

THE ADJACENCY SPECTRUM OF TWO VARIANTS OF RESTRICTED JOIN OF TWO GRAPHS

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Abstract The subdivision graph $S(G)$ of a graph G is the graph obtained by inserting a new vertex into each edge of G or replacing each edge by a path of length 2. Let $I(G)$ denote the set of new vertices in $S(G)$. The partial complement of $S(G)$, denoted by $\overline{S}(G)$ is the graph obtained by deleting edges from $\overline{S}(G)$ between the distinct vertices of $V(G)$ and those between the distinct vertices of $I(G)$. For vertex disjoint graphs G, H and $A \subseteq V(G), B \subseteq V(H)$, the restricted join of two graphs G and H , denoted by $G \nabla_{[A,B]} H$ is the graph obtained from disjoint copies of G and H by connecting every vertex of A to every vertex of B . The PCS_{vertex} -join of G and H is defined as $G \nabla_v H \simeq \overline{S}(G) \nabla_{[V(G), V(H)]} H$. The PCS_{edge} -join of G and H is defined as $G \nabla_e H \simeq \overline{S}(G) \nabla_{[I(G), V(H)]} H$. In this paper, the adjacency spectrum of $G \nabla_v H$ and $G \nabla_e H$ is determined. This concept is used to construct an infinite family of integral graphs.

1 Introduction

The graphs discussed in this paper are finite, simple, and undirected. Let G be a graph on s vertices $\{v_1, v_2, \dots, v_s\}$ and t edges. Here, $A(G)$ denotes the adjacency matrix of G and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s$ its eigenvalues. The eigenvalues of $A(G)$ are known as the eigenvalues of G . The spectrum of a graph G , $Spec_G$ is the sequence of eigenvalues of G that is expressed as $Spec_G = (\lambda_1^{m_1}, \lambda_2^{m_2}, \dots, \lambda_k^{m_k})$ in the sense that λ_j occurs m_j times for $j = 1, 2, \dots, k$ [2]. The energy of G is given by $\mathcal{E}_G = \sum_{i=1}^s |\lambda_i| = \sum_{j=1}^k m_j |\lambda_j|$ [1]. An integral graph is defined as a graph whose spectrum consists only of integers. The structural properties of graphs have been studied in relation to the graph spectrum [2, 3, 9]. The energy and spectrum of graphs are widely researched.

For a spanning subgraph Γ of G , its G -complement, denoted as $\overline{\Gamma}_G$ is the graph whose vertex set $V(\overline{\Gamma}_G) = V(\Gamma)$ and edge set $E(\overline{\Gamma}_G) = E(G) \setminus E(\Gamma)$. If $G = K_n$, then $\overline{\Gamma}_G = \overline{\Gamma}$. The subdivision graph $S(G)$ of a graph G is the graph formed by inserting a new vertex into each edge of G [2]. The set of new vertices in $S(G)$ is denoted by $I(G)$. Clearly $S(G)$ is isomorphic to a bipartite graph whose vertex set is bipartitioned into the nonempty sets $V(G)$ and $I(G)$. The partial complement of $S(G)$ of a graph G , denoted by $\overline{S}(G)$ is the graph obtained by taking the complement of $S(G)$ and deleting adjacency between the distinct vertices of $V(G)$ and those between the distinct vertices of $I(G)$ [4]. Let \mathcal{G} be the complete bipartite graph on $V(G) \cup I(G)$ with bipartition $V(G)$ and $I(G)$, $\overline{S}(G)$ is nothing but \mathcal{G} -complement of $S(G)$.

Let G and H be the two vertex-disjoint graphs with vertex sets $V(G)$ and $V(H)$, respectively. The join of G and H , denoted by $G \nabla H$ is the graph obtained by joining each vertex of G with every vertex of H . Let $A \subseteq V(G)$ and $B \subseteq V(H)$ be the subsets. The restricted join of G and H

denoted by $G \nabla_{[A,B]} H$ is the graph with the vertex set $V(G) \cup V(H)$ and the edge set consisting all the edges of G and H together with edges connecting each vertex in A with every vertex of B . Recently, several restricted join variations, including the subdivision-vertex join, subdivision-edge join, R -vertex join, R -edge join, central-vertex join, and central-edge join, were presented, and their spectra were examined.

