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On three dimensional locally semisymmetric almost Ricci soliton

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Abstract *The object of the present paper is to characterize the locally ϕ- semisymmetry in three dimensional almost Ricci soliton trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. Locally Ricci semisymmetric, Locally projectively ϕ-semisymmetric and Locally conformally ϕ- semisymmetric almost Ricci soliton trans-Sasakian manifolds of dimension three have also been studied with respect to generalized Tanaka Webster Okumura connection.*

1 Introduction

The notion of almost Ricci soliton was introduced by Pigola et. al.[31], where they modified the concept of Ricci soliton by adding the condition on the parameter λ to be a variable function, more precisely, we say that a Riemannian manifold (M^n, g) admits an almost Ricci soliton, if there exists a complete vector field V , called potential vector field and a smooth soliton function $\lambda : M^n \to R$ satisfying

$$Ric + \frac{1}{2}\pounds_V g = \lambda g \tag{1.1}$$

where Ric and £ stand, respectively, for the Ricci tensor and Lie derivative. We shall refer to this equation as the fundamental equation of an almost Ricci soliton (M^n, g, V, λ). It will be called expanding, steady or shrinking, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Otherwise it will be called indefinite. For some related study see [27] and [33].

Let M be an n-dimensional, $n \geq 3$, connected smooth Riemannian manifold endowed with the Riemannian metric g. Let ∇, R, S and r be the Levi-Civita connection, curvature tensor, Ricci tensor and the scalar curvature of M respectively. The manifold M is called locally symmetric due to cartan ([7], [8]) if the global geodesic symmetry at $p \in M$ is an isometry, which is equivalent to the fact that $\nabla R = 0$. Generalizing the concept of local symmetry , the notion of semisymmetry was introduced by Cartan [9] and fully classified by Szabo ([28], [29], [30]). The manifold M is said to be semisymmetric if

$$(R(U,V).R)(X,Y)Z = 0, \tag{1.2}$$

for all vector fields X, Y, Z, U, V on M, where $R(U,V)$ is considered as the derivation of the tensor algebra at each point of M.

In 1977 Takahashi [32] introduced the notion of local ϕ- symmetry on Sasakian manifolds. A Sasakian manifold is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{1.3}$$

for any vector fields X, Y, Z, W on M, where ϕ is the structure tensor of the manifold M. The concept of locally ϕ- symmetry on various structures and their generalizations or extension *are studied in ([11], [22], [23], [24], [25]). By extending the notion of semisymmetry and generalizing the concept of local ϕ- symmetry of Takahashi [32] the authors in the paper [26] introduced the notion of locally ϕ-semisymmetric Sasakian manifolds. A Sasakian manifold M, $n \geq 3$, is said to be locally ϕ-semisymmetric if,*

$$\phi^2\{(R(U,V).R)(X,Y)Z\} = 0, \tag{1.4}$$

for all horizontal vector fields X, Y, Z, U, V on M.
In 1985 J. A. Oubina [19] introduced a new class of almost contact metric manifolds, called trans-Sasakian manifold, which includes Sasakian, Kenmotso and Cosymplectic structures. The authors in the papers [2], [3], [6] and [10] have studied such manifolds and obtained some interesting results . It is known that [15] trans-Sasakian structure of type (0, 0), (0, β) and (α, 0) are Cosymplectic, β-Kenmotsu and α-Sasakian respectively. The local classification of trans-Sasakian manifold is given by J. C. Marrero [18] and it is proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either Cosymplectic or α-Sasakian or β-Kenmotsu manifold. In the present paper we have studied three dimensional trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. The notion of generalized Tanaka Webster Okumura connection was introduced and studied by the authors in the paper [14]. In the present paper we have studied three dimensional locally ϕ-semisymmetric trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. The present paper is organized as follows:

After introduction in Section 1 we give some preliminaries in Section 2. Section 3 is devoted to the study of three dimensional locally Ricci semi-symmetric almost Ricci soliton trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. Sections 4 and 5 are devoted to the study of three dimensional locally ϕ- semisymmetric and locally projectively ϕ- semisymmetric Ricci soliton trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection. Finally, Section 6 is completed with the study of three dimensional locally conformally ϕ- semisymmetric Ricci soliton trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection.

2 Preliminaries

Let M be a $(2n+1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on M such that [5]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1. \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \quad X, Y \in T(M) \tag{2.2}$$

Then also

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi). \tag{2.3}$$

$$g(\phi X, X) = 0. \tag{2.4}$$

$$g(\phi X, Y) = -g(X, \phi Y) \tag{2.5}$$

An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a trans-Sasakian manifold [19] if ($M^{2n+1} \times R, J, G$) belongs to the class W_4 [13] of the Hermitian manifolds, where J is the almost complex structure on $M^{2n+1} \times R$ defined by [12]

$$J(Z, f\frac{d}{dt}) = (\phi Z - f\xi, \eta(Z)\frac{d}{dt}), \tag{2.6}$$

for any vector field Z on M^{2n+1} and smooth function f on $M^{2n+1} \times R$ and G is the Hermitian metric on the product $M^{2n+1} \times R$. This may be expressed by the condition [19]

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.7}$$

for some smooth functions α and β on M^{2n+1}, and we say that the trans-Sasakian structure is of type (α, β). From equation (2.7), it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.8}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y)\xi + \beta g(\phi X, \phi Y). \tag{2.9}$$

