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# GEOMETRIC PROPERTIES OF p-VALENT ANALYTIC FUNCTIONS USING MULTIPLIER TRANSFORMATION

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**Abstract** *In the present paper, we derive certain sufficient conditions for parabolic starlikeness/ starlikeness, uniform convexity/ convexity of multivalent as well as univalent analytic functions using multiplier transformation. We also obtain some sandwich-type results regarding these functions. For illustration of the results, we have plotted the images of open unit disk under certain functions using Mathematica 7.0.*

## 1 Introduction

*Let $\mathcal{A}_p$ denote the class of functions of the form*

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ (p \in \mathbb{N};\ z \in \mathbb{E}),$$

*which are analytic and $p$-valent in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$.*
*We write, $\mathcal{A}_1 = \mathcal{A}$, the class of analytic functions $f$, normalized by the conditions $f(0) = f'(0) - 1 = 0$.*
*Let the functions $f$ and $g$ be analytic in $\mathbb{E}$. We say that $f$ is subordinate to $g$ written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function $\phi$ in $\mathbb{E}$ (i.e. $\phi$ is regular in $|z| < 1$, $\phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that*

$$f(z) = g(\phi(z)),\ |z| < 1.$$

*Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, $u$ an analytic function in $\mathbb{E}$ with $(u(z), zu'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and $h$ be univalent in $\mathbb{E}$. Then the function $u$ is said to satisfy first order differential subordination if*

$$\Phi(u(z), zu'(z); z) \prec h(z),\ \Phi(u(0), 0; 0) = h(0). \tag{1.1}$$

*A univalent function $q$ is called dominant of the differential subordination (1.1) if $u(0) = q(0)$ and $u \prec q$ for all $u$ satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of $\mathbb{E}$.*
*Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic and univalent function in domain $\mathbb{C}^2 \times \mathbb{E}$, $h$ be analytic function in $\mathbb{E}$, $u$ be analytic and univalent in $\mathbb{E}$ with $(u(z), zu'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then $u$ is called the solution of the first order differential superordination if*

$$h(z) \prec \Psi(u(z), zu'(z); z), h(0) = \Psi(u(0), 0; 0). \tag{1.2}$$

*An analytic function $q$ is called a subordinant of the differential superordination (1.2) if $q \prec u$ for all u satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants $q$ of*

*(1.2), is said to be the best subordinant of (1.2). The best subordinant is unique up to a rotation of* $\mathbb{E}$.

*A function* $f \in \mathcal{A}_p$ *is said to be* $p$*-valent starlike of order* $\alpha$ $(0 \leq \alpha < p)$ *in* $\mathbb{E}$ *if*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in \mathbb{E}.$$

*We denote by* $S_p^*(\alpha)$*, the class of all such functions. Write* $S_p^*(0) = S_p^*$*, the class of* $p$*-valent starlike functions.*

*A function* $f \in \mathcal{A}_p$ *is said to be* $p$*-valent convex of order* $\alpha$ $(0 \leq \alpha < p)$ *in* $\mathbb{E}$ *if*

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in \mathbb{E}.$$

*Let the class of such functions be denoted by* $\mathcal{K}_p(\alpha)$*. Obviously,* $K_p(0) = K_p$*, the class of* $p$*-valent convex functions.*

*A function* $f \in \mathcal{A}_p$ *is said to be parabolic* $p$*-valent starlike in* $\mathbb{E}$ *if*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - p\right|, \ z \in \mathbb{E}. \tag{1.3}$$

*The class of such functions is denoted by* $S_{\mathcal{P}}^p$*. Write* $S_{\mathcal{P}}^1 = S_{\mathcal{P}}$*, the class of parabolic starlike functions.*

*A function* $f \in \mathcal{A}_p$ *is said to be uniformly* $p$*-valent convex in* $\mathbb{E}$*, if*

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)} - (p-1)\right|, \ z \in \mathbb{E}, \tag{1.4}$$

*and let* $UCV_p$ *denote the class of all such functions. Write* $UCV_1 = UCV$*, the class of uniformly convex functions.*

*Rønning [25] and Ma and Minda [22] studied the domain* $\Omega$ *and the function* $q(z)$ *defined below:*

$$\Omega = \left\{u + iv : u > \sqrt{(u-1)^2 + v^2}\right\}.$$

*Clearly the function*

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \tag{1.5}$$

*maps the unit disk* $\mathbb{E}$ *onto the domain* $\Omega$*. Hence the conditions (1.3) and (1.4) are respectively, equivalent to*

$$\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right) \prec q(z).$$

*and*

$$\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec q(z).$$

*For* $f \in \mathcal{A}_p$*, we define the multiplier transformation* $I_p(n, \lambda)$ *as*

$$I_p(n, \lambda)[f](z) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k, \ where\ \lambda \geq 0, \ n \in \mathbb{Z}.$$

*The operator* $I_1(n, 0)$ *is the well-known Sălăgean [26] derivative operator* $D^n\, f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $f \in \mathcal{A}$*. In 1992, Uralegaddi and Somanatha [32] investigated the operator* $I_1(n, 1)$*. In 1993, Jung et al. [16] studied the transformation*

$$I_1(-1, \lambda)[f](z) = z + \sum_{k=2}^{\infty}\left(\frac{1+\lambda}{k+\lambda}\right) a_k z^k, \ where\ \lambda > -1, \ f \in \mathcal{A}.$$

*In 2003, Cho and Srivastava [14], Cho and Kim [11] investigated the operator $I_1(n,\ \lambda)$. In 2005, Aghalary et al. [1] studied the operator $I_p(n,\ \lambda)$. In 2008, Singh et al. ([27], [28]) and in 2012, Billing ([2], [3], [4], [5], [6], ) investigated the operator $I_p(n,\ \lambda)$ and obtained certain sufficient conditions for starlike and convex functions. Let $\mathcal{S}_n(\alpha)$ denote the class of functions $f \in \mathcal{A}$ for which*

$$Re\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right) > \alpha,\ z \in \mathbb{E},\ 0 \le \alpha < 1.$$

*In 1989, Owa, Shen and Obradović [24] investigated this class and proved the following result.*

**Theorem 1.1.** *For $n \in \mathbb{N}_0$, if $f \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{n+1}[f](z)}{D^n[f](z)} - 1\right|^{1-\beta} \left|\frac{D^{n+2}[f](z)}{D^{n+1}[f](z)} - 1\right|^{\beta} < (1-\alpha)^{1-2\beta}\left(1 - \frac{3}{2}\alpha + \alpha^2\right)^{\beta},\ z \in \mathbb{E},$$

*for some $\alpha$ $(0 \le \alpha \le 1/2)$ and $\beta$ $(0 \le \beta \le 1)$ then*

$$Re\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right) > \alpha \ i.e.\ f \in \mathcal{S}_n(\alpha).$$

*Later on, Li and Owa [21] extended this result by proving the result given below.*

