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CUBIC FORMS AND B-MINIMAL LIGHTLIKE HYPERSURFACES WITH ITS BLASCHKE STRUCTURE

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Abstract In this paper, we introduce the notion of cubic form L on lightlike hypersurfaces with its Blaschke structure. We investigate relations between some properties of the cubic form and some geometric aspects of lightlike hypersurfaces. The expression for the variation of the volume functional of the second fundamental form of a lightlike hypersurface in a semi-Euclidean space involves the so-called *mean curvature of the second fundamental form* are given.

1 Introduction

The theory of hypersurfaces, defined as submanifolds of codimension one, is one of the fundamental theories of submanifolds. As it is known, the main difference between the geometry of hypersurface in Riemannian manifold and in semi-Riemannian manifold is that in the latter case the induced metric tensor field by the semi-Riemannian metric on the ambient space is not necessarily non-degenerate. If the induced metric tensor field is degenerate, the classical theory of Riemannian and semi-Riemannian hypersurfaces fails since the normal bundle and the tangent bundle of the hypersurface have a non zero intersection.

In [3], C. Atindogbe introduced the Blaschke structure for a class of lightlike hypersurfaces in the Lorentz space $\mathbb{R}_1^{n+2}$, $(n \geq 1)$. More precisely, for a 1-degenerate lightlike hypersurface immersion, he shows the existence of a unique (up to sign) normalized null transversal vector field that is equi-affine and satisfies some apolarity conditions. In the present paper, we introduce the notion of cubic form L on these lightlike hypersurfaces with its Blaschke structure and make a systematic study of this cubic form. We investigate relations between some properties of the cubic form L and some geometric aspects of lightlike hypersurfaces. For some related study see [1] and [7]. In Section 3 we summarize notions and basic facts concerning Blaschke type normalization on lightlike hypersurfaces M. In Section 4 we show the existence of a unique connection $\widetilde{\nabla}$ in addition to the induced connection ∇ on Blaschke lightlike hypersurface. We investigate the case where the cubic form is identically zero. In Section 5 we study the symmetry properties of the first covariant derivative of the cubic form. Section 6 is devoted to the case that the Blaschke structure is conformal. Finally, in Section 7 , we study the variation of the volume of such hypersurfaces. The expression for the variation of the volume functional of the second fundamental form of a lightlike hypersurface in a semi-Euclidean space involves the so-called *mean curvature of the second fundamental form* are given.

2 Preliminaries on Lightlike hypersurfaces

Let $(\overline{M}, \overline{g})$ be a $(m+2)$-dimensional semi-Riemannian manifold of index ν, $(0 < \nu < m+2)$. Consider a hypersurface M of $\overline{M}$ and denote by g the tensor field induced by $\overline{g}$ on M. We say that M is a lightlike (degenerate, null) hypersurface if $rank(g) = m$. Then the normal vector bundle $TM^{\perp}$ intersects the tangent bundle along a nonzero differentiable distribution called the radical distribution of M and denoted by $Rad(TM)$:

$$Rad(TM) : x \mapsto Rad(T_xM) = T_xM \cap T_xM^{\perp}. \tag{2.1}$$

A $screen\ distribution$ $S(TM)$ on M is a non-degenerate vector bundle complementary to $TM^{\perp}$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following result has an important role in the study of the geometry of lightlike hypersurfaces.

Theorem 2.1. *[6] Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of tr(TM) on $\mathcal{U}$ satisfying*

$$\overline{g}(N, \xi) = 1 \textit{ and } \overline{g}(N, N) = \overline{g}(N, W) = 0, \tag{2.2}$$

for all $W \in \Gamma(S(TM)_{|\mathcal{U}})$.

With this theorem we may write the following decomposition:

$$T\overline{M}_{|M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM), \tag{2.3}$$

where $\perp$ denotes an orthogonal direct sum and $\oplus$ a direct sum. Throughout the paper, we denoted by $\Gamma(E)$ the $C^{\infty}(M)$-module of smooth sections of a vector bundle E over M, while $C^{\infty}(M)$ represents the algebra of a smooth functions on M. Also, all manifolds are supposed to be smooth, paracompact and connected.

Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, $\overline{\nabla}$ be the Levi-Civita connection of $\overline{M}$, ∇ the induced connection on (M, g). Gauss and Weingarten formulas provide the following relations (see details in [6], section 4.2)

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.4}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.5}$$

for all $X, Y \in \Gamma(TM)$ and $V \in tr(TM)$, where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$ while h is a $\Gamma(tr(TM))$-valued symmetric $C^{\infty}(M)$-bilinear form on $\Gamma(TM)$ and ∇^t is a linear connection on $tr(TM)$. It is easy to see that ∇ is a torsion-free connection. Define a symmetric $C^{\infty}(M)$-bilinear form B and a 1-form τ on the coordinate neighborhood $\mathcal{U} \subset M$ by

$$B(X, Y) = \overline{g}(h(X, Y), \xi), \tag{2.6}$$

$$\tau(X) = \overline{g}(\nabla^t_X N, \xi), \tag{2.7}$$

for all $X, Y \in \Gamma(TM_{|\mathcal{U}})$. Then, on $\mathcal{U}$, equations (2.4) and (2.5) become,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.8}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.9}$$

respectively. It is important to stress the fact that the local second fundamental form B in Eq.(2.8) does not depend on the choice of the screen distribution and satisfies,

$$B(X, \xi) = 0, \tag{2.10}$$

for all $X \in \Gamma(TM|_{\mathcal{U}})$. Let P be the projection morphism of TM to $S(TM)$ with respect to the decomposition (2.2). We obtain: for all $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$,

$$\nabla_X PY = \overset{*}{\nabla}_X PY + \overset{*}{h}(X, PY), \tag{2.11}$$

$$\nabla_X U = -\overset{*}{A}_U X + \overset{*}{\nabla^t}_X U, \tag{2.12}$$

where $\overset{*}{\nabla}_X PY$ and $\overset{*}{A}_U X$ belong to $\Gamma(S(TM))$, $\overset{*}{\nabla}$ and $\overset{*}{\nabla}{}^t$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(TM^{\perp})$ respectively, $\overset{*}{h}$ is a $\Gamma(TM^{\perp})$-valued $C^{\infty}(M)$-bilinear form on $\Gamma(TM)\times\Gamma(S(TM))$, $\overset{*}{A}_U$ is a $\Gamma(S(TM))$-valued $C^{\infty}(M)$-linear operator on $\Gamma(S(TM))$. $\overset{*}{h}$ and $\overset{*}{A}_U$ are the second fundamental form and the shape operator of the screen distribution $S(TM)$ respectively. Define on $\mathcal{U}$ the following relations

$$C(X,PY) = \overline{g}(\overset{*}{h}(X,PY),N), \tag{2.13}$$

$$\epsilon(X) = \overline{g}(\overset{*}{\nabla}{}^t{}_X \xi, N). \tag{2.14}$$

One shows that $\epsilon(X) = -\tau(X)$. Thus, locally (2.11) and (2.12) become

$$\nabla_X PY = \overset{*}{\nabla}_X PY + C(X,PY)\xi, \tag{2.15}$$

$$\nabla_X \xi = -\overset{*}{A}_\xi X - \tau(X)\xi, \tag{2.16}$$

respectively. The linear connection $\overset{*}{\nabla}$ is a metric connection on $\Gamma(S(TM))$. But, in general, the induced connection ∇ on M is not compatible with the induced metric g. Indeed, we have:

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{2.17}$$

for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$, where

$$\eta(X) = \overline{g}(X,N), \tag{2.18}$$

for all $X \in \Gamma(TM|_{\mathcal{U}})$. Finally, it is straightforward to verify that

$$B(X,Y) = g(\overset{*}{A}_\xi X, Y), \quad g(A_N Y, N) = 0, \tag{2.19}$$

$$C(X,PY) = g(A_N X, Y), \quad \overset{*}{A}_\xi \xi = 0, \tag{2.20}$$

for $X, Y \in \Gamma(TM|_{\mathcal{U}})$.

We denote the curvature tensor associated with $\overline{\nabla}$ and ∇ by $\overline{R}$ and R, respectively. Then we have (see [6]): for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$

$$\overline{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{2.21}$$

$$\begin{aligned} g\Big(R(X,Y)PZ,PW\Big) &= g\Big(\overset{*}{R}(X,Y)PZ,PW\Big) + C(X,PZ)B(Y,PW) \\ &\quad - C(Y,PZ)B(X,PW), \end{aligned} \tag{2.22}$$

$$\overline{g}\Big(\overline{R}(X,Y)\xi,N\Big) = C(Y,\overset{*}{A}_\xi X) - C(X,\overset{*}{A}_\xi Y) - 2d\tau(X,Y). \tag{2.23}$$