The Ricci operator, Ricci tensor and curvature tensor for three dimensional trans-Sasakian manifolds have been studied in the paper [10]. We have the following for three dimensional trans-Sasakian manifolds[10]:

$$2\alpha\beta + \xi\alpha = 0, \tag{2.10}$$

$$S(X,\xi) = \{2(\alpha^2 - \beta^2)\}\eta(X) - \xi - (\phi X)\alpha\beta, \tag{2.11}$$

$$\begin{aligned} S(X,Y) &= \{\tfrac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\}g(X,Y) \\ &- \{\tfrac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\}\eta(X)\eta(Y) \\ &- \{Y\beta + (\phi Y)\alpha\}\eta(X) - \{X\beta + (\phi X)\alpha\}\eta(Y), \end{aligned} \tag{2.12}$$

$$\begin{aligned} R(X,Y)Z &= \{\tfrac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)\}\{g(Y,Z)X - g(X,Z)Y\} \\ &- g(Y,Z)[\{\tfrac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\}\eta(X)\xi \\ &- \eta(X)(\phi grad\alpha - grad\beta) + \{X\beta + (\phi X)\alpha\}\xi] \\ &+ g(X,Z)[\{\tfrac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\}\eta(Y)\xi \\ &- \eta(Y)(\phi grad\alpha - grad\beta) + \{Y\beta + (\phi Y)\alpha\}\xi] \\ &- [\{Z\beta + (\phi Z)\alpha\}\eta(Y) + \{Y\beta + (\phi Y)\alpha\}\eta(Z) \\ &+ \{\tfrac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\}\eta(Y)\eta(Z)]X \\ &+ [\{Z\beta + (\phi Z)\alpha\}\eta(X) + \{X\beta + (\phi X)\alpha\}\eta(Z) \\ &+ \{\tfrac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\}\eta(X)\eta(Z)]Y, \end{aligned} \tag{2.13}$$

where S is the Ricci tensor of type $(0,2)$, R is the curvature tensor of type $(1,3)$ and r is the scalar curvature of the manifold M.
Now if X, Y and Z are orthogonal to ξ, then relations (2.12) and (2.13) are changes to

$$S(X,Y) = \{\tfrac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\}g(X,Y), \tag{2.14}$$

$$\begin{aligned} R(X,Y)Z &= \{\tfrac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)\}\{g(Y,Z)X - g(X,Z)Y\} \\ &- g(Y,Z)\{X\beta + (\phi X)\alpha\}\xi + g(X,Z)\{Y\beta + (\phi Y)\alpha\}\xi. \end{aligned} \tag{2.15}$$

Again from (2.15) we have

$$R(\xi,Y)Z = \{\tfrac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)\}g(Y,Z)\xi - g(Y,Z)(X\beta)\xi. \tag{2.16}$$

$$R(X,Y)\xi = 0, \tag{2.17}$$

where X, Y and Z are orthogonal to ξ.
The generalized Tanaka Webster Okumura connection [14] $\tilde{\nabla}$ and the Levi-Civita connection ∇ are related by

$$\tilde{\nabla}_X Y = \nabla_X Y + A(X,Y), \tag{2.18}$$

for all vectors fields X, Y on M. Here

$$A(X,Y) = \alpha\{g(X,\phi Y)\xi + \eta(Y)\phi X\} + \beta\{g(X,Y)\xi - \eta(Y)X\} - l\eta(X)\phi Y, \tag{2.19}$$

where l is a real constant. The Torsion $\tilde{T}$ of the gTWO-connection $\tilde{\nabla}$ is given by

$$\tilde{T}(X,Y) = \alpha\{2g(X.\phi Y)\xi - \eta(X)\phi Y + \eta(Y)\phi X\} + \eta(X)(\beta Y - l\phi Y) - \eta(Y)(\beta X - l\phi X). \tag{2.20}$$

Again relation between the curvature tensors $\tilde{R}$ and R with respect to the generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ respectively is given by [1]

$$\begin{aligned}
& \tilde{R}(X,Y)Z = R(X,Y)Z + \alpha\{g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z) + g(X,\phi\nabla_Y Z)\xi \\
+ & \eta(\nabla_Y Z)\phi X - g(Y,\phi\nabla_X Z)\xi - \eta(\nabla_X Z)\phi Y - \eta(Z)\phi[X,Y]\} \\
+ & \beta\{\eta(\nabla_X Z)Y - \eta(\nabla_Y Z)X + \eta(Z)[X,Y]\} \\
- & l\{\eta(X)\phi\nabla_Y Z + \eta(Y)\phi\nabla_X Z + \eta([X,Y])\phi Z\} \\
+ & \{\alpha g(Y,\phi Z) + \beta g(Y,Z)\}\{\nabla_X\xi + \alpha\phi X + \beta(\eta(X)\xi - X)\} \\
- & \{\alpha g(X,\phi Z) + \beta g(X,Z)\}\{\nabla_Y\xi + \alpha\phi Y + \beta(\eta(Y)\xi - Y)\} \\
+ & (\alpha\phi Y - \beta Y)[\nabla_X\eta(Z) + \alpha\{g(X,\phi\eta(Z))\xi + \eta(\eta(Z))\phi X\} \\
+ & \beta\{g(X,\eta(Z))\xi - \eta(\eta(Z))X\} - l\eta(X)\phi\eta(Z)] \\
- & (\alpha\phi X - \beta X)[\nabla_Y\eta(Z) + \alpha\{g(Y,\phi\eta(Z))\xi + \eta(\eta(Z))\phi Y\} \\
+ & \beta\{g(Y,\eta(Z))\xi - \eta(\eta(Z))Y\} - l\eta(Y)\phi\eta(Z)] \\
+ & \alpha\eta(Z)[\nabla_X\phi Y - \nabla_Y\phi X + \beta\{g(X,\phi Y) - g(Y,\phi X)\}\xi + l\{\eta(X)Y - \eta(Y)X\}] \\
- & \beta\eta(Z)[\nabla_X Y - \nabla_Y X + \alpha\{g(X,\phi Y)\xi + \eta(Y)\phi X - g(Y,\phi X)\xi \\
+ & \eta(X)\phi Y\} - \beta\{\eta(Y)X - \eta(X)Y\} + l\{\eta(Y)\phi X - \eta(X)\phi Y\}] \\
- & l\eta(Y)[\nabla_X\phi Z + \alpha\{\eta(X)\eta(Z) - g(X,Z)\}\xi + \beta g(X,\phi Z)\xi \\
+ & l\{\eta(X)Z - \eta(X)\eta(Z)\xi\}] + l\eta(X)[\nabla_Y\phi Z + \alpha\{\eta(Y)\eta(Z) - g(Y,Z)\}\xi \\
+ & \beta g(Y,\phi Z)\xi + l\{\eta(Y)Z - \eta(Y)\eta(Z)\xi\}] \\
- & l\phi Z[\nabla_X\eta(Y) - \nabla_Y\eta(X) + \alpha\{g(X,\phi\eta(Y))\xi + \eta(\eta(Y))\phi X) - g(Y,\phi\eta(X))\xi \\
- & \eta(\eta(X))\phi Y\} + \beta\{g(X,\eta(Y))\xi - \eta(\eta(Y))X - g(Y,\eta(X))\xi + \eta(\eta(X))Y\} \\
+ & l\{\eta(Y)\phi\eta(X) - \eta(X)\phi\eta(Y)\}].
\end{aligned} \tag{2.21}$$