**Theorem 1.2.** *For $n \in \mathbb{N}_0$, if $f \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{n+1}[f](z)}{D^n[f](z)} - 1\right|^{\gamma} \left|\frac{D^{n+2}[f](z)}{D^{n+1}[f](z)} - 1\right|^{\beta} < \begin{cases} (1-\alpha)^{\gamma}\left(\frac{3}{2} - \alpha\right)^{\beta}, & 0 \le \alpha \le \frac{1}{2}, \\ 2^{\beta}(1-\alpha)^{\beta+\gamma}, & \frac{1}{2} \le \alpha < 1 \end{cases}$$

*for some $\alpha$ $(0 \le \alpha < 1)$, $\beta \ge 0$ and $\gamma \ge 0$ with $\beta + \gamma > 0$, then $f \in \mathcal{S}_n(\alpha),\ n \in \mathbb{N}_0$.*

*Let $\mathcal{S}_n(p,\ \lambda,\ \alpha)$ denote the class of functions $f \in \mathcal{A}_p$ for which*

$$Re\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right) > \frac{\alpha}{p},\ z \in \mathbb{E},\ 0 \le \alpha < p.$$

*In 2008, Singh et al. [27] studied the above class and proved the following sufficient condition for a multivalent function to be a member of this class.*

**Theorem 1.3.** *Let $f \in \mathcal{A}_p$ satisfies*

$$\left|\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)} - 1\right|^{\gamma} \left|\frac{I_p(n+2,\ \lambda)[f](z)}{I_p(n+1,\ \lambda)[f](z)} - 1\right|^{\beta} < M(p,\ \lambda,\ \alpha,\ \beta,\ \gamma),\ z \in \mathbb{E},$$

*for some real numbers $\alpha,\ \beta$ and $\gamma$ such that $0 \le \alpha < p,\ \beta \ge 0,\ \gamma \ge 0,\ (\beta+\gamma) > 0$, then $f \in \mathcal{S}_n(p,\ \lambda,\ \alpha),$ where $n \in \mathbb{N}_0$ and*

$$M(p,\ \lambda,\ \alpha,\ \beta,\ \gamma) = \begin{cases} \left(1 - \frac{\alpha}{p}\right)^{\gamma}\left(1 - \frac{\alpha}{p} + \frac{1}{2(p+\lambda)}\right)^{\beta}, & 0 \le \alpha \le \frac{p}{2}, \\ \left(1 - \frac{\alpha}{p}\right)^{\gamma+\beta}\left(1 + \frac{1}{(p+\lambda)}\right)^{\beta}, & \frac{p}{2} \le \alpha < p. \end{cases}$$

*In 2013, Billing ([3], [5]) also proved the following results and obtained sufficient conditions for starlikeness and convexity of univalent functions.*

**Theorem 1.4.** *Let $\alpha,\ \beta$ be real numbers such that $\alpha > \frac{2}{1-\beta},\ 0 \le \beta < 1$ and let*

$$0 < M \equiv M(\alpha,\ \beta,\ \gamma,\ p) = \frac{(\alpha+p+\lambda)[\alpha(1-\beta)-2]}{\alpha[1+(1-\beta)(p+\lambda)]}$$

*If $f \in \mathcal{A}_p$ satisfies the differential inequality*

$$\left|(1-\alpha)\frac{I_p(n,\ \lambda)f(z)}{z^p}+\alpha\frac{I_p(n+1,\ \lambda)f(z)}{z^p}-1\right|<M(\alpha,\ \beta,\ \gamma,\ p)$$

*then*

$$Re\left(\frac{I_p(n+1,\ \lambda)f(z)}{I_p(n,\ \lambda)f(z)}\right)>\beta,\ z\in\mathbb{E}.$$

**Theorem 1.5.** *Let $\alpha$ be a non-zero complex number such that $\Re(\alpha)>0$ and $h(z)$ be analytic and convex in $\mathbb{E}$ with $h(0)=1$. If $f\in\mathcal{A}_p$ satisfies*

$$\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\left\{1+\alpha\left(\frac{I_p(n+2,\ \lambda)[f](z)}{I_p(n+1,\ \lambda)[f](z)}-\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)\right\}$$

$$\prec\frac{p+\lambda}{\alpha}\int_0^1 t^{\frac{p+\lambda}{\alpha}-1}h(zt)\,dt,$$

*then*

$$\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\prec h(z).$$

*For more contributions in this direction, we refer to Cho et al. ([12], [13]), Dattoli [15], Srivastava et al. ([29], [30]), Tang et al. [31] etc.*
*For $f\in\mathcal{A}_p$, we define a class $\mathcal{S}_n(p,\ \lambda)$ consisting of functions which satisfy*

$$Re\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)>\left|\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}-p\right|,\ z\in\mathbb{E}, \tag{1.6}$$

*where $\lambda\geq 0,\ n\in\mathbb{N}_0$. Note that $\mathcal{S}_0(p,\ 0)=\mathcal{S}^p_{\mathcal{P}}$ and $\mathcal{S}_1(p,\ 0)=UCV_p$. Condition (1.6) is equivalent to*

$$\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\prec q(z)$$

*where $q$ is given by equation (1.5).*
*In 2018, Brar and Billing ([7], [8], [9]) proved the following results and obtained sufficient conditions for starlikeness, convexity and uniform close-to-convexity of functions $f\in\mathcal{A}_p$ in a parabolic region.*

**Theorem 1.6.** *Let $\beta$ and $\gamma$ be complex numbers such that $\beta\neq 0$. Let $q(z)\neq 0$, be a univalent function in $\mathbb{E}$ such that*
$(i) Re\left[1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}\right]>0$

$(ii) Re\left[1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}+\left(1+\frac{\gamma}{\beta}\right)(p+\lambda)q(z)\right]>0.$

*If $f\in\mathcal{A}_p$ satisfies*

$$\left\{\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right\}^{\gamma}\left\{\frac{I_p(n+2,\ \lambda)[f](z)}{I_p(n+1,\ \lambda)[f](z)}\right\}^{\beta}\prec(q(z))^{\gamma}\left\{q(z)+\frac{zq'(z)}{(p+\lambda)q(z)}\right\}^{\beta},$$

*then*

$$\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\prec q(z),\ \lambda\geq 0,\ n\in\mathbb{N}_0,$$

*and $q(z)$ is the best dominant.*

**Theorem 1.7.** *Let $\alpha$ be a non-zero complex number and $q(z)\neq 0$, be a univalent function in $\mathbb{E}$ such that*
$(i) Re\left[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right]>0$

$(ii) Re\left[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{p+\lambda}{\alpha}q(z)\right]>0.$
*If $f\in\mathcal{A}_p$ satisfies*

$$(1-\alpha)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)+\alpha\left(\frac{I_p(n+2,\ \lambda)[f](z)}{I_p(n+1,\ \lambda)[f](z)}\right)\prec q(z)+\frac{\alpha}{p+\lambda}\frac{zq'(z)}{q(z)},$$

*then*

$$\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\prec q(z),\ \lambda\geq 0,\ n\in\mathbb{N}_0,$$

