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Integral Inequalities for a Polynomial with Restricted Zeros

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Abstract. Suppose $P(z)$ is a polynomial of degree n having a multiple zero at some point within the unit disc. In this paper, we prove some Zygmund-type inequalities for this type of polynomial while having all other zeros either in $|z| \geq k \geq 1$ or $|z| \leq k \leq 1$. The results obtained generalize as well as extend some already known Bernstein-type inequalities.

1 Introduction

Let $\mathscr{P}_n$ be the class of polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n with real or complex coefficients. For $P \in \mathscr{P}_n$ and $1 \leq q < \infty$, we define the following

$$||P||_q = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}$$

and

$$||P||_\infty = \max_{|z|=1} |P(z)|.$$

If $P \in \mathscr{P}_n$, then we have for $1 \leq q < \infty$,

$$||P'||_q \leq n||P||_q \tag{1.1}$$

Inequality (1.1) is a famous result due to Zygmund [10]. Arestov [1] proved that (1.1) remains true for $0 < q < 1$ as well. If we let $q \to \infty$ in (1.1), then we have the following elegant result due to Bernstein [4].

$$||P'||_\infty \leq n||P||_\infty. \tag{1.2}$$

If we confine our study to the class of polynomials which do not vanish in $|z| < 1$, then for each $q > 0$, inequality (1.1) can be replaced [5, 6, 9] by

$$||P'||_q \leq \frac{n}{||1 + z^n||_q} ||P||_q. \tag{1.3}$$

Govil and Rahman [6] extended inequality (1.3) to the class of polynomials $P \in \mathscr{P}_n$ having no zeros in the disk $|z| < k$, $k \geq 1$ and proved

$$||P'||_q \leq \frac{n}{||k + z^n||_q} ||P||_q, \tag{1.4}$$

for $0 < q < \infty$.

Aziz and Shah [2] considered the class of polynomials $\mathscr{P}_n$ having a zero of order s at the origin and all other zeros outside or on the circle $|z| = k$, $k \geq 1$ and obtained for every q, $1 \leq q < \infty$ the following result

$$\|P'\|_q \leq \left\{ s + \frac{n-s}{\|k+z\|_q} \right\} \|P\|_q. \tag{1.5}$$

In the same paper for the polynomial $P \in \mathscr{P}_n$ with $s-$fold zero at origin and rest of the zeros within or on the circle $|z| = k$, $k \geq 1$, they proved;

$$\|P'\|_q \geq \frac{n+sk}{1+k} \|P\|_q. \tag{1.6}$$

In this paper, instead of assuming a zero of order s at the origin, we assume that there is a zero at a point z_0 of order s with $|z_0| < 1$ and prove the following generalizations of (1.5) and (1.6). The essence of our result lies in the fact that the applications of the field are enhanced by taking the restriction on the zeros in this particular way. These applications range from providing insights into chemical reactions, trajectories and motion planning for robotics, to analyzing orbital mechanics and predicting celestial events.

2 Lemma

Lemma 2.1 ([6]). *If $P(z)$ is a polynomial of degree at most n and $P^*(z) = z^n \overline{P(1/\overline{z})}$, then*

$$\max_{|z|=1} |(P^*(z))^{'}| + \max_{|z|=1} |P^{'}(z)| \leq n \max_{|z|=1} |P(z)|. \tag{2.1}$$

The result is best possible and equality holds for $P(z) = az^n$, a is a complex number.

3 Main Results

Theorem 3.1. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, except a zero of order s at a point $z_0 \in \mathbb{C}$, with $|z_0| < 1$, then for every $q \geq 1$,*

$$\|P'\|_q \leq A \left\{ \frac{s}{1-|z_0|} + \frac{n-s}{\|z+k\|_q} \right\} \|P\|_q, \tag{3.1}$$

where $A = \left(\frac{k+|z_0|}{k-|z_0|} \right)^s$.