We suppose that X, Y, Z, $\nabla_X Z$ and $\nabla_Y Z$ are orthogonal to ξ. Then (2.21) becomes

$$\begin{aligned}
\tilde{R}(X,Y)Z &= R(X,Y)Z + \alpha\{g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z) + g(X,\phi\nabla_Y Z) \\
&- g(Y,\phi\nabla_X Z)\}\xi + \beta g([X,Y],Z)\xi \\
&+ \{\alpha g(Y,\phi Z) + \beta g(Y,Z)\}\{\nabla_X\xi + \alpha\phi X + \beta - X\} \\
&- \{\alpha g(X,\phi Z) + \beta g(X,Z)\}\{\nabla_Y\xi + \alpha\phi Y + \beta - Y\},
\end{aligned} \tag{2.22}$$

3 Locally Ricci semi-symmetric three dimensional almost Ricci soliton trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection

In this section we have studied locally Ricci semi-symmetric almost Ricci soliton trans-Sasakian manifolds of dimension three with respect to generalized Tanaka Webster Okumura connection.

Definition 3.1. A trans-Sasakian manifold of dimension three is said to be locally Ricci-semisymmetric with respect to generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ if

$$(\tilde{R}(U,V).\tilde{S})(X,Y) = 0, \tag{3.1}$$

for all horizontal vector fields X, Y, U, V on M.

Thus for a trans-Sasakian manifold of dimension three we obtain by using (2.8) in (2.22)

$$\begin{aligned}
\tilde{R}(X,Y)Z &= R(X,Y)Z + \alpha\{g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z) + g(X,\phi\nabla_Y Z) \\
&- g(Y,\phi\nabla_X Z)\}\xi + \beta g([X,Y],Z)\xi,
\end{aligned} \tag{3.2}$$

where X, Y and Z are orthogonal to ξ.
Applying ϕ^2 on both side of (3.2) we get

$$\phi^2\{\tilde{R}(X,Y)Z\} = \phi^2\{R(X,Y)Z\} \tag{3.3}$$

Now taking inner product on both side of (3.2) by W we obtain

$$g(\tilde{R}(X,Y)Z,W) = g(R(X,Y)Z,W). \tag{3.4}$$

From relation (3.4) we obtain

$$\tilde{S}(X,W) = S(X,W), \tag{3.5}$$

where W is orthogonal to ξ.
We have the following for almost Ricci soliton (1.1)

$$Ric + \frac{1}{2}\pounds_V g = \lambda g \tag{3.6}$$

In view of (3.5) and (3.6) we get

$$\tilde{S}(X,Y) = (\lambda - \beta)g(X,Y) + \beta\eta(X)\eta(Y) \tag{3.7}$$

If X and Y are orthogonal to ξ then we have from (3.7)

$$\tilde{S}(X,Y) = Ag(X,Y) \tag{3.8}$$

where $A = (\lambda - \beta) = W(r)$
Again we know that the Ricci operator $\tilde{Q}$ with generalized Tanaka Webster Okumura connection is given by

$$\tilde{S}(X,Y) = g(\tilde{Q}X,Y). \tag{3.9}$$

Thus combining (3.8) and (3.9) we get

$$\tilde{Q}X = AX. \tag{3.10}$$

We know that

$$(\tilde{R}(U,V).\tilde{S})(X,Y) = (\tilde{R}(U,V).S)(X,Y). \tag{3.11}$$

$$(\tilde{R}(U,V).\tilde{S})(X,Y) = -S(\tilde{R}(U,V)X,Y) - S(X,\tilde{R}(U,V)Y) \tag{3.12}$$

In view of (2.14), (3.2) and (3.8) we get from (3.12)

$$(\tilde{R}(U,V).\tilde{S})(X,Y) = -A\{g(R(U,V)X,Y) + g(X,R(U,V)Y)\} \tag{3.13}$$

Again using (2.15) in relation (3.13) we get

$$\begin{aligned}(\tilde{R}(U,V).\tilde{S})(X,Y) &= -A\{\tfrac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)\}[g(V,X)g(U,Y) \\ &- g(U,X)g(V,Y) + g(U,X)g(V,Y) - g(V,X)g(U,Y)] \\ &= 0.\end{aligned} \tag{3.14}$$

Thus we are in a position to state the following:

Theorem 3.2. *An almost Ricci soliton trans-Sasakian manifolds of dimension three is locally Ricci semisymmetric with respect to generalized Tanaka Webster Okumura connection.*

4 Locally ϕ-semisymmetric three dimensional almost Ricci soliton trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection

In this section we have studied locally ϕ-semisymmetric almost Ricci soliton trans-Sasakian manifolds of dimension three with respect to generalized Tanaka Webster Okumura connection.