Indulal introduced and studied subdivision-vertex join and subdivision-edge join of two regular graphs and determined their graph spectra [5]. In addition to constructing many families of new integral graphs, Liu and Zhang also found the Laplacian and signless Laplacian spectra of the subdivision-vertex join and subdivision-edge join of two graphs [12]. X. Liu and P. Lu introduced and determined the spectrum of subdivision-vertex neighbourhood corona and subdivision-edge neighbourhood corona of two graphs [11]. R -vertex join and R -edge join of two graphs have been introduced by Arpita Das and Pratima Panigrahi and also determined its spectrum [7]. Jie Lan and Bo Zhou studied spectrum of R -vertex corona, R -edge corona, R -vertex neighbourhood corona and R -edge neighbourhood corona of two graphs [10]. Jahfar and Chithra defined central-vertex join (CV -join) and central-edge join (CE -join) of two graphs and studied their spectra [8]. Motivated by these ideas we have defined two types of graph products $G \nabla_v H$ and $G \nabla_e H$ and determined their adjacency spectrum.

Definition 1.1. The PCS_{vertex} - join of G and H , denoted by $G \nabla_v H$ is the graph formed from a restricted join of $\overline{S}(G)$ and H with $A = V(G)$ and $B = V(H)$. That is, $G \nabla_v H = \overline{S}(G) \nabla_{[V(G),V(H)]} H$.

Definition 1.2. The PCS_{edge} - join of G and H , denoted by $G \nabla_e H$ is the graph formed from a restricted join of $\overline{S}(G)$ and H with $A = I(G)$ and $B = V(H)$. That is, $G \nabla_e H = \overline{S}(G) \nabla_{[I(G),V(H)]} H$.

In addition, an infinite family of integral graphs can be generated by applying these two joins.

2 Adjacency spectrum of the PCS_{vertex} -join of two graphs G and H

Theorem 2.1. Let G be a k_1 - regular graph having s_1 vertices, t_1 edges, and $k_1, \lambda_2, \lambda_3, \dots, \lambda_{s_1}$ be the eigenvalues of $A(G)$. Let H be a k_2 -regular graph having s_2 vertices, t_2 edges, and $k_2, \mu_2, \mu_3, \dots, \mu_{s_2}$ be the eigenvalues of $A(H)$. Then, the adjacency spectrum of $G \nabla_v H$ consists of $\pm \sqrt{\lambda_j + k_1}$, $j = 2, 3, \dots, s_1$; 0 of multiplicity $t_1 - s_1$; μ_j , $j = 2, 3, \dots, s_2$, along with the three roots of the polynomial $x^3 - k_2x^2 - (2k_1 - s_1k_1 - 2t_1 + t_1s_1 + s_1s_2)x + k_2(s_1 - 2)(t_1 - k_1)$.

Proof. The complement of the incidence matrix $R(G)$ of G is denoted by $\overline{R(G)}$. After a proper ordering of the vertices, $A(G \nabla_v H)$ has the following form

$$\begin{bmatrix} O_{s_1 \times s_1} & \overline{R(G)} & J_{s_1 \times s_2} \\ \overline{R(G)}^T & O_{t_1 \times t_1} & O_{t_1 \times s_2} \\ J_{s_2 \times s_1} & O_{s_2 \times t_1} & A(H) \end{bmatrix}$$

Let $X_1 = J_{n_1 \times 1}, X_2, \dots, X_{s_1}$ be the orthogonal set of eigenvectors of $A(G)$ corresponding to the eigenvalues $k_1 = \lambda_1, \lambda_2, \dots, \lambda_{s_1}$ respectively.