Proof. Consider a polynomial $P \in \mathscr{P}_n$ which has $s-$fold zero at z_0 with $|z_0| < 1$ and rest of the $n-s$ zeros in $|z| \geq k$, $k \geq 1$, so that we can write

$$P(z) = (z - z_0)^s h(z),$$

where $h(z)$ is a polynomial of degree $n-s$ having all its zeros in $|z| \geq k$, $k \geq 1$. Applying inequality (1.4) to the polynomial $h(z)$, we have for $q > 0$,

$$\begin{aligned} \|h'\|_q &\leq \frac{n-s}{\|k+z^n\|_q} \|h\|_q \\ &\leq \frac{n-s}{\|k+z\|_q} \|h\|_q. \end{aligned} \tag{3.2}$$

Now

$$P(z) = (z - z_0)^s h(z),$$

implies

$$\begin{aligned} P'(z) &= s(z-z_0)^{s-1} h(z) + (z-z_0)^s h'(z) \\ &= (z-z_0)^s \left\{ s \frac{h(z)}{(z-z_0)} + h'(z) \right\}. \end{aligned}$$

For $k \geq 1$ and $|z| = 1$, we have

$$\begin{aligned}|P'(z)| &\leq (1+|z_0|)^s\Big\{s\frac{|h(z)|}{(1-|z_0|)} + |h'(z)|\Big\}\\ &= (1+|z_0|)^s\left\{\frac{s|h(z)|}{1-|z_0|} + |h'(z)|\right\}\\ &\leq (k+|z_0|)^s\left\{\frac{s|h(z)|}{1-|z_0|} + |h'(z)|\right\}.\end{aligned}$$

Therefore for the points $z = e^{i\theta}$, $0 \leq \theta < 2\pi$ and $q > 1$, we have

$$\begin{aligned}\int_0^{2\pi} |P'(e^{i\theta})|^q d\theta &\leq \int_0^{2\pi} (k+|z_0|)^{sq}\left\{\frac{s|h(e^{i\theta})|}{1-|z_0|} + |h'(e^{i\theta})|\right\}^q d\theta\\ &= (k+|z_0|)^{sq}\int_0^{2\pi}\left\{\frac{s}{1-|z_0|}|h(e^{i\theta})| + |h'(e^{i\theta})|\right\}^q d\theta.\end{aligned}$$

Using Minkowski inequality, for every $q \geq 1$, we get

$$\begin{aligned}\left\{\int_0^{2\pi}|P'(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} &\leq (k+|z_0|)^s\left[\left\{\int_0^{2\pi}\left(\frac{s}{1-|z_0|}\right)^q|h(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} + \left\{\int_0^{2\pi}|h'(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}\right]\\ &= (k+|z_0|)^s\left[\left(\frac{s}{1-|z_0|}\right)\left\{\int_0^{2\pi}|h(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} + \left\{\int_0^{2\pi}|h'(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}\right].\end{aligned}$$

This gives

$$||P'||_q \leq (k+|z_0|)^s\left[\frac{s}{1-|z_0|}||h||_q + ||h'||_q\right]. \tag{3.3}$$

Using (3.2) in (3.3), we get

$$\begin{aligned}||P'||_q &\leq (k+|z_0|)^s\left[\frac{s}{1-|z_0|}||h||_q + \frac{n-s}{||k+z||_q}||h||_q\right]\\ &= (k+|z_0|)^s\left[\frac{s}{1-|z_0|} + \frac{n-s}{||k+z||_q}\right]||h||_q. \qquad (3.4)\end{aligned}$$

Also

$$h(z) = \frac{P(z)}{(z-z_0)^s}.$$

That is for $|z| = k$, $k \geq 1$,

$$|h(z)| \leq \frac{|P(z)|}{(k-|z_0|)^s}.$$

Now for points $z = e^{i\theta}$, $0 \leq \theta < 2\pi$ and $q > 1$,

$$||h||_q \leq \frac{1}{(k-|z_0|)^s}||P||_q. \tag{3.5}$$

Using (3.5) in (3.4), we obtain

$$||P'||_q \leq \left(\frac{k+|z_0|}{k-|z_0|}\right)^s\left[\frac{s}{1-|z_0|} + \frac{n-s}{||k+z||_q}\right]||P||_q.$$

This completes proof of Theorem 3.1. □

If we let $q \to \infty$, then we get from Theorem 3.1 the following generalization of Erdös- Lax Theorem.