*and $q$ is the best dominant.*

**Theorem 1.8.** *Let $\alpha$ be a non-zero complex number and $\lambda\geq 0,\ n\in\mathbb{N}_0$. Let $q(z)\neq 0$, be a univalent function in $\mathbb{E}$ such that*
$(i) Re\left[1+\frac{zq''(z)}{q'(z)}\right]>0$
$(ii) Re\left[1+\frac{zq''(z)}{q'(z)}+\frac{p+\lambda}{\alpha}\right]>0.$
*If $f\in\mathcal{A}_p$ satisfies*

$$\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\left[1+\alpha\left(\frac{I_p(n+2,\ \lambda)[f](z)}{I_p(n+1,\ \lambda)[f](z)}-\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)\right]$$
$$\prec q(z)+\frac{\alpha}{p+\lambda}zq'(z),$$

*then*

$$\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\prec q(z),$$

*and $q$ is the best dominant.*

**Theorem 1.9.** *Let $\alpha$ be a non-zero complex number and $\lambda\geq 0,\ n\in\mathbb{N}_0$. Let $q(z)\neq 0$, be a univalent function in $\mathbb{E}$ such that*
$(i) Re\left[1+\frac{zq''(z)}{q'(z)}\right]>0$
$(ii) Re\left[1+\frac{zq''(z)}{q'(z)}+\frac{p+\lambda}{\alpha}\right]>0.$
*If $f\in\mathcal{A}_p$ satisfies*

$$(1-\alpha)\frac{I_p(n,\ \lambda)[f](z)}{z^p}+\alpha\frac{I_p(n+1,\ \lambda)[f](z)}{z^p}\prec q(z)+\frac{\alpha}{p+\lambda}zq'(z),$$

*then*

$$\frac{I_p(n,\ \lambda)[f](z)}{z^p}\prec q(z),$$

*and $q$ is the best dominant.*

*This work is inspired by various authors. For some related study see ([17], [18], [19], [20]).*

*In the present paper, we study the class $\mathcal{S}_n(p,\ \lambda)$ using differential subordination and differential superordination involving multiplier transformation $I_p(n,\ \lambda)$. As particular cases to our main results, we obtain sufficient conditions for starlikeness and convexity of multivalent as well as univalent analytic functions in a parabolic region. We also deduce some sandwich-type results regarding these functions.*

## 2 Preliminaries

*To prove our main results we shall use the following definition and lemmas of Miller-Mocanu and Bulboacă.*

**Definition 2.1.** ([23], Definition 2, p.817 )Denote by $Q$, the set of all functions $f(z)$ that are analytic and injective on $\bar{\mathbb{E}} \setminus \mathbb{E}(f)$, where

$$\mathbb{E}(f) = \left\{ \zeta \in \partial\mathbb{E} \; : \; \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{E} \setminus \mathbb{E}(f)$.

**Lemma 2.2.** *([23], Theorem 3.4h, p.132). Let $q$ be univalent in $\mathbb{E}$ and let $\theta$ and $\varphi$ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\varphi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\varphi[q(z)],\ h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either*

*(i) $h$ is convex, or*

*(ii) $Q_1$ is starlike.*
*In addition, assume that*

*(iii) $Re\left(\frac{zh'(z)}{Q_1(z)}\right) > 0$ for all $z \in \mathbb{E}$.*

*If $p$ is analytic in $\mathbb{E}$, with $p(0) = q(0),\ p(\mathbb{E}) \subset \mathbb{D}$ and*

$$\theta[p(z)] + zp'(z)\varphi[p(z)] \prec \theta[q(z)] + zq'(z)\varphi[q(z)],$$

*then $p(z) \prec q(z)$ and $q$ is the best dominant.*

**Lemma 2.3.** *([10]). Let $q$ be univalent in $\mathbb{E}$ and let $\theta$ and $\varphi$ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\varphi[q(z)],\ h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either*

*(i) $Q_1$ is starlike and*

*(ii) $Re\left(\frac{\theta'(q(z))}{\varphi(q(z))}\right) > 0$ for all $z \in \mathbb{E}$.*

*If $p \in \mathcal{H}[q(0),\ 1] \cap Q$ with $p(\mathbb{E}) \subset \mathbb{D}$ and $\theta[p(z)] + zp'(z)\varphi[p(z)]$ is univalent in $\mathbb{E}$ and*

$$\theta[q(z)] + zq'(z)\varphi[q(z)] \prec \theta[p(z)] + zp'(z)\varphi[p(z)],$$

*then $q(z) \prec p(z)$ and $q$ is the best subordinant.*

## 3 A Subordination Theorem Involving Multiplier Transformation

*In what follows, all the powers taken are principal ones.*

**Theorem 3.1.** *Let $\beta$ and $\gamma$ be complex numbers such that $\beta \neq 0$. Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

$(i) Re\left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)}\right] > 0$ *and*

$(ii) Re\left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} + \frac{a}{c}\left(1 + \frac{\gamma}{\beta}\right)q(z) + \frac{b}{c}\left(2 + \frac{\gamma}{\beta}\right)q^2(z)\right] > 0.$

*If $f \in \mathcal{A}_p$ satisfies*

$$\Gamma\left[\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)},\ z\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)';\ z\right] \prec \Gamma\left(q(z),\ zq'(z);\ z\right), \tag{3.1}$$

*where $a,\ b,\ c\ (\neq 0)$ are real numbers and*

$$\Gamma(w,\ zw';\ z) = w^{\gamma}\left(aw + bw^2 + c\frac{zw'}{w}\right)^{\beta},\ w \in \mathbb{D} = \mathbb{C}\backslash\{0\},$$

*then*

$$\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)} \prec q(z),\ \lambda \geq 0,\ n \in \mathbb{N}_0,$$

*and* $q(z)$ *is the best dominant.*

*Proof.* Let the functions $\theta$ and $\varphi$ be defined as:

$$\theta(w) = aw^{\frac{\gamma}{\beta}+1} + bw^{\frac{\gamma}{\beta}+2} \text{ and } \varphi(w) = cw^{\frac{\gamma}{\beta}-1}$$

Clearly, the functions $\theta$ and $\varphi$ are analytic in domain $\mathbb{D} = \mathbb{C}\backslash\{0\}$ and $\varphi(w) \neq 0$ in $\mathbb{D}$. Therefore,

$$Q_1(z) = \varphi(q(z))zq'(z) = c(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

and

$$h(z) = \theta(q(z)) + Q_1(z) = a(q(z))^{\frac{\gamma}{\beta}+1} + b(q(z))^{\frac{\gamma}{\beta}+2} + c(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

On differentiating, we get

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)} + \frac{a}{c}\left(1 + \frac{\gamma}{\beta}\right)q(z) + \frac{b}{c}\left(2 + \frac{\gamma}{\beta}\right)q^2(z).$$

In view of the given conditions $(i)$ and $(ii)$, we see that $Q_1$ is starlike and $Re\left(\frac{zh'(z)}{Q_1(z)}\right) > 0$. Thus conditions of Lemma [2.2], are satisfied. In view of (3.1), we have

$$\theta\left[\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right] + z\left[\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right]'\varphi\left[\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right]$$
$$\prec \theta[q(z)] + zq'(z)\varphi[q(z)]$$

Therefore, the proof, now follows from the Lemma [2.2].