For $l = 2, 3, \dots, s_1$, $\eta = \begin{bmatrix} \sqrt{\lambda_l + k_1} X_l \\ \overline{R(G)}^T X_l \\ O \end{bmatrix}$ is an eigenvector of $A(G \nabla_v H)$ that corresponds to

the eigenvalue $\sqrt{\lambda_l + k_1}$, because

$$A(G \nabla_v H) \cdot \eta = \begin{bmatrix} O_{s_1 \times s_1} & \overline{R(G)} & J_{s_1 \times s_2} \\ \overline{R(G)}^T & O_{t_1 \times t_1} & O_{t_1 \times s_2} \\ J_{s_2 \times s_1} & O_{s_2 \times t_1} & A(H) \end{bmatrix} \cdot \begin{bmatrix} \sqrt{\lambda_l + k_1} X_l \\ \overline{R(G)}^T X_l \\ O \end{bmatrix}$$

$$\begin{aligned}
 &= \begin{bmatrix} \lambda_l + k_1 X_l \\ \sqrt{\lambda_l + k_1} \overline{R(G)}^T X_l \\ O \end{bmatrix} \\
 &= \sqrt{\lambda_l + k_1} \cdot \begin{bmatrix} \sqrt{\lambda_l + k_1} X_l \\ \overline{R(G)}^T X_l \\ O \end{bmatrix}.
 \end{aligned}$$

Similarly $\eta' = \begin{bmatrix} -\sqrt{\lambda_l + k_1} X_l \\ \overline{R(G)}^T X_l \\ O \end{bmatrix}$ is an eigenvector of $A(G \nabla_v H)$ with an eigenvalue $-\sqrt{\lambda_l + k_1}$.

By the Lemma 2.2, we have $\overline{R(G)} Z_n = 0$ where Z_n are $t_1 - s_1$ linearly independent vectors of $L(G)$ corresponding to an eigenvalue -2 , $n = 1, 2, \dots, t_1 - s_1$. As

$$\begin{aligned}
 A(G \nabla_v H) \cdot \vartheta &= \begin{bmatrix} O_{s_1 \times s_1} & \overline{R(G)} & J_{s_1 \times s_2} \\ \overline{R(G)}^T & O_{t_1 \times t_1} & O_{t_1 \times s_2} \\ J_{s_2 \times s_1} & O_{s_2 \times t_1} & A(H) \end{bmatrix} \cdot \begin{bmatrix} O \\ Z_n \\ O \end{bmatrix} \\
 &= \begin{bmatrix} O \\ O \\ O \end{bmatrix} = 0 \cdot \vartheta
 \end{aligned}$$

at this point, $\vartheta = \begin{bmatrix} O \\ Z_n \\ O \end{bmatrix}$ is an eigenvector of $A(G \nabla_v H)$ with an eigenvalue 0. Now $\nu_m \neq k_2$ is an

eigenvalue of $A(H)$ with eigenvector Y_m . For $m = 2, 3, \dots, s_2$, $\psi = \begin{bmatrix} O \\ O \\ Y_m \end{bmatrix}$ is an eigenvector of $A(G \nabla_v H)$ that corresponds to an eigenvalue ν_m , because

$$\begin{aligned}
 A \cdot \psi &= \begin{bmatrix} O_{s_1 \times s_1} & \overline{R(G)} & J_{s_1 \times s_2} \\ \overline{R(G)}^T & O_{t_1 \times t_1} & O_{t_1 \times s_2} \\ J_{s_2 \times s_1} & O_{s_2 \times t_1} & A(H) \end{bmatrix} \cdot \begin{bmatrix} O \\ O \\ Y_m \end{bmatrix} \\
 &= \begin{bmatrix} O \\ O \\ \nu_m Y_m \end{bmatrix} \\
 &= \nu_m \cdot \begin{bmatrix} O \\ O \\ Y_m \end{bmatrix}.
 \end{aligned}$$

Thus we have obtained $t_1 + s_1 + s_2 - 3$ eigenvalues. The three remaining eigenvalues need to be found using the equitable partition of a matrix. The adjacency matrix $A(G \nabla_v H)$ satisfies equitable partition with the quotient matrix is given by

$$Q_{A(G \nabla_v H)} = \begin{bmatrix} 0 & t_1 - k_1 & s_2 \\ s_1 - 2 & 0 & 0 \\ s_1 & 0 & k_2 \end{bmatrix}.$$

The characteristic equation of Q is given by $x^3 - k_2 x^2 - (2k_1 - s_1 k_1 - 2t_1 + t_1 s_1 + s_1 s_2)x + k_2(s_1 - 2)(t_1 - k_1) = 0$. The three remaining eigenvalues of $A(G \nabla_v H)$ are that of $Q_{A(G \nabla_v H)}$ [3, 6]. □

Corollary 2.2. Let G a graph on s_1 vertices and t_1 edges with regularity k_1 and $H = \overline{K_{s_2}}$. Then the adjacency spectrum of $G \nabla_v H$ consists of

- $\pm \sqrt{\lambda_j + k_1}$, $j = 2, 3, \dots, s_1$
- 0 of multiplicity $t_1 + s_2 - s_1$
- $\pm \sqrt{(s_1 - 2)(t_1 - k_1) + s_1 s_2}$.