Definition 4.1. A trans-sasakian manifold of dimension three is said to be locally ϕ-semisymmetric with respect to generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ if

$$\phi^2\{(\tilde{R}(U,V).\tilde{R})(X,Y)Z\} = 0, \tag{4.1}$$

for all horizontal vector fields X, Y, Z, U, V on M.

Again we have that

$$
\begin{aligned}
[(\tilde{R}(U,V).\tilde{R})(X,Y)Z] &= \tilde{R}(U,V)\tilde{R}(X,Y)Z - \tilde{R}(\tilde{R}(U,V)X,Y)Z - \tilde{R}(X,\tilde{R}(U,V)Y)Z \\
&- \tilde{R}(X,Y)\tilde{R}(U,V)Z
\end{aligned} \tag{4.2}
$$

Using (3.2) and (3.8) in first and last terms of right hand side of (4.2) we get

$$
\begin{aligned}
\tilde{R}(U,V)\tilde{R}(X,Y)Z &= W(r)R(U,V)R(X,Y)Z + [\alpha\{g(V,\nabla_U\phi\tilde{R}(X,Y)Z) \\
&- g(U,\nabla_V\phi\tilde{R}(X,Y)Z) + g(U,\phi\nabla_V\tilde{R}(X,Y)Z) \\
&- g(V,\phi\nabla_U\tilde{R}(X,Y)Z)\} + \beta g([U,V],\tilde{R}(X,Y)Z)]\xi,
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
\tilde{R}(X,Y)\tilde{R}(U,V)Z &= W(r)R(X,Y)R(U,V)Z + [\alpha\{g(Y,\nabla_X\phi\tilde{R}(U,V)Z) \\
&- g(X,\nabla_Y\phi\tilde{R}(U,V)Z) + g(X,\phi\nabla_Y\tilde{R}(U,V)Z) \\
&- g(Y,\phi\nabla_X\tilde{R}(U,V)Z)\} + \beta g([X,Y],\tilde{R}(U,V)Z)]\xi.
\end{aligned} \tag{4.4}
$$

Similarly, using (3.2) and (3.8) in second and third terms of right hand side of (4.2) we get

$$
\begin{aligned}
\tilde{R}(\tilde{R}(U,V)X,Y)Z &= W(r)R(R(U,V)X,Y)Z + [\alpha\{g(V,\nabla_U\phi X) - g(U,\nabla_V\phi X) \\
&+ g(U,\phi\nabla_V X) - g(V,\phi\nabla_U X)\} + \beta g([U,V],X)]R(\xi,Y)Z \\
&+ [\alpha\{g(Y,\nabla_{\tilde{R}(U,V)X}\phi Z) - g(\tilde{R}(U,V)X,\nabla_Y\phi Z) \\
&+ g(\tilde{R}(U,V)X,\phi\nabla_Y Z) - g(Y,\phi\nabla_{\tilde{R}(U,V)X}Z\} \\
&+ \beta g([\tilde{R}(U,V)X,Y],Z)]\xi,
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
\tilde{R}(X,\tilde{R}(U,V)Y)Z &= -\tilde{R}(\tilde{R}(U,V)Y,X)Z = -W(r)R(R(U,V)Y,X)Z \\
&- [\alpha\{g(V.\nabla_U\phi Y) - g(U,\nabla_V\phi Y) + g(U,\phi\nabla_V Y) - g(V,\phi\nabla_U Y)\} \\
&+ \beta g([U,V],Y)]R(\xi,X)Z - [\alpha\{g(X,\nabla_{\tilde{R}(U,V)Y}\phi Z) \\
&- g(\tilde{R}(U,V)Y,\nabla_X\phi Z) + g(\tilde{R}(U,V)Y,\phi\nabla_X Z) - g(X,\phi\nabla_{\tilde{R}(U,V)Y}Z\} \\
&+ \beta g([\tilde{R}(U,V)Y,X],Z)]\xi.
\end{aligned} \tag{4.6}
$$

Using (4.3), (4.4), (4.5) and (4.6) in relation (4.2) and by straight forward calculation we get

$$
\begin{aligned}
[(\tilde{R}(U,V).\tilde{R})(X,Y)Z] &= W(r)[(R(U,V).(X,Y)Z] \\
&+ [\alpha\{g(V,\nabla_U\phi\tilde{R}(X,Y)Z) - g(U,\nabla_V\phi\tilde{R}(X,Y)Z) \\
&+ g(U,\phi\nabla_V\tilde{R}(X,Y)Z) - g(V,\phi\nabla_U\tilde{R}(X,Y)Z)\} \\
&+ \beta g([U,V],\tilde{R}(X,Y)Z)]\xi \\
&+ [\alpha\{g(V.\nabla_U\phi X) - g(U,\nabla_V\phi X) \\
&+ g(U,\phi\nabla_V X) - g(V,\phi\nabla_U X)\} \\
&+ \beta g([U,V],X)]R(\xi,Y)Z \\
&+ [\alpha\{g(Y,\nabla_{\tilde{R}(U,V)X}\phi Z) - g(\tilde{R}(U,V)X,\nabla_Y\phi Z) \\
&+ g(\tilde{R}(U,V)X,\phi\nabla_Y Z) - g(Y,\phi\nabla_{\tilde{R}(U,V)X}Z\} \\
&+ \beta g([\tilde{R}(U,V)X,Y],Z)]\xi \\
&- [\alpha\{g(V.\nabla_U\phi Y) - g(U,\nabla_V\phi Y) \\
&+ g(U,\phi\nabla_V Y) - g(V,\phi\nabla_U Y)\} \\
&+ \beta g([U,V],Y)]R(\xi,X)Z \\
&- [\alpha\{g(X,\nabla_{\tilde{R}(U,V)Y}\phi Z) - g(\tilde{R}(U,V)Y,\nabla_X\phi Z) \\
&+ g(\tilde{R}(U,V)Y,\phi\nabla_X Z) - g(X,\phi\nabla_{\tilde{R}(U,V)Y}Z\} \\
&+ \beta g([\tilde{R}(U,V)Y,X],Z)]\xi \\
&+ [\alpha\{g(Y,\nabla_X\phi\tilde{R}(U,V)Z) - g(X,\nabla_Y\phi\tilde{R}(U,V)Z) \\
&+ g(X,\phi\nabla_Y\tilde{R}(U,V)Z) - g(Y,\phi\nabla_X\tilde{R}(U,V)Z)\} \\
&+ \beta g([X,Y],\tilde{R}(U,V)Z)]\xi.
\end{aligned} \tag{4.7}
$$