Corollary 3.2. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, except $s-$fold zero at a point $z_0 \in \mathbb{C}$, with $|z_0| < 1$, then*

$$\|P'\|_\infty \leq A\left\{\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right\}\|P\|_\infty, \tag{3.6}$$

where $A = \left(\dfrac{k+|z_0|}{k-|z_0|}\right)^s$.

Remark 3.3. If we take $z_0 = 0$ in Theorem 3.1, we get a result due to Aziz and Shah [2].

Remark 3.4. If $s = 0$ in Theorem 3.1, we get a result due to Govil and Rahman [6].

Theorem 3.5. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ in $|z| > k$, $k \leq 1$, except $s-$fold zero at a point $z_0 \in \mathbb{C}$, with $|z_0| < 1$, then for every $q > 0$,*

$$\|P'\|_q \geq \left\{\frac{s}{1+|z_0|} + \frac{n-s}{1+k}\right\}\|P\|_q. \tag{3.7}$$

Proof. Since $P(z) \neq 0$ in $|z| > k$, $k \leq 1$ and all zeros of $P(z)$ lie in $|z| \leq k$, $k \leq 1$ with $s-$fold zero at z_0, with $|z_0| < 1$, therefore we have

$$P(z) = (z - z_0)^s h(z),$$

where $h(z)$ is a polynomial of degree $n - s$ having all zeros in $|z| \leq k$, $k \leq 1$. This gives

$$\frac{zP'(z)}{P(z)} = \frac{zs}{z - z_0} + \frac{zh'(z)}{h(z)}. \tag{3.8}$$

If $z_1, z_2, z_3, ..., z_{n-s}$ are the zeros of $h(z)$, then $|z_j| \leq k$, $k \leq 1$ and from (3.8), we have

$$\begin{aligned} Re\left(\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right) &= Re\left(\frac{se^{i\theta}}{e^{i\theta} - z_0}\right) + Re\left(\frac{e^{i\theta}h'(e^{i\theta})}{h(e^{i\theta})}\right) \\ &= Re\left(\frac{se^{i\theta}}{e^{i\theta} - z_0}\right) + \sum_{j=1}^{n-s} Re\left(\frac{e^{i\theta}}{e^{i\theta} - z_j}\right) \\ &= Re\left(\frac{s}{1 - z_0e^{-i\theta}}\right) + \sum_{j=1}^{n-s} Re\left(\frac{1}{1 - z_je^{-i\theta}}\right) \end{aligned} \tag{3.9}$$

for the points $e^{i\theta}$ which are not the zeros of $h(z)$. Now, if $|w| \leq k$, $k \leq 1$, and $1 \leq \mu \leq n$, then it can be easily verified that

$$Re\left(\frac{1}{1 - z_je^{-i\theta}}\right) = Re\left(\frac{1}{1-w}\right) \geq \frac{1}{1+k} \tag{3.10}$$

and for $|z_0| < 1$,

$$Re\left(\frac{s}{1 - z_0e^{-i\theta}}\right) \geq \frac{s}{1+|z_0|}. \tag{3.11}$$

Using (3.10) and (3.11) in (3.9), we get

$$|P'(e^{i\theta})| \geq \left\{\frac{s}{1+|z_0|} + \frac{n-s}{1+k}\right\}|P(e^{i\theta})| \tag{3.12}$$

for the points which are not the zeros of $P(z)$. Since inequality (3.12) is trivially true for points $e^{i\theta}$, which are the zeros of $P(z)$, we conclude for $|z| = 1$,

$$|P'(z)| \geq \left\{\frac{s}{1+|z_0|} + \frac{n-s}{1+k}\right\}|P(z)|. \tag{3.13}$$

