,.
E-mail: nirmalkumarsrmvcas@gmail.com

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