Applying ϕ^2 on both side of (4.7) we get

$$\begin{array}{rcl}\phi^2[(\tilde{R}(U,V).\tilde{R})(X,Y)Z] & = & W(r)\phi^2[(R(U,V).(X,Y)Z] \\ & + & [\alpha\{g(V.\nabla_U\phi X)-g(U,\nabla_V\phi X) \\ & + & g(U,\phi\nabla_V X)-g(V,\phi\nabla_U X)\} \\ & + & \beta g([U,V],X)]\{\phi^2 R(\xi,Y)Z\} \\ & - & [\alpha\{g(V.\nabla_U\phi Y)-g(U,\nabla_V\phi Y) \\ & + & g(U,\phi\nabla_V Y)-g(V,\phi\nabla_U Y)\} \\ & + & \beta g([U,V],Y)]\{\phi^2 R(\xi,X)Z\}\end{array} \tag{4.8}$$

Using (2.16) in (4.8) we get

$$\phi^2[(\tilde{R}(U,V).\tilde{R})(X,Y)Z] = W(r)\phi^2[(R(U,V).(X,Y)Z] \tag{4.9}$$

Thus we are in a position to state the following:

Theorem 4.2. *An almost Rici soliton trans-Sasakian manifold of dimension three is locally ϕ-semisymmetric with respect to generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ if and only if it is so with respect to Levi-civita connection ∇.*

5 Locally projectively ϕ-semisymmetric three dimensional almost Ricci soliton trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection

In this section we have studied locally projectively ϕ- semisymmetric almost Ricci soliton trans-Sasakian manifolds of dimension three with respect to generalized Tanaka Webster Okumura connection.

Definition 5.1. For a (2n+1)- dimensional $(n > 1)$ manifold the Weyl projective curvature tensor $\tilde{P}$ with respect to generalized Tanaka Webster Okumura connection will be given by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{2n}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y]. \tag{5.1}$$

In view of (3.2) and (3.8) we get from (5.1)

$$\begin{array}{rcl}\tilde{P}(X,Y)Z & = & R(X,Y)Z+\alpha\{g(Y,\nabla_X\phi Z)-g(X,\nabla_Y\phi Z)+g(X,\phi\nabla_Y Z) \\ & - & g(Y,\phi\nabla_X Z)\}\xi+\beta g([X,Y],Z)\xi-\frac{A}{2n}\{g(Y,Z)X-g(X,Z)Y\},\end{array} \tag{5.2}$$

Definition 5.2. A (2n+1)-dimensional (n>1) trans-Sasakian manifold is said to be a Locally projectively ϕ-semisymmetric with respect to generalized Tanaka Webster Okumura connection if the relation

$$\phi^2[(\tilde{R}(X,Y).\tilde{P})(Z,U)V] = 0, \tag{5.3}$$

holds for all horizontal vector fields X, Y, Z, U and V on M.

Let M be a 3-dimensional connected trans-Sasakian manifold. Now we know that

$$\begin{array}{rcl}[(\tilde{R}(X,Y).\tilde{P})(Z,U)V] & = & \tilde{R}(X,Y)\tilde{P}(Z,U)V-\tilde{P}(\tilde{R}(X,Y)Z,U)V-\tilde{P}(Z,\tilde{R}(X,Y)U)V \\ & - & \tilde{P}(Z,U)\tilde{R}(X,Y)V\end{array} \tag{5.4}$$

Using (3.2) and (5.2) in the first term of right hand side of (5.4) we have

$$\begin{array}{rcl}\tilde{R}(X,Y)\tilde{P}(Z,U)V & = & R(X,Y)R(Z,U)V+\alpha\{g(Y,\nabla_X\phi R(Z,U)V) \\ & - & g(X,\nabla_Y\phi R(Z,U)V)+g(X,\phi\nabla_Y R(Z,U)V) \\ & - & g(Y,\phi\nabla_X R(Z,U)V)\}\xi+\beta g([X,Y],R(Z,U)V)\xi \\ & + & \alpha\{g(U,\nabla_Z\phi V)-g(Z,\nabla_U\phi V) \\ & + & g(Z,\phi\nabla_U V)-g(U,\phi\nabla_Z V)\}\xi+\beta g([Z,U],V)\xi \\ & - & \frac{A}{2n}[g(U,V)\tilde{R}(X,Y)Z-g(Z,V)\tilde{R}(X,Y)U]\end{array} \tag{5.5}$$