□

*By setting* $\lambda = 0,\ p = 1$ *in Theorem 3.1, we obtain the following result involving Salagean operator.*

**Theorem 3.2.** *Let* $\beta$ *and* $\gamma$ *be complex numbers such that* $\beta \neq 0$ *and* $q(z) \neq 0$*, be a univalent function in* $\mathbb{E}$*, which satisfy conditions* $(i)$ *and* $(ii)$ *of Theorem 3.1. If* $f \in \mathcal{A},\ z \in \mathbb{E}$*, satisfies*

$$\left\{\frac{D^{n+1}[f](z)}{D^n[f](z)}\right\}^\gamma \left\{(a-c)\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right) + b\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right)^2 + c\left(\frac{D^{n+2}[f](z)}{D^{n+1}[f](z)}\right)\right\}^\beta$$
$$\prec (q(z))^\gamma\left\{aq(z) + bq^2(z) + c\frac{zq'(z)}{q(z)}\right\}^\beta \qquad (3.2)$$

*where* $a,\ b$ *and* $c\ (\neq 0)$ *are real numbers, then*

$$\frac{D^{n+1}[f](z)}{D^n[f](z)} \prec q(z),\ n \in \mathbb{N}_0,$$

*and* $q(z)$ *is the best dominant.*

## 4 Applications to Parabolic Starlikeness and Uniform Convexity

**Remark 4.1.** Selecting $q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$, $\beta = \gamma = 1$ in Theorem 3.1, then after having some calculations, we get

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}q^2(z) = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} + \frac{2a}{c}\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right] + \frac{3b}{c}\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2.$$

Thus for $a,\ b,\ c\ (\neq 0) \in \mathbb{R}$ such that $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0$, we notice that $q(z)$ satisfy conditions $(i)$ and $(ii)$ of Theorem 3.1. Thus, we derive the following result:

**Theorem 4.2.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0$. If $f \in \mathcal{A}_p$ satisfies*

$$(a - cp - c\lambda)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)^2 + b\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)^3 + c(p+\lambda)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)\left(\frac{I_p(n+2,\ \lambda)[f](z)}{I_p(n+1,\ \lambda)[f](z)}\right)$$
$$\prec a\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2 + b\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3 + \frac{4c\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),\ \textit{where } \lambda \geq 0,\ n \in \mathbb{N}_0,$$

*then*

$$\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

*i.e.* $f \in S_n(p,\ \lambda)$.

*Taking $\lambda = n = 0$ in above Theorem, we get the following result.*

**Corollary 4.3.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0$. If $f \in \mathcal{A}_p$ satisfies*

$$(a - cp)\left(\frac{zf'(z)}{pf(z)}\right)^2 + b\left(\frac{zf'(z)}{pf(z)}\right)^3 + c\left(\frac{zf'(z)}{pf(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right)$$
$$\prec a\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2 + b\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3 + \frac{4c\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

*then* $f \in S^p_{\mathcal{P}}$.

*Setting $p = 1$ in above corollary, we have*

**Corollary 4.4.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0$. If $f \in \mathcal{A}$ satisfies*

$$(a-c)\left(\frac{zf'(z)}{f(z)}\right)^2 + b\left(\frac{zf'(z)}{f(z)}\right)^3 + c\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)$$

$$\prec a\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2 + b\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3$$

$$+\frac{4c\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

*then $f \in S_{\mathcal{P}}$.*

*Setting $\lambda = 0,\ n = 1$ in Theorem 4.2, we obtain the result given below.*

**Corollary 4.5.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{a-cp}{p^2}\left(1+\frac{zf''(z)}{f'(z)}\right)^2 + \frac{b}{p^3}\left(1+\frac{zf''(z)}{f'(z)}\right)^3 + \frac{c}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)$$

$$\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec a\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2$$

$$+b\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3 + \frac{4c\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

*then $f \in UCV_p$.*

*Taking $p = 1$ in above corollary, we get*

**Corollary 4.6.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0$. If $f \in \mathcal{A}$ satisfies*

$$(a-c)\left(1+\frac{zf''(z)}{f'(z)}\right)^2 + b\left(1+\frac{zf''(z)}{f'(z)}\right)^3 + c\left(1+\frac{zf''(z)}{f'(z)}\right)$$

$$\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec a\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2$$

$$+b\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3 + \frac{4c\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

*then $f \in UCV$.*

## 5 Applications to Starlikeness and Convexity

**Remark 5.1.** When we select $q(z) = e^z,\ \beta = \gamma = 1$ in Theorem 3.1, a little calculation yields that

$$1+\frac{zq''(z)}{q'(z)} = 1+z$$

and

$$1+\frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}\,q^2(z) = 1+z+\frac{2a}{c}e^z + \frac{3b}{c}e^{2z}.$$

For $a,\ b,\ c\ (\neq 0) \in \mathbb{R}$ such that $\frac{a}{c} \geq 0.4$ and $\frac{b}{c} = 1$, we see that $q(z)$ satisfy conditions $(i)$ and $(ii)$ of Theorem 3.1. Hence, we obtain the following result:

**Theorem 5.2.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\dfrac{a}{c} \geq 0.4$ and $\dfrac{b}{c} = 1.$ If $f \in \mathcal{A}_p$ satisfies*

$$(a-cp-c\lambda)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)^2 + b\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)^3$$
$$+c(p+\lambda)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)\left(\frac{I_p(n+2,\ \lambda)[f](z)}{I_p(n+1,\ \lambda)[f](z)}\right)$$
$$\prec ae^{2z} + be^{3z} + cze^{z},$$

*then* $\dfrac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)} \prec e^z,\ \lambda \geq 0,\ n \in \mathbb{N}_0.$

*By taking $\lambda = n = 0$ in above Theorem, we obtain the result given below.*

**Corollary 5.3.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\dfrac{a}{c} \geq 0.4$ and $\dfrac{b}{c} = 1.$ If $f \in \mathcal{A}_p$ satisfies*

$$(a-cp)\left(\frac{zf'(z)}{pf(z)}\right)^2 + b\left(\frac{zf'(z)}{pf(z)}\right)^3 + c\left(\frac{zf'(z)}{pf(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$\prec ae^{2z} + be^{3z} + cze^{z},$$

*then $f \in S_p^*$.*

*Selecting $p = 1$ in above corollary, we have*

**Corollary 5.4.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\dfrac{a}{c} \geq 0.4$ and $\dfrac{b}{c} = 1.$ If $f \in \mathcal{A}$ satisfies*

$$(a-c)\left(\frac{zf'(z)}{f(z)}\right)^2 + b\left(\frac{zf'(z)}{f(z)}\right)^3 + c\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$\prec ae^{2z} + be^{3z} + cze^{z},$$

*then $f \in S^*$.*

*Setting $\lambda = 0,\ n = 1$ in Theorem 5.2, we derive the result given below.*