3 Adjacency spectrum of the PCS_{edge} -join of two graphs G and H

Theorem 3.1. Let G be a k_1 -regular graph of order s_1 and size t_1 with the eigenvalues of $A(G)$ are $\lambda_1 = k_1, \lambda_2, \lambda_3, \dots, \lambda_{s_1}$. Let H be a k_2 -regular graph of order s_2 and size t_2 with eigenvalues of $A(H)$ are $\mu_1 = k_2, \mu_2, \mu_3, \dots, \mu_{s_2}$. Then, the adjacency spectrum of $G \nabla_e H$ consists of the following numbers: $\pm \sqrt{\lambda_j + k_1}$, $j = 2, 3, \dots, s_1$; 0 of multiplicity $t_1 - s_1$; μ_j , $j = 2, 3, \dots, s_2$, combined with the three roots of the polynomial

$$x^3 - k_2 x^2 - ((s_1 - 2)(t_1 - k_1) + s_2 t_1)x + k_2(s_1 - 2)(t_1 - k_1).$$

Proof. By the order of vertices of $G \nabla_e H$, its adjacency matrix of form is given by

$$\begin{bmatrix} O_{s_1 \times s_1} & \overline{R} & O_{s_1 \times s_2} \\ \overline{R}^T & O_{t_1 \times t_1} & J_{t_1 \times s_2} \\ O_{s_2 \times s_1} & J_{s_2 \times t_1} & A(H) \end{bmatrix},$$

where R is the incidence matrix of G .

Let $X_1 = J_{n_1 \times 1}, X_2, \dots, X_{s_1}$ be the orthogonal set of eigenvectors of $A(G)$ corresponding to the eigenvalues $k_1 = \lambda_1, \lambda_2, \dots, \lambda_{s_1}$ respectively.

Now, $\eta = \begin{bmatrix} \sqrt{\lambda_j + k_1} X_j \\ \overline{R}^T X_j \\ O \end{bmatrix}$, $j = 2, 3, \dots, s_1$, is an eigenvector of $A(G \nabla_e H)$ with the eigen-

value $\sqrt{\lambda_j + k_1}$. This is due to

$$\begin{aligned} A(G \nabla_e H) \cdot \eta &= \begin{bmatrix} O_{s_1 \times s_1} & \overline{R} & O_{s_1 \times s_2} \\ \overline{R}^T & O_{t_1 \times t_1} & J_{t_1 \times s_2} \\ O_{s_2 \times s_1} & J_{s_2 \times t_1} & A(H) \end{bmatrix} \cdot \begin{bmatrix} \sqrt{\lambda_j + k_1} X_j \\ \overline{R}^T X_j \\ O \end{bmatrix} = \begin{bmatrix} \lambda_j + k_1 X_j \\ \sqrt{\lambda_j + k_1} \overline{R}^T X_j \\ O \end{bmatrix} \\ &= \sqrt{\lambda_j + k_1} \cdot \begin{bmatrix} \sqrt{\lambda_j + k_1} X_j \\ \overline{R}^T X_j \\ O \end{bmatrix}. \end{aligned}$$

Similarly $\eta' = \begin{bmatrix} -\sqrt{\lambda_j + k_1} X_j \\ \overline{R}^T X_j \\ O \end{bmatrix}$ is an eigenvector of $A(G \nabla_e H)$ corresponding to the eigenvalue

$-\sqrt{\lambda_j + k_1}$.

Now, -2 is an eigenvalue of $A(L(G))$ with multiplicity $t_1 - s_1$ times. Let $Y_l, l = 1, 2, \dots, t_1 - s_1$ be a set of eigenvectors of $A(L(G))$ with the eigenvalue -2 . Then by the Lemma 1.2.34, we have $RY_l = 0$.