Using (3.2), (5.1) and (5.2) in the second term of right hand side of (5.4) we get

$$
\begin{aligned}
\tilde{P}(\tilde{R}(X,Y)Z,U)V &= R(R(X,Y)Z,U)V \\
&+ [\alpha\{g(U,\nabla_{R(X,Y)Z}\phi V) - g(R(X,Y)Z,\nabla_U\phi V) \\
&+ g(R(X,Y)Z,\phi\nabla_U V) - g(U,\phi\nabla_{R(X,Y)Z})V\} \\
&+ \beta g([R(X,Y)Z,U],V)]\xi \\
&+ [\alpha\{g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z) + g(X,\phi\nabla_Y Z) \\
&- g(Y,\phi\nabla_X Z)\} + \beta g([X,Y],Z)]\tilde{R}(\xi,U)V \\
&- \frac{A}{2n}[g(U,V)\tilde{R}(X,Y)Z - g(R(X,Y)Z,V)U].
\end{aligned} \tag{5.6}
$$

Similarly, using (3.2), (5.1) and (5.2) in the third and fourth terms of right hand side of (5.4) we have the following:

$$
\begin{aligned}
\tilde{P}(Z,\tilde{R}(X,Y)U)V &= R(Z,R(X,Y)U)V \\
&+ [\alpha\{g(R(X,Y)Z,\nabla_Z\phi V) - g(Z,\nabla_{R(X,Y)Z}\phi V \\
&- g(R(X,Y)Z,\phi\nabla_Z V) + g(Z,\phi\nabla_{R(X,Y)Z}V\} \\
&+ \beta g([R(X,Y)Z,U],V)]\xi \\
&+ [\alpha\{g(Y,\nabla_X\phi U) - g(X,\nabla_Y\phi U) + g(X,\phi\nabla_Y U) \\
&- g(Y,\phi\nabla_X U)\} + \beta g([X,Y],U)]\tilde{R}(Z,\xi)V \\
&- \frac{A}{2n}[g(R(X,Y)U,V)Z - g(Z,V)\tilde{R}(X,Y)U]
\end{aligned} \tag{5.7}
$$

$$
\begin{aligned}
\tilde{P}(Z,U)\tilde{R}(X,Y)V &= R(Z,U)R(X,Y)V \\
&+ \alpha\{g(U,\nabla_Z\phi R(X,Y)V) - g(Z,\nabla_U\phi R(X,Y)V) \\
&+ g(Z,\phi\nabla_U R(X,Y)V) - g(U,\phi\nabla_Z R(X,Y)V)\}\xi \\
&+ \beta g([Z,U],R(X,Y)V)\xi \\
&- \frac{A}{2n}\{g(U,R(X,Y)V)Z - g(Z,R(X,Y)V)U\}.
\end{aligned} \tag{5.8}
$$

In view of (5.5), (5.6), (5.7) and (5.8) we get from (5.4) by straight forward calculation

$$
\begin{aligned}
[(\tilde{R}(X,Y).\tilde{P})(Z,U)V] &= [R(X,Y).R)(Z,U)V] \\
&+ \alpha\{g(Y,\nabla_X\phi R(Z,U)V) \\
&- g(X,\nabla_Y\phi R(Z,U)V) + g(X,\phi\nabla_Y R(Z,U)V) \\
&- g(Y,\phi\nabla_X R(Z,U)V)\}\xi + \beta g([X,Y],R(Z,U)V)\xi \\
&+ \alpha\{g(U,\nabla_Z\phi V) - g(Z,\nabla_U\phi V) \\
&+ g(Z,\phi\nabla_U V) - g(U,\phi\nabla_Z V)\}\xi + \beta g([Z,U],V)\xi \\
&+ \alpha\{g(U,\nabla_{R(X,Y)Z}\phi V) - g(R(X,Y)Z,\nabla_U\phi V) \\
&+ g(R(X,Y)Z,\phi\nabla_U V) - g(U,\phi\nabla_{R(X,Y)Z})V\} \\
&+ \beta g([R(X,Y)Z,U],V)]\xi \\
&+ [\alpha\{g(Y,\nabla_X\phi Z) - g(X,\nabla_Y\phi Z) + g(X,\phi\nabla_Y Z) \\
&- g(Y,\phi\nabla_X Z)\} + \beta g([X,Y],Z)]\tilde{R}(\xi,U)V \\
&+ [\alpha\{g(R(X,Y)Z,\nabla_Z\phi V) - g(Z,\nabla_{R(X,Y)Z}\phi V \\
&- g(R(X,Y)Z,\phi\nabla_Z V) + g(Z,\phi\nabla_{R(X,Y)Z}V\} \\
&+ \beta g([R(X,Y)Z,U],V)]\xi \\
&+ [\alpha\{g(Y,\nabla_X\phi U) - g(X,\nabla_Y\phi U) + g(X,\phi\nabla_Y U) \\
&- g(Y,\phi\nabla_X U)\} + \beta g([X,Y],U)]\tilde{R}(Z,\xi)V \\
&+ \alpha\{g(U,\nabla_Z\phi R(X,Y)V) - g(Z,\nabla_U\phi R(X,Y)V) \\
&+ g(Z,\phi\nabla_U R(X,Y)V) - g(U,\phi\nabla_Z R(X,Y)V)\}\xi \\
&+ \beta g([Z,U],R(X,Y)V)\xi
\end{aligned} \tag{5.9}
$$