**Corollary 5.5.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\dfrac{a}{c} \geq 0.4$ and $\dfrac{b}{c} = 1.$ If $f \in \mathcal{A}_p$ satisfies*

$$\frac{a-cp}{p^2}\left(1+\frac{zf''(z)}{f'(z)}\right)^2 + \frac{b}{p^3}\left(1+\frac{zf''(z)}{f'(z)}\right)^3 + \frac{c}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec ae^{2z} + be^{3z} + cze^{z},$$

*then $f \in \mathcal{K}_p$.*

*Setting $p = 1$ in above corollary, we have*

**Corollary 5.6.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\dfrac{a}{c} \geq 0.4$ and $\dfrac{b}{c} = 1.$ If $f \in \mathcal{A}$ satisfies*

$$(a-c)\left(1+\frac{zf''(z)}{f'(z)}\right)^2 + b\left(1+\frac{zf''(z)}{f'(z)}\right)^3 + c\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec ae^{2z} + be^{3z} + cze^{z},$$

*then $f \in \mathcal{K}$.*

**Remark 5.7.** By taking $q(z) = 1 + tz,\ 0 < t \leq 1,\ \beta = \gamma = 1$ in Theorem 3.1, then after having some calculations we have

$$1 + \frac{zq''(z)}{q'(z)} = 1$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}\, q^2(z) = 1 + \frac{2a}{c}(1 + tz) + \frac{3b}{c}(1 + tz)^2.$$

Thus for $a,\ b,\ c\ (\neq 0) \in \mathbb{R}$ such that $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0$, we observe that $q(z)$ satisfy conditions $(i)$ and $(ii)$ of Theorem 3.1. Therefore, we immediately, arrive at the following result:

**Theorem 5.8.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0.$ If $f \in \mathcal{A}_p$ satisfies*

$$(a - cp - c\lambda)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)^2 + b\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)^3 + c(p+\lambda)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)\left(\frac{I_p(n+2,\ \lambda)[f](z)}{I_p(n+1,\ \lambda)[f](z)}\right) \prec a(1+tz)^2 + b(1+tz)^3 + ctz,$$

*then* $\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)} \prec 1 + tz,\ 0 < t \leq 1,\ \lambda \geq 0,\ n \in \mathbb{N}_0.$

*Setting $\lambda = n = 0$ in above Theorem, we get the following result.*

**Corollary 5.9.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0.$ If $f \in \mathcal{A}_p$ satisfies*

$$(a - cp)\left(\frac{zf'(z)}{pf(z)}\right)^2 + b\left(\frac{zf'(z)}{pf(z)}\right)^3 + c\left(\frac{zf'(z)}{pf(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec a(1+tz)^2 + b(1+tz)^3 + ctz;\ 0 < t \leq 1,$$

*then $f \in S_p^*$.*

*Selecting $p = 1$ in above corollary, we have*

**Corollary 5.10.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0.$ If $f \in \mathcal{A}$ satisfies*

$$(a - c)\left(\frac{zf'(z)}{f(z)}\right)^2 + b\left(\frac{zf'(z)}{f(z)}\right)^3 + c\left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec a(1+tz)^2 + b(1+tz)^3 + ctz,\ 0 < t \leq 1,$$

*then $f \in S^*$.*

*Setting $\lambda = 0,\ n = 1$ in Theorem 5.8, we derive the result given below.*

**Corollary 5.11.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0.$ If $f \in \mathcal{A}_p$ satisfies*

$$\frac{a - cp}{p^2}\left(1 + \frac{zf''(z)}{f'(z)}\right)^2 + \frac{b}{p^3}\left(1 + \frac{zf''(z)}{f'(z)}\right)^3 + \frac{c}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)\left(1 + \frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)}\right) \prec a(1+tz)^2 + b(1+tz)^3 + ctz,$$

*then $f \in \mathcal{K}_p,\ 0 < t \leq 1.$*

*Taking $p = 1$ in above corollary, we have*

**Corollary 5.12.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0$. If $f \in \mathcal{A}$ satisfies*

$$(a-c)\left(1+\frac{zf''(z)}{f'(z)}\right)^2+b\left(1+\frac{zf''(z)}{f'(z)}\right)^3+c\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec a(1+tz)^2+b(1+tz)^3+ctz,$$

*then*

$$1+\frac{zf''(z)}{f'(z)} \prec 1+tz;\ 0<t\leq 1.$$

*i.e. $f \in \mathcal{K}$.*

**Remark 5.13.** By selecting $q(z)=1+\frac{2}{3}z^2,\ \beta=\gamma=1$ in Theorem 3.1, we have

$$1+\frac{zq''(z)}{q'(z)}=2$$

and

$$1+\frac{zq''(z)}{q'(z)}+\frac{2a}{c}q(z)+\frac{3b}{c}\,q^2(z)=2+\frac{2a}{c}\left(1+\frac{2}{3}z^2\right)+\frac{3b}{c}\left(1+\frac{2}{3}z^2\right)^2.$$

For $a,\ b,\ c\ (\neq 0) \in \mathbb{R}$ such that $\frac{a}{c} \geq -0.6$ and $\frac{b}{c} \geq 0$, we notice that $q(z)$ satisfy conditions $(i)$ and $(ii)$ of Theorem 3.1. Hence, we obtain the following result:

**Theorem 5.14.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq -0.6$ and $\frac{b}{c} \geq 0$. If $f \in \mathcal{A}_p$ satisfies*

$$(a-cp-c\lambda)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)^2+b\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)^3$$
$$+c(p+\lambda)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)\left(\frac{I_p(n+2,\ \lambda)[f](z)}{I_p(n+1,\ \lambda)[f](z)}\right)$$
$$\prec a\left(1+\frac{2}{3}z^2\right)^2+b\left(1+\frac{2}{3}z^2\right)^3+\frac{4}{3}cz^2,$$

*then* $\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)} \prec e^z,\ \lambda \geq 0,\ n \in \mathbb{N}_0.$

*By taking $\lambda=n=0$ in above Theorem, we obtain the result given below.*

**Corollary 5.15.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq -0.6$ and $\frac{b}{c} \geq 0$. If $f \in \mathcal{A}_p$ satisfies*

$$(a-cp)\left(\frac{zf'(z)}{pf(z)}\right)^2+b\left(\frac{zf'(z)}{pf(z)}\right)^3+c\left(\frac{zf'(z)}{pf(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$\prec a\left(1+\frac{2}{3}z^2\right)^2+b\left(1+\frac{2}{3}z^2\right)^3+\frac{4}{3}cz^2,$$

*then $f \in S_p^*$.*

*Selecting $p=1$ in above corollary, we have*

**Corollary 5.16.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq -0.6$ and $\frac{b}{c} \geq 0$. If $f \in \mathcal{A}$ satisfies*

$$(a-c)\left(\frac{zf'(z)}{f(z)}\right)^2+b\left(\frac{zf'(z)}{f(z)}\right)^3+c\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$\prec a\left(1+\frac{2}{3}z^2\right)^2+b\left(1+\frac{2}{3}z^2\right)^3+\frac{4}{3}cz^2,$$