For $l = 1, 2, \dots, t_1 - s_1$, $\vartheta = \begin{bmatrix} O \\ Y_l \\ O \end{bmatrix}$ is an eigenvector with the eigenvalue 0. This is due to

$$A(G \nabla_e H) \cdot \vartheta = \begin{bmatrix} O_{s_1 \times s_1} & \overline{R} & O_{s_1 \times s_2} \\ \overline{R}^T & O_{t_1 \times t_1} & J_{t_1 \times s_2} \\ O_{s_2 \times s_1} & J_{s_2 \times t_1} & A(H) \end{bmatrix} \cdot \begin{bmatrix} O \\ Y_l \\ O \end{bmatrix}$$

$$= \begin{bmatrix} O \\ O \\ O \end{bmatrix} = 0.\vartheta.$$

Now $\mu_p \neq k_2$ is an eigenvalue of $A(H)$ with an eigenvector Z_p .

For $p = 2, 3, \dots, s_2$, $\psi = \begin{bmatrix} O \\ O \\ Z_p \end{bmatrix}$ is an eigenvector of $A(G \nabla_e H)$ that corresponds to the eigenvalue μ_p , because

$$\begin{aligned} A(G \nabla_e H) \cdot \psi &= \begin{bmatrix} O_{s_1 \times s_1} & \bar{R} & O_{s_1 \times s_2} \\ \bar{R}^T & O_{t_1 \times t_1} & J_{t_1 \times s_2} \\ O_{s_2 \times s_1} & J_{s_2 \times t_1} & A(H) \end{bmatrix} \cdot \begin{bmatrix} O \\ O \\ Z_p \end{bmatrix} \\ &= \begin{bmatrix} O \\ O \\ \mu_p Z_p \end{bmatrix} = \mu_p \cdot \begin{bmatrix} O \\ O \\ Z_p \end{bmatrix}. \end{aligned}$$

Thus, the $t_1 + s_1 + s_2 - 3$ eigenvalues are obtained. The adjacency matrix $\mathcal{A}(G \nabla_e H)$ satisfies equitable partition with the quotient matrix is given by

$$Q_{A(G \nabla_e H)} = \begin{bmatrix} 0 & t_1 - k_1 & 0 \\ s_1 - 2 & 0 & s_2 \\ 0 & t_1 & k_2 \end{bmatrix}.$$

The characteristic equation of $Q_{A(G \nabla_e H)}$ is given by $x^3 - k_2x^2 - ((s_1 - 2)(t_1 - k_1) + s_2t_1)x + k_2(s_1 - 2)(t_1 - k_1) = 0$. Then according to the Lemma 1.2.30, the remaining three eigenvalues of $A(G \nabla_e H)$ are those of $Q_{A(G \nabla_e H)}$. \square

Corollary 3.2. Let G be a graph of order s_1 and size t_1 with regularity k_1 and $H = \overline{K_{s_2}}$. Then the following make up the adjacency spectrum of $G \nabla_e H$

- $\pm \sqrt{\lambda_j + k_1}$, $j = 2, 3, \dots, s_1$
- 0 of multiplicity $t_1 + s_2 - s_1$
- $\pm \sqrt{(s_1 - 2)(t_1 - k_1) + s_2t_1}$

4 A class of integral graphs

Theorem 4.1. Let G be a k_1 -regular graph with s_1 vertices and t_1 edges and be the eigen values of the adjacency matrix of G , $\lambda_1 = k_1, \lambda_2, \lambda_3, \dots, \lambda_{s_1}$. Let H be a k_2 -regular graph with s_2 vertices and t_2 and be the eigenvalues of the adjacency matrix of H , $\mu_1 = k_2, \mu_2, \mu_3, \dots, \mu_{s_2}$. Then, the $G \nabla_v H$ is integral if and only if $\pm \sqrt{\lambda_j + k_1}$, $j = 2, 3, \dots, s_1$, the three roots of the cubic equation $x^3 - k_2x^2 - ((s_1 - 2)(t_1 - k_1) + s_1s_2)x + k_2(s_1 - 2)(t_1 - k_1) = 0$ are integers and H is integral.