Applying ϕ^2 on both side of (5.9) we get

$$\begin{aligned}\phi^2[(\tilde{R}(X,Y).\tilde{P})(Z,U)V] &= \phi^2[R(X,Y).R)(Z,U)V] \\ &+ [\alpha\{g(Y,\nabla_X\phi Z)-g(X,\nabla_Y\phi Z)+g(X,\phi\nabla_Y Z) \\ &- g(Y,\phi\nabla_X Z)\}+\beta g([X,Y],Z)]\{\phi^2\tilde{R}(\xi,U)V\} \\ &+ [\alpha\{g(Y,\nabla_X\phi U)-g(X,\nabla_Y\phi U)+g(X,\phi\nabla_Y U) \\ &- g(Y,\phi\nabla_X U)\}+\beta g([X,Y],U)]\{\phi^2\tilde{R}(Z,\xi)V\}\end{aligned} \tag{5.10}$$

In view of (2.16) and (3.2) we get from (5.10)

$$\phi^2[(\tilde{R}(X,Y).\tilde{P})(Z,U)V] = \phi^2[R(X,Y).R)(Z,U)V] \tag{5.11}$$

Thus we are in a position to state the following:

Theorem 5.3. *An almost Ricci soliton trans-Sasakian manifold of dimension three is locally projectively ϕ-semisymmetric with respect to generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ if and only if it is so with respect to Levi-civita connection ∇.*

6 Locally conformally ϕ-semisymmetric three dimensional almost Ricci soliton trans-Sasakian manifolds with respect to generalized Tanaka Webster Okumura connection

In this section we have studied locally conformally ϕ- semisymmetric almost Ricci soliton trans-Sasakian manifolds of dimension three with respect to generalized Tanaka Webster Okumura connection.

Definition 6.1. For a $(2n+1)$ dimensional Riemannian manifold the Weyl conformal curvature tensor with respect to generalized Tanaka Webster Okumura connection is defined by

$$\begin{aligned}\tilde{C}(X,Y)Z &= \tilde{R}(X,Y)Z-\tfrac{1}{2n-1}\{\tilde{S}(Y,Z)X-\tilde{S}(X,Z)Y+g(Y,Z)\tilde{Q}X-g(X,Z)\tilde{Q}Y\} \\ &+ \tfrac{\tilde{r}}{2n(2n-1)}\{g(Y,Z)X-g(X,Z)Y\}.\end{aligned} \tag{6.1}$$

Using (3.8) and (3.10) in (6.1) we get

$$\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z+\{\tfrac{r}{2n(2n-1)}-\tfrac{2A}{2n-1}\}\{g(Y,Z)X-g(X,Z)Y\} \tag{6.2}$$

or,

$$\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z+N\{g(Y,Z)X-g(X,Z)Y\}, \tag{6.3}$$

where $N=\{\frac{r}{2n(2n-1)}-\frac{2A}{2n-1}\}$ and $A=(\lambda-\beta)$.
Using (3.2) in (6.3) we get

$$\begin{aligned}\tilde{C}(X,Y)Z &= R(X,Y)Z+[\alpha\{g(Y,\nabla_X\phi Z)-g(X,\nabla_Y\phi Z)+g(X,\phi\nabla_Y Z)-g(Y,\phi\nabla_X Z)\} \\ &+ \beta g([X,Y],Z)]\xi+N\{g(Y,Z)X-g(X,Z)Y\},\end{aligned} \tag{6.4}$$

Definition 6.2. A trans-Sasakian manifold of dimension three is said to be locally conformally ϕ-semisymmetric with respect to generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ if

$$\phi^2\{(\tilde{R}(U,V).\tilde{C})(X,Y)Z\}=0, \tag{6.5}$$

for all horizontal vector fields X, Y, Z, U, V on M.

We know that

$$\begin{aligned}[(\tilde{R}(U,V).\tilde{C})(X,Y)Z] &= \tilde{R}(U,V)\tilde{C}(X,Y)Z-\tilde{C}(\tilde{R}(U,V)X,Y)Z-\tilde{C}(X,\tilde{R}(U,V)Y)Z \\ &- \tilde{C}(X,Y)\tilde{R}(U,V)Z\end{aligned} \tag{6.6}$$

Using (3.2), (2.17) and (6.3) in each terms of right hand side of (6.6) and by straightforward calculation we obtain the followings:

$$\begin{aligned}\tilde{R}(U,V)\tilde{C}(X,Y)Z &= R(U,V)C(X,Y)Z+[\alpha\{g(Y,\nabla_X\phi Z)-g(X,\nabla_Y\phi Z)+g(X,\phi\nabla_Y Z)\\ &- g(Y,\phi\nabla_X Z)\}+\beta g([X,Y],Z)]\xi+[\alpha\{g(V,\nabla_U\phi\tilde{C}(X,Y)Z)\\ &- g(U,\nabla_V\phi\tilde{C}(X,Y)Z)+g(U,\phi\nabla_V\tilde{C}(X,Y)Z)-g(V,\phi\nabla_U\tilde{C}(X,Y)Z)\}\\ &+ \beta g([U,V],\tilde{C}(X,Y)Z)]\xi+N\{g(Y,Z)R(U,V)X-g(X,Z)R(U,V)Y\},\end{aligned} \tag{6.7}$$

$$\begin{aligned}\tilde{C}(\tilde{R}(U,V)X,Y)Z &= C(R(U,V)X,Y)Z+[\alpha\{g(V,\nabla_U\phi X)-g(U,\nabla_V\phi X)+g(U,\phi\nabla_V X)\\ &- g(V,\phi\nabla_U X)\}+\beta g([U,V],X)]R(\xi,Y)Z+[\alpha\{g(Y,\nabla_{\tilde{R}(U,V)X}\phi Z)\\ &- g(\tilde{R}(U,V)X,\nabla_Y\phi Z)+g(\tilde{R}(U,V)X,\phi\nabla_Y Z)-g(Y,\phi\nabla_{\tilde{R}(U,V)X}Z)\\ &+ \beta g([\tilde{R}(U,V)X,Y],Z)]\xi+N\{g(Y,Z)\tilde{R}(U,V)X-g(R(U,V)X,Z)Y\},\end{aligned} \tag{6.8}$$