3 Basic facts on Blaschke type normalization on lightlike hypersurfaces

In the following section, we give a brief review on Blaschke structure on lightlike hypersurfaces. A full discussion of the content can be found in [3]. Consider a lightlike hypersurface immersion $f: M \to \mathbb{R}_1^{n+2}$ and let g denote the metric tensor field induced on $f(M)$. We have

$$g(X,Y) = \langle f_* X, f_* Y\rangle|_{f(x)},$$

for all X, Y tangent to M and $x \in M$, where $\langle\, , \rangle := \bar{g}$ denote the Lorentz metric on $\mathbb{R}_1^{n+2}$ and f_* the linear tangent map. Let $\{\xi, N\}$ a pair such that in the Theorem 2.1, we know Eq. (2.10) that B is degenerate and his rank is independent of choice of this pair (see [3]). We say that a lightlike hypersurface is **1-degenerate** if B has the maximal rank n, which we assume from now on.

Let $Q = TM/TM^{\perp}$ denote the factor bundle by the characteristic line bundle, and for $\bar{X}, \bar{Y} \in \Gamma(Q)$,

$$\overline{B^N}(\bar{X}, \bar{Y}) = B^N(X, Y). \tag{3.1}$$

As $B(\xi,) = 0$, $\overline{B^N}$ is well defined. Furthermore, it is non-degenerate on Q. Let us define a metric volume form on M relative to B^N by

$$\omega_{B^N} = \sqrt{|det(\overline{B^N_{ab}})|} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n, \tag{3.2}$$

where $\overline{B^N_{ab}} = \overline{B^N}(\overline{\partial_a}, \overline{\partial_b}) = B^N(\partial_a, \partial_b) = B^N_{ab}$. The $(n+1)$-form ω_{B^N} is indeed invariant under admissible coordinate changes (see [3]). For a lightlike hypersurface immersion $f : M \to \mathbb{R}^{n+2}_1$, let N be a null transversal vector field and consider the the parallel volume form on $\mathbb{R}^{n+2}_1$ given by the standard determinant det. The induced Volume form (with respect to N) denotes by θ^N is given by

$$\theta^N(X_0, \cdots, X_n) = det[f_*(X_0), \cdots f_*(X_n), N]. \tag{3.3}$$

For more information about construction of $(\overline{B^N_{ab}})$, ω_{B^N} and θ^N see [3]. By an appropriate choice of null transversal vector field N witch satisfies the following two goals:

(B_1) (∇^N, θ^N) is an equi-affine structure, i.e $\nabla^N \theta^N = 0$,

(B_2) θ^N coincide with the volume element ω_{B^N} relative to the 1-degenerate second fundamental form B^N.

We have:

Theorem 3.1. *[3] Let $f : M \to \mathbb{R}^{n+1}_1$ be a 1-degenerate lightlike hypersurface (isometric) immersion. For each point $x_0 \in M$, there is a null transversal vector field defined in a neighborhood of x_0 satisfying conditions (B_1) and (B_2) above, such a null transversal vector field is unique up to sign and gives rise to a normalization of the null characteristic vector field.*

Definition 3.2. [3] A null transversal vector field N satisfying (B_1) and (B_2) is called a Blaschke null transversal vector field. Locally it is uniquely determined up to sign. For each point $x \in M$, the line through x in the direction of the Blaschke null transversal vector N_x is independent of the choice of the sign for N and is called Blaschke null transversal through x. The triplet (∇^N, B^N, A_N) is called the Blaschke structure on the 1-degenerate lightlike hypersurface (M, g). The later with this structure will be denoted (M, g, N_{Bla}). The unique null vector field ξ with $\langle \xi, N \rangle = 1$ is called the Blaschke normalized null characteristic (radical) vector field.

We note that there are many examples of 1-degenerate lightlike hypersurfaces with Blaschke structure see [3].

4 Cubic forms on a 1-degenerate lightlike hypersurface with its Blaschke structure

In this section, we consider a cubic form L on a lightlike hypersurface with its Blaschke structure. We study some geometric aspects of these lightlike hypersurfaces relative to the cubic form. We first prove the following Lemma.