Theorem 4.2. Let G be a k_1 -regular graph with s_1 vertices and t_1 edges and be the eigenvalues of the adjacency matrix of G , $\lambda_1 = k_1, \lambda_2, \lambda_3, \dots, \lambda_{s_1}$. Let H be a k_2 -regular graph with s_2 vertices and t_2 edges and be the eigenvalues of the adjacency matrix of H , $\mu_1 = k_2, \mu_2, \mu_3, \dots, \mu_{s_2}$. Then, the $G \nabla_e H$ is integral if and only if $\pm \sqrt{\lambda_j + k_1}$, $j = 2, 3, \dots, s_1$, the three roots of the cubic equation $x^3 - k_2x^2 - ((s_1 - 2)(t_1 - k_1) + s_2t_1)x + k_2(s_1 - 2)(t_1 - k_1) = 0$ are integers and H is integral.

Specifically, if $H = \overline{K_{s_2}}$ then $k_2 = 0$; thereby using the Corollaries 2.2 and 3.2, we get these theorems.

Theorem 4.3. $G\nabla_v \overline{K_{s_2}}$ is integral if and only if $\pm\sqrt{\lambda_j + k_1}$, $j = 2, 3, \dots, s_1$ and $\pm\sqrt{(s_1 - 2)(t_1 - k_1) + s_1 s_2}$ are integers.

Theorem 4.4. $G\nabla_e \overline{K_{s_2}}$ is integral if and only if $\pm\sqrt{\lambda_j + k_1}$, $j = 2, 3, \dots, s_1$ and $\pm\sqrt{(s_1 - 2)(t_1 - k_1) + s_2 t_1}$ are integers.

The following Propositions provide infinite families of integral graphs of these two joins.

Proposition 4.5. Let \mathbf{P} denote the Petersen graph, $r \equiv 4$ or $6 \pmod{10}$ and $s_2 = \frac{r^2 - 96}{10}$. Then $P\nabla_v \overline{K_{s_2}}$ is integral.

Proposition 4.6. Let \mathbf{P} denote the Petersen graph and r be an integer such that $s_2 = \frac{r^2 - 96}{15}$. Then $P\nabla_e \overline{K_{s_2}}$ is integral.

Remark 4.7.

- (i) $P\nabla_v \overline{K_n}$ is integral if $n = 10, 16, 48, 58, 106, \dots$.
- (ii) $P\nabla_e \overline{K_n}$ is integral if $n = 23, 32, 80, 95, 167, 188, \dots$.

Proposition 4.8. $K_{s_1} \nabla_v \overline{K_{s_2}}$ is integral if and only if there are two integers t and r such that $s_1 = t^2 + 2$ and $s_2 = \frac{2r^2 - (t^6 + t^4)}{2(t^2 + 2)}$.

Proposition 4.9. $K_{s_1} \nabla_e \overline{K_{s_2}}$ is integral if and only if there are two integers t and r such that $s_1 = t^2 + 2$ and $s_2 = \frac{2r^2 - (t^6 + t^4)}{t^4 + 3t^2 + 2}$.

Remark 4.10.

- (i) The possible values of s_2 in $K_3 \nabla_v \overline{K_{s_2}}$ are $s_2 = 3r^2 + 2r$ or $3r^2 + 2r + 1$ where r is a positive integer.
- (ii) The possible values of s_2 in $K_4 \nabla_e \overline{K_{s_2}}$ are $s_2 = 15r^2 + 10r - 1$ or $15r^2 + 20r + 4$ where r is a positive integer.

Proposition 4.11. $K_{s_1, s_1} \nabla_v \overline{K_{s_2}}$ is integral if and only if there are two integers t and r such that $s_1 = t^2$ and $s_2 = \frac{r^2 - 2t^2(t^2 - 1)^2}{2t^2}$.

Proposition 4.12. $K_{s_1, s_1} \nabla_e \overline{K_{s_2}}$ is integral if and only if there are two integers t and r such that $s_1 = t^2$ and $s_2 = \frac{r^2 - 2t^2(t^2 - 1)^2}{t^4}$.

5 Conclusion

The new joins of graphs called the PCS_{vertex} -join and the PCS_{edge} -join are defined and their spectra are obtained. The spectrum of these joins is necessary in order to construct new infinite families of integral graphs. The following research will concentrate on identifying the Laplacian and the signless Laplacian spectrum of these two joins.

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