Therefore, for every $q > 0$,

$$\|P'\|_q \geq \left\{\frac{s}{1+|z_0|} + \frac{n-s}{1+k}\right\}\|P\|_q.$$

This completes proof of Theorem 3.5. □

If $s = 0$, Theorem 3.5 reduces to the following result

Corollary 3.6. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ in $|z| > k$, $k \leq 1$, then for every $q > 0$,*

$$\|P'\|_q \geq \frac{n}{1+k}\|P\|_q. \tag{3.14}$$

The result is best possible and equality holds for the polynomial $P(z) = \left(\frac{z+k}{1+k}\right)^n$.

Remark 3.7. If we let $q \to \infty$ in inequality (3.7), we get a result due to Ahanger and Shah [3].

Remark 3.8. If we take $z_0 = 0$ in Theorem 3.5, then we get a result due to Aziz and Shah [2].

Theorem 3.9. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, except s–fold zero at a point $z_0 \in \mathbb{C}$, with $|z_0| \leq 1 - \frac{s(1+k)}{s+nk}$, $nk + s > 0$, then for every $q \geq 1$,*

$$\|P^{'}\|_q \leq n(2\pi)^{\frac{1}{q}} \max_{|z|=1} |P(z)| + \left\{\frac{s}{1-|z_0|} - \frac{(s+nk)}{1+k}\right\}\|P\|_q. \tag{3.15}$$

Proof. Since all the zeros of $P(z)$ lie in $|z| \geq k$, except s–fold zero at a point z_0 with $|z_0| < 1$, $0 \leq s < n$, therefore

$$P(z) = (z - z_0)^s u(z),$$

where $u(z)$ is a polynomial of degree $n - s$ having all its zeros in $|z| \geq k$. If $z_1, z_2, z_3, \ldots, z_{n-s}$ are the zeros of $u(z)$, then $|z_j| \geq k$, $k > 1$, $j = 1, 2, 3, \ldots, n - s$. Therefore, we have

$$\frac{zP^{'}(z)}{P(z)} = \frac{sz}{z - z_0} + \sum_{j=0}^{n-s} \frac{z}{z - z_j}.$$

This gives

$$Re\left(\frac{zP^{'}(z)}{P(z)}\right) = Re\left(\frac{sz}{z - z_0}\right) + Re\left(\sum_{j=0}^{n-s} \frac{z}{z - z_j}\right).$$

For the points $z = e^{i\theta}$, $0 \leq \theta < 2\pi$ which are not the zeros of $P(z)$, we have

$$\begin{aligned} Re\left(\frac{e^{i\theta}P^{'}(e^{i\theta})}{P(e^{i\theta})}\right) &= Re\left(\frac{se^{i\theta}}{e^{i\theta - z_0}}\right) + Re\left(\sum_{j=0}^{n-s} \frac{e^{i\theta}}{e^{i\theta} - z_j}\right) \\ &= Re\left(\frac{se^{i\theta}}{e^{i\theta - z_0}}\right) + Re\left(\sum_{j=0}^{n-s} \frac{1}{1 - e^{-i\theta}z_j}\right). \end{aligned}$$

Using the fact that $|w| \geq k > 1$,

$$Re\left(\frac{1}{1-w}\right) \leq \frac{1}{1+k}$$

and

$$Re\left(\frac{se^{i\theta}}{e^{i\theta - z_0}}\right) \leq \left|\frac{se^{i\theta}}{e^{i\theta} - z_0}\right| \leq \frac{s}{1 - |z_0|},$$

we obtain, for $0 \le \theta < 2\pi$,

$$Re\left(\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right) \le \frac{s}{1-|z_0|} + \frac{n-s}{1+k}.$$