*then $f \in S^*$.*

*Setting $\lambda = 0,\ n = 1$ in Theorem 5.14, we derive the result given below.*

**Corollary 5.17.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq -0.6$ and $\frac{b}{c} \geq 0$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{a-cp}{p^2}\left(1+\frac{zf''(z)}{f'(z)}\right)^2+\frac{b}{p^3}\left(1+\frac{zf''(z)}{f'(z)}\right)^3+\frac{c}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right)\prec a\left(1+\frac{2}{3}z^2\right)^2+b\left(1+\frac{2}{3}z^2\right)^3+\frac{4}{3}cz^2,$$

*then $f \in \mathcal{K}_p$.*

*Setting $p = 1$ in above corollary, we have*

**Corollary 5.18.** *Let $a,\ b$ and $c \neq 0$ are real numbers such that $\frac{a}{c} \geq -0.6$ and $\frac{b}{c} \geq 0$. If $f \in \mathcal{A}$ satisfies*

$$(a-c)\left(1+\frac{zf''(z)}{f'(z)}\right)^2+b\left(1+\frac{zf''(z)}{f'(z)}\right)^3+c\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right)\prec a\left(1+\frac{2}{3}z^2\right)^2+b\left(1+\frac{2}{3}z^2\right)^3+\frac{4}{3}cz^2,$$

*then $f \in \mathcal{K}$.*

## 6 A Superordination Theorem Involving Multiplier Transformation

**Theorem 6.1.** *Let $\beta$ and $\gamma$ be complex numbers such that $\beta \neq 0$. Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

$(i) Re\left[1+\frac{zq''(z)}{q'(z)}+\left(\frac{\gamma}{\beta}-1\right)\frac{zq'(z)}{q(z)}\right] > 0$ *and*

$(ii) Re\left[\frac{a}{c}\left(1+\frac{\gamma}{\beta}\right)q(z)+\frac{b}{c}\left(2+\frac{\gamma}{\beta}\right)q^2(z)\right] > 0.$

*Suppose $\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)} \in \mathcal{H}[q(0),\ 1]] \cap Q$, and*

$\Gamma\left[\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)},\ z\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)';\ z\right]$ *is univalent in $\mathbb{E}$. If $f \in \mathcal{A}_p$ satisfies the differential superordination*

$$\Gamma\left(q(z),\ zq'(z);\ z\right) \prec \Gamma\left[\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)},\ z\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)';\ z\right], \tag{6.1}$$

*where $a,\ b,\ c\ (\neq 0)$ are real numbers and*

$$\Gamma(w,\ zw';\ z) = w^{\gamma}\left(aw+bw^2+c\frac{zw'}{w}\right)^{\beta},\ w \in \mathbb{D} = \mathbb{C}\backslash\{0\},$$

*then*

$$q(z) \prec \frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)},\ \lambda \geq 0,\ n \in \mathbb{N}_0,$$

*and $q(z)$ is the best subordinant.*

*Proof.* Let the functions $\theta$ and $\varphi$ be defined as:

$$\theta(w) = aw^{\frac{\gamma}{\beta}+1}+bw^{\frac{\gamma}{\beta}+2} \text{ and } \varphi(w) = cw^{\frac{\gamma}{\beta}-1}$$

Clearly, the functions $\theta$ and $\varphi$ are analytic in domain $\mathbb{D} = \mathbb{C}\backslash\{0\}$ and $\varphi(w) \neq 0$ in $\mathbb{D}$. Therefore,

$$Q_1(z) = \varphi(q(z))zq'(z) = c(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

and

$$h(z) = \theta(q(z)) + Q_1(z) = a(q(z))^{\frac{\gamma}{\beta}+1} + b(q(z))^{\frac{\gamma}{\beta}+2} + c(q(z))^{\frac{\gamma}{\beta}-1}zq'(z)$$

On differentiating, we get

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq'(z)}{q(z)}$$

and

$$\frac{\theta'(q(z))}{\varphi(q(z))} = \frac{zh'(z)}{Q_1(z)} - \frac{zQ_1'(z)}{Q_1(z)} = \frac{a}{c}\left(1 + \frac{\gamma}{\beta}\right)q(z) + \frac{b}{c}\left(2 + \frac{\gamma}{\beta}\right)q^2(z).$$

In view of the given conditions $(i)$ and $(ii)$, we see that $Q_1$ is starlike and $Re\left(\frac{\theta'(q(z))}{\varphi(q(z))}\right) > 0$. Thus by (6.1), we have

$$\theta[q(z)] + zq'(z)\varphi[q(z)] \prec$$
$$\theta\left[\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right] + z\left[\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right]'\varphi\left[\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right]$$

Therefore, the proof, now follows from the Lemma [2.3].

□

## 7 Some Sandwich-type Results

*On combining Theorem 3.1 and Theorem 6.1, we get the following sandwich-type theorem.*

**Theorem 7.1.** *Let $\beta$ and $\gamma$ be complex numbers such that $\beta \neq 0$. Let $q_1(z) \neq 0$ and $q_2(z) \neq 0$ be univalent functions in $\mathbb{E}$ such that*

$$(i) Re\left[1 + \frac{zq_i''(z)}{q_i'(z)} + \left(\frac{\gamma}{\beta} - 1\right)\frac{zq_i'(z)}{q_i(z)}\right] > 0 \ and$$

$$(ii) Re\left[\frac{a}{c}\left(1 + \frac{\gamma}{\beta}\right)q_i(z) + \frac{b}{c}\left(2 + \frac{\gamma}{\beta}\right)q_i^2(z)\right] > 0;\ i = 1,\ 2.$$

*Suppose* $\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)} \in \mathcal{H}[q(0),\ 1]] \cap Q$, *and* $\Gamma\left[\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)},\ z\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)';\ z\right]$ *is univalent in $\mathbb{E}$. If $f \in \mathcal{A}_p$ satisfies*

$$\Gamma\left(q_1(z),\ zq_1'(z);\ z\right) \prec \Gamma\left[\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)},\ z\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)';\ z\right]$$
$$\prec \Gamma\left(q_2(z),\ zq_2'(z);\ z\right),$$

*where $a,\ b,\ c\ (\neq 0)$ are real numbers and*

$$\Gamma(w,\ zw';\ z) = w^{\gamma}\left(aw + bw^2 + c\frac{zw'}{w}\right)^{\beta},\ w \in \mathbb{D} = \mathbb{C}\backslash\{0\},$$

*then*

$$q_1(z) \prec \frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)} \prec q_2(z),\ \lambda \geq 0,\ n \in \mathbb{N}_0,$$

*and $q_1(z)$ and $q_2(z)$ are the best subordinant and the best dominant respectively.*

*When we select* $\beta = \gamma = a = b = c = 1,\ q_1(z) = 1 + tz;\ 0 < t \leq 0.4$ *and* $q_2(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ *in Theorem 7.1, we get the following result:*