Lemma 4.1. *Let (M, g, N) be a lightlike hypersurface of $\mathbb{R}^{n+1}_1$ with its Blaschke structure and ∇ be the induced connection on M. There exists a unique torsion-free connection $\tilde{\nabla}$ on M such that*

$$B(\tilde{\nabla}_X Y, Z) = \frac{1}{2}(\nabla_X B)(Y, Z) + B(\nabla_X Y, Z), \tag{4.1}$$

$$\eta(\tilde{\nabla}_X Y) = \eta(\nabla_X Y). \tag{4.2}$$

Moreover,

$$\tilde{\nabla} B = 0. \tag{4.3}$$

Proof. We first prove that $\widetilde{\nabla}$ is a connection. Let X_1, X_2, Y_1, Y_2, Y be in $\Gamma(TM)$ and f belongs to $C^\infty(M)$. Using (4.1), by direct calculation we have

$$B(\widetilde{\nabla}_{X_1+X_2} Y, Z) = B(\widetilde{\nabla}_{X_1} Y + \widetilde{\nabla}_{X_2} Y, Z),$$

for all $Z \in \Gamma(TM)$. Then there exists a smooth function α on M such that

$$\widetilde{\nabla}_{X_1+X_2} Y - \widetilde{\nabla}_{X_1} Y - \widetilde{\nabla}_{X_2} Y = \alpha\xi.$$

Using (4.2), we get $\alpha = 0$. Then

$$\widetilde{\nabla}_{X_1+X_2} Y = \widetilde{\nabla}_{X_1} Y + \widetilde{\nabla}_{X_2} Y. \tag{4.4}$$

By using the same process, we prove the other conditions satisfying connection. Since ∇ is a torsion-free connection, it is easy to see from (4.1) and (4.2) that $\widetilde{\nabla}$ is torsion-free connection. To prove that $\widetilde{\nabla}$ is unique, suppose that $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ are two connections on M satisfying (4.1) and (4.2). Using (4.1) for $\widetilde{\nabla}$ and $\widetilde{\nabla}'$, we have,

$$B(\widetilde{\nabla}_X Y, Z) = B(\widetilde{\nabla}'_X Y, Z).$$

Then

$$B(\widetilde{\nabla}_X Y - \widetilde{\nabla}'_X Y, Z) = 0,$$

for all $Z \in \Gamma(TM)$. Then, there exists $\alpha \in C^\infty(M)$ such that

$$\widetilde{\nabla}_X Y - \widetilde{\nabla}'_X Y = \alpha\xi.$$

By (4.2), $\alpha = 0$. That is $\widetilde{\nabla} = \widetilde{\nabla}'$.

$$\begin{aligned}
(\widetilde{\nabla}_X B)(Y, Z) &= X \cdot B(Y, Z) - B(\widetilde{\nabla}_X Y, Z) - B(Y, \widetilde{\nabla}_X Z) \\
&= X \cdot B(Y, Z) - \frac{1}{2}(\nabla_X B)(Y, Z) - B(\nabla_X Y, Z) \\
&\quad -\frac{1}{2}(\nabla_X B)(Y, Z) + B(Y, \nabla_X Z) \\
&= (\nabla_X B)(Y, Z) - (\nabla_X B)(Y, Z) = 0.
\end{aligned}$$

Then, we have the result. □

We consider a degenerate hypersurface $f : M \to \mathbb{R}^{n+2}_1$ with its Blaschke structure. We known from equation (2.17) that the induced connection ∇ is compatible with the induced metric g if and only if M is totally geodesic or equivalently $B = 0$. Many geometric properties has been studied in the case of totally geodesic hypersurfaces. In this case, the induced connection ∇ is Levi-Civita connection, then this simplifies the computations. But totally geodesic hypersurfaces are few in general [9]. For this, all hypersurfaces M considering in the sequel are not totally geodesic. We consider B as a degenerate metric on M. In addition to the induced connection ∇ on M, we may consider the Levi-Civita connection $\widetilde{\nabla}$ for the metric B given by the previous lemma. We define a **cubic form** L for a 1-degenerate lightlike hypersurface by

$$L(X, Y, Z) := (\nabla_X B)(Y, Z).$$

From equation $(6-2)$ in [3], we have $(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z)$, then we see that the cubic form L is symmetric in X and Y. Since B is symmetric, it follows that the cubic form is symmetric in all the three variables.

The difference tensor of type $(1, 2)$ between the induced connection ∇ and the Levi-Civita connection $\widetilde{\nabla}$ for the metric B is denoted by

$$K(X, Y) = \nabla_X Y - \widetilde{\nabla}_X Y. \tag{4.5}$$

Since both ∇ and $\tilde{\nabla}$ have zero torsion, we have