$$\begin{aligned}\tilde{C}(X,\tilde{R}(U,V)Y)Z &= C(X,R(U,V)Y+[\alpha\{g(V,\nabla_U\phi Y)-g(U,\nabla_V\phi Y)+g(U,\phi\nabla_V Y)\\ &- g(V,\phi\nabla_U Y)\}+\beta g([U,V],Y)]R(X,\xi)Z+[\alpha\{g(\tilde{R}(U,V)Y,\nabla_X\phi Z)\\ &- g(X,\nabla_{\tilde{R}(U,V)Y}\phi Z)+g(X,\phi\nabla_{\tilde{R}(U,V)Y}Z)-g(\tilde{R}(U,V)Y,\phi\nabla_X Z)\}\\ &+ \beta g([X,\tilde{R}(U,V)Y],Z)]\xi+N\{g(R(U,V)Y,Z)X-g(X,Z)\tilde{R}(U,V)Y\},\end{aligned} \tag{6.9}$$

$$\begin{aligned}\tilde{C}(X,Y)\tilde{R}(U,V)Z &= R(X,Y)R(U,V)Z+[\alpha\{g(Y,\nabla_X\phi\tilde{R}(U,V)Z)-g(X,\nabla_Y\phi\tilde{R}(U,V)Z)\\ &+ g(X,\phi\nabla_Y\tilde{R}(U,V)Z)-g(Y,\phi\nabla_X\tilde{R}(U,V)Z)\}+\beta g([X,Y],\tilde{R}(U,V)Z)]\xi\\ &+ N\{g(R(U,V)Z,Y)X-g(R(U,V)Z,X)Y\}.\end{aligned} \tag{6.10}$$

In view of (6.7), (6.8), (6.9) and (6.10) and by straight forward calculation we get from (6.6)

$$\begin{aligned}[(\tilde{R}(U,V).\tilde{C})(X,Y)Z] &= [(R(U,V).C)(X,Y)Z]\\ &+ [\alpha\{g(Y,\nabla_X\phi Z)-g(X,\nabla_Y\phi Z)+g(X,\phi\nabla_Y Z)\\ &- g(Y,\phi\nabla_X Z)\}+\beta g([X,Y],Z)]\xi\\ &+ [\alpha\{g(V,\nabla_U\phi\tilde{C}(X,Y)Z)-g(U,\nabla_V\phi\tilde{C}(X,Y)Z)\\ &+ g(U,\phi\nabla_V\tilde{C}(X,Y)Z)-g(V,\phi\nabla_U\tilde{C}(X,Y)Z)\}\\ &+ \beta g([U,V],\tilde{C}(X,Y)Z)]\xi\\ &+ N\{g(Y,Z)R(U,V)X-g(X,Z)R(U,V)Y\}\\ &+ [\alpha\{g(V,\nabla_U\phi X)-g(U,\nabla_V\phi X)+g(U,\phi\nabla_V X)\\ &- g(V,\phi\nabla_U X)\}+\beta g([U,V],X)]R(\xi,Y)Z\\ &+ [\alpha\{g(Y,\nabla_{\tilde{R}(U,V)X}\phi Z)-g(\tilde{R}(U,V)X,\nabla_Y\phi Z)\\ &+ g(\tilde{R}(U,V)X,\phi\nabla_Y Z)-g(Y,\phi\nabla_{\tilde{R}(U,V)X}Z)\\ &+ \beta g([\tilde{R}(U,V)X,Y],Z)]\xi+Ng(Y,Z)\tilde{R}(U,V)X\\ &+ [\alpha\{g(V,\nabla_U\phi Y)-g(U,\nabla_V\phi Y)+g(U,\phi\nabla_V Y)\\ &- g(V,\phi\nabla_U Y)\}+\beta g([U,V],Y)]R(X,\xi)Z\\ &+ [\alpha\{g(\tilde{R}(U,V)Y,\nabla_X\phi Z)-g(X,\nabla_{\tilde{R}(U,V)Y}\phi Z)\\ &+ g(X,\phi\nabla_{\tilde{R}(U,V)Y}Z)-g(\tilde{R}(U,V)Y,\phi\nabla_X Z)\}\\ &+ \beta g([X,\tilde{R}(U,V)Y],Z)]\xi-Ng(X,Z)\tilde{R}(U,V)Y\}\\ &+ [\alpha\{g(Y,\nabla_X\phi\tilde{R}(U,V)Z)-g(X,\nabla_Y\phi\tilde{R}(U,V)Z)\\ &+ g(X,\phi\nabla_Y\tilde{R}(U,V)Z)-g(Y,\phi\nabla_X\tilde{R}(U,V)Z)\}\\ &+ \beta g([X,Y],\tilde{R}(U,V)Z)]\xi.\end{aligned} \tag{6.11}$$

Applying ϕ^2 on both side of (6.11) and then using (2.3), (2.16) and (3.3) we get

$$\phi^2[(\tilde{R}(U,V).\tilde{C})(X,Y)Z]=\phi^2[(R(U,V).C)(X,Y)Z]. \tag{6.12}$$

Thus we are in a position to state the following:

Theorem 6.3. *An almost Ricci soliton trans-Sasakian manifold of dimension three is locally conformally ϕ-semisymmetric with respect to generalized Tanaka Webster Okumura connection $\tilde{\nabla}$ if and only if it is so with respect to Levi-civita connection ∇.*

7 Conclusion remarks

This paper aims is to obtain new result on an almost Ricci soliton trans-Sasakian manifold of dimension three admitting generalized Tanaka Webster Okumura connection in various geometrical structure. The results of this work are significant as well as interesting and capable to develop its study in the future.

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