**Theorem 7.2.** *If* $f \in \mathcal{A}_p$ *satisfies*

$$(1+tz)^2 + (1+tz)^3 + tz \prec (1-p-\lambda)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)^2$$

$$+\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)^3 + (p+\lambda)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)\left(\frac{I_p(n+2,\ \lambda)[f](z)}{I_p(n+1,\ \lambda)[f](z)}\right)$$

$$\prec \left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2 + \left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3$$

$$+\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),\ where\ \lambda \geq 0,\ n \in \mathbb{N}_0,$$

*then*

$$1 + tz \prec \frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2;\ 0 < t \leq 0.4.$$

*Taking* $\lambda = n = 0$ *in above Theorem, we get the following result.*

**Corollary 7.3.** *If* $f \in \mathcal{A}_p$ *satisfies*

$$(1+tz)^2 + (1+tz)^3 + tz \prec (1-p)\left(\frac{zf'(z)}{pf(z)}\right)^2 + \left(\frac{zf'(z)}{pf(z)}\right)^3$$

$$+\left(\frac{zf'(z)}{pf(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2$$

$$+\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3 + \frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

*then*

$$1 + tz \prec \frac{zf'(z)}{pf(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2;\ 0 < t \leq 0.4.$$

*Setting* $p = 1$ *in above corollary, we have*

**Corollary 7.4.** *If* $f \in \mathcal{A}$ *satisfies*

$$(1+tz)^2 + (1+tz)^3 + tz \prec \left(\frac{zf'(z)}{f(z)}\right)^3 + \left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)$$

$$\prec \left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2 + \left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3$$

$$+\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

*then*

$$1 + tz \prec \frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2;\ 0 < t \leq 0.4.$$

*Setting* $\lambda = 0,\ n = 1$ *in Theorem 7.2, we obtain the result given below.*

**Corollary 7.5.** *If $f \in \mathcal{A}_p$ satisfies*

$$(1+tz)^2+(1+tz)^3+tz \prec \frac{1-p}{p^2}\left(1+\frac{zf''(z)}{f'(z)}\right)^2+\frac{1}{p^3}\left(1+\frac{zf''(z)}{f'(z)}\right)^3$$
$$+\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right)$$
$$\prec \left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2+\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3$$
$$+\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

*then*

$$1+tz \prec \frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2;\ 0<t\le 0.4.$$

*Taking $p=1$ in above corollary, we get*

**Corollary 7.6.** *If $f \in \mathcal{A}$ satisfies*

$$(1+tz)^2+(1+tz)^3+tz \prec \left(1+\frac{zf''(z)}{f'(z)}\right)^3$$
$$+\left(1+\frac{zf''(z)}{f'(z)}\right)\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec \left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2$$
$$+\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3+\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

*then*

$$1+tz \prec 1+\frac{zf''(z)}{f'(z)} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2;\ 0<t\le 0.4.$$

*In particular, taking $t=0.3$ in Corollary 7.4 and Corollary 7.6, we get the following two examples respectively.*

**Example 7.7.** If $f \in \mathcal{A}$ satisfies

$$(1+0.3z)^2+(1+0.3z)^3+0.3z \prec \left(\frac{zf'(z)}{f(z)}\right)^3+\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$\prec \left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2+\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3$$
$$+\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

then

$$1+0.3z \prec \frac{zf'(z)}{f(z)} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

**Example 7.8.** If $f \in \mathcal{A}$ satisfies

$$(1+0.3z)^2+(1+0.3z)^3+0.3z \prec \left(1+\frac{zf''(z)}{f'(z)}\right)^3$$
$$+\left(1+\frac{zf''(z)}{f'(z)}\right)\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec \left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2$$

$$+\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3+\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

then

$$1+0.3z\prec 1+\frac{zf''(z)}{f'(z)}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

*For illustration, using Mathematica 7.0, we plot the images of unit disk* $\mathbb{E}$ *under the functions* $w_1(z)=(1+0.3z)^2+(1+0.3z)^3+0.3z$ *and*

$$w_2(z)=\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2+\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^3$$

$$+\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

*in Figure 7.1. The images of unit disk* $\mathbb{E}$ *under the functions* $q_1(z)=1+0.3z$ *and* $q_2(z)=1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ *are given in Figure 7.2. In view of Example 7.7, we conclude that when the differential operator*

$$\left(\frac{zf'(z)}{f(z)}\right)^3+\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)$$

*takes values in the light shaded region in Figure 7.1, then* $\frac{zf'(z)}{f(z)}$ *takes values in the light shaded portion in Figure 7.2. Thus* $f(z)$ *is parabolic starlike in* $\mathbb{E}$*. Similarly, from Example 7.8, we can say that when the differential operator*

$$\left(1+\frac{zf''(z)}{f'(z)}\right)^3+\left(1+\frac{zf''(z)}{f'(z)}\right)\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right)$$

*takes values in the light shaded portion in Figure 7.1, then* $1+\frac{zf''(z)}{f'(z)}$ *takes values in the light shaded region in Figure 7.2. Hence* $f(z)$ *is uniformly convex in* $\mathbb{E}$*.*

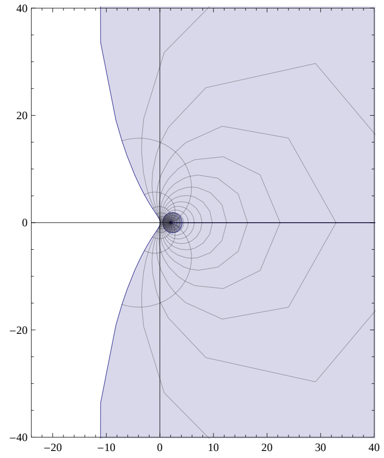


Figure 7.1

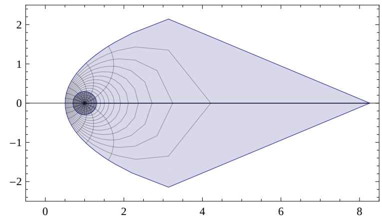


Figure 7.2

*By taking* $a = 2,\ \beta = \gamma = b = c = 1,\ q_1(z) = 1 + tz;\ 0 < t \leq 0.6$ *and* $q_2(z) = e^z$ *in Theorem 7.1, we have*

**Theorem 7.9.** *If* $f \in \mathcal{A}_p$ *satisfies*

$$2(1+tz)^2 + (1+tz)^3 + tz \prec (2-p-\lambda)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)^2$$

$$+\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)^3 + (p+\lambda)\left(\frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)}\right)\left(\frac{I_p(n+2,\ \lambda)[f](z)}{I_p(n+1,\ \lambda)[f](z)}\right)$$

$$\prec 2e^{2z} + e^{3z} + ze^z,\ where\ \lambda \geq 0,\ n \in \mathbb{N}_0,$$

*then*

$$1 + tz \prec \frac{I_p(n+1,\ \lambda)[f](z)}{I_p(n,\ \lambda)[f](z)} \prec e^z;\ 0 < t \leq 0.6.$$

*By taking* $\lambda = n = 0$ *in above Theorem, we obtain the result given below.*

**Corollary 7.10.** *If* $f \in \mathcal{A}_p$ *satisfies*

$$2(1+tz)^2 + (1+tz)^3 + tz \prec (2-p)\left(\frac{zf'(z)}{pf(z)}\right)^2 + \left(\frac{zf'(z)}{pf(z)}\right)^3$$

$$+\left(\frac{zf'(z)}{pf(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 2e^{2z} + e^{3z} + ze^z,$$

*then*

$$1 + tz \prec \frac{zf'(z)}{pf(z)} \prec e^z;\ 0 < t \leq 0.6.$$

*Selecting* $p = 1$ *in above corollary, we have*

**Corollary 7.11.** *If* $f \in \mathcal{A}$ *satisfies*

$$2(1+tz)^2 + (1+tz)^3 + tz \prec \left(\frac{zf'(z)}{f(z)}\right)^2 + \left(\frac{zf'(z)}{f(z)}\right)^3$$

$$+\left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 2e^{2z} + e^{3z} + ze^z,$$

*then*

$$1 + tz \prec \frac{zf'(z)}{f(z)} \prec e^z;\ 0 < t \leq 0.6.$$

*Setting* $\lambda = 0,\ n = 1$ *in Theorem 7.9, we derive the result given below.*

**Corollary 7.12.** *If* $f \in \mathcal{A}_p$ *satisfies*

$$2(1+tz)^2 + (1+tz)^3 + tz \prec \frac{2-p}{p^2}\left(1 + \frac{zf''(z)}{f'(z)}\right)^2 + \frac{1}{p^3}\left(1 + \frac{zf''(z)}{f'(z)}\right)^3$$

$$+\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)\left(1 + \frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)}\right) \prec 2e^{2z} + e^{3z} + ze^z,$$

*then*

$$1 + tz \prec \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec e^z;\ 0 < t \leq 0.6.$$

*Setting $p = 1$ in above corollary, we have*

**Corollary 7.13.** *If $f \in \mathcal{A}$ satisfies*

$$2(1+tz)^2 + (1+tz)^3 + tz \prec \left(1+\frac{zf''(z)}{f'(z)}\right)^2 + \left(1+\frac{zf''(z)}{f'(z)}\right)^3$$
$$+\left(1+\frac{zf''(z)}{f'(z)}\right)\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec 2e^{2z} + e^{3z} + ze^z,$$

*then*

$$1+tz \prec 1+\frac{zf''(z)}{f'(z)} \prec e^z;\ 0 < t \leq 0.6.$$

*Particularly, selecting $t = 0.2$ in Corollary 7.11 and Corollary 7.13, we obtain the following two examples respectively.*

**Example 7.14.** If $f \in \mathcal{A}$ satisfies

$$2(1+0.2z)^2 + (1+0.2z)^3 + 0.2z \prec \left(\frac{zf'(z)}{f(z)}\right)^2 + \left(\frac{zf'(z)}{f(z)}\right)^3$$
$$+\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 2e^{2z} + e^{3z} + ze^z,$$

then

$$1+0.2z \prec \frac{zf'(z)}{f(z)} \prec e^z.$$

**Example 7.15.** If $f \in \mathcal{A}$ satisfies

$$2(1+0.2z)^2 + (1+0.2z)^3 + 0.2z \prec \left(1+\frac{zf''(z)}{f'(z)}\right)^2 + \left(1+\frac{zf''(z)}{f'(z)}\right)^3$$
$$+\left(1+\frac{zf''(z)}{f'(z)}\right)\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec 2e^{2z} + e^{3z} + ze^z,$$

then

$$1+0.2z \prec 1+\frac{zf''(z)}{f'(z)} \prec e^z.$$

*Again, using Mathematica 7.0, we plot the images of unit disk $\mathbb{E}$ under the functions $w_3(z) = 2(1+0.2z)^2 + (1+0.2z)^3 + 0.2z$ and $w_4(z) = 2e^{2z} + e^{3z} + ze^z$, which are given by Figure 7.3.*

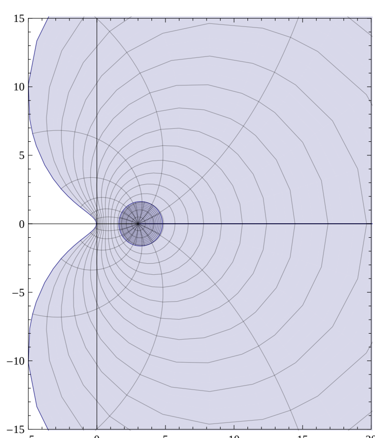


Figure 7.3

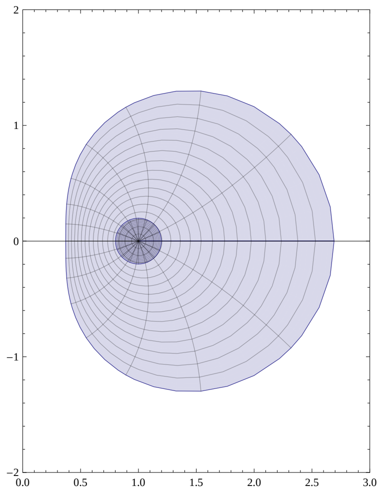


Figure 7.4

*The images of unit disk* $\mathbb{E}$ *under the functions* $q_3(z) = 1 + 0.2z$ *and* $q_4(z) = e^z$ *are given by Figure 7.4. Therefore, from Example 7.14, we notice that the differential operator* $\dfrac{zf'(z)}{f(z)}$ *takes values in the light shaded region of Figure 7.4, when the differential operator*

$$\left(\frac{zf'(z)}{f(z)}\right)^2 + \left(\frac{zf'(z)}{f(z)}\right)^3 + \left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

*takes values in the light shaded region of Figure 7.3. Thus the function* $f(z)$ *is starlike in* $\mathbb{E}$*. Similarly, in Example 7.15, we observe that the differential operator* $1 + \dfrac{zf''(z)}{f'(z)}$ *takes values in the light shaded region of Figure 7.4, when the operator*

$$\left(1 + \frac{zf''(z)}{f'(z)}\right)^2 + \left(1 + \frac{zf''(z)}{f'(z)}\right)^3 + \left(1 + \frac{zf''(z)}{f'(z)}\right)\left(1 + \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)}\right)$$

*takes values in the light shaded region of Figure 7.3 and hence the function* $f(z)$ *is convex in* $\mathbb{E}$*.*

## 8 Conclusion

*In this paper, we successfully established sufficient conditions for starlikeness and convexity of both multivalent and univalent analytic functions by employing multiplier transformations. Through the use of differential subordination and superordination, we examined the class* $\mathcal{S}_n(p, \lambda)$ *and derived significant results, including specific criteria for starlikeness and convexity within a parabolic region. Additionally, we presented sandwich-type results that further broaden the understanding of these function classes. The graphical illustrations generated using Mathematica 7.0 substantiate the theoretical findings, offering visual confirmation of the behavior of the analytic functions within the open unit disk. This research contributes to the field by enhancing the analytical tools available for studying the geometric properties of analytic functions in complex analysis.*

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