Now, for $P^*(z) = z^n\overline{P(1/\overline{z})}$, it can be easily verified for $|z| = 1$ that

$$|(P^*(z))'| = |nP(z) - zP'(z)|.$$

This implies for $|z| = 1$,

$$\begin{aligned}\left|\frac{(P^*(z))'}{P(z)}\right| &= \left|n - \frac{zP'(z)}{P(z)}\right| \\ &\ge n - \left|\frac{zP'(z)}{P(z)}\right| \\ &\ge n - Re\left(\frac{zP'(z)}{P(z)}\right) \\ &\ge n - \left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right).\end{aligned}$$

That is

$$|(P^*(z))'| \ge \left\{n - \left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right)\right\}|P(z)|. \tag{3.16}$$

Since

$$|z_0| \le 1 - \frac{s(1+k)}{s+nk} \quad \text{and} \quad nk + s > 0,$$

it can be easily verified that

$$n - \left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right) \ge 0.$$

Therefore for $|z| = 1$, using Lemma 2.1, we get

$$\begin{aligned}\left\{n - \left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right)\right\}|P(z)| + |P'(z)| &\le |(P^*(z))'| + |P'(z)| \\ &\le n \max_{|z|=1} |P(z)|.\end{aligned}$$

Thus for $|z| = 1$, we have

$$|P'(z)| \le n \max_{|z|=1} |P(z)| - \left\{n - \left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right)\right\}|P(z)|.$$

This gives for $z = e^{i\theta}$, $0 \le \theta < 2\pi$ and $q > 0$,

$$\begin{aligned}\left\{\int_0^{2\pi} |P'(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} &\le \left\{\int_0^{2\pi}\left[n \max_{|z|=1} |P(z)| + \left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k} - n\right)|P(z)|\right]^q d\theta\right\}^{\frac{1}{q}} \\ &= n \max_{|z|=1} |P(z)|\left(\int_0^{2\pi} 1 d\theta\right)^{\frac{1}{q}} + \left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k} - n\right)\left\{\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \\ &= (2\pi)^{\frac{1}{q}} n \max_{|z|=1} |P(z)| - \left\{n - \left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right)\right\}\left\{\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}\end{aligned}$$

That is

$$\begin{aligned}||P^{'}||_q &\leq n(2\pi)^{\frac{1}{q}} \max_{|z|=1}|P(z)| - \left(\frac{s+nk}{1+k} - \frac{s}{1-|z_0|}\right)||P||_q \\ &= n(2\pi)^{\frac{1}{q}} \max_{|z|=1}|P(z)| - \left(\frac{(n-s)k-(s+nk)|z_0|}{(1+k)(1-|z_0|)}\right)||P||_q\end{aligned}$$

This completes proof of Theorem 3.9 □

For $s = 0$, Theorem 3.9 gives the following interesting corollary

Corollary 3.10. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for every $q \geq 1$,*

$$||P^{'}||_q \leq n(2\pi)^{\frac{1}{q}} \max_{|z|=1}|P(z)| - \left(\frac{nk}{1+k}\right)||P||_q. \tag{3.17}$$

Remark 3.11. If $q \to \infty$ in (3.17), then Corollary 3.1 gives the result due to Malik [8].

For $z_0 = 0$, Theorem 3.9 gives

Corollary 3.12. *If $P \in \mathscr{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, except $s-$fold zero at a point z_0 with $|z_0| < 1$, then for every $q \geq 1$,*

$$||P^{'}||_q \leq n(2\pi)^{\frac{1}{q}} \max_{|z|=1}|P(z)| - \left(\frac{(n-s)k}{1+k}\right)||P||_q. \tag{3.18}$$

Remark 3.13. If $q \to \infty$ in (3.18), then Corollary 3.2 gives the result due to Aziz and Shah [2].

4 Conclusion remarks

This paper aims is to obtain new integral mean estimates for polynomials having $s-$fold zeros at some generalized point z_0 within a given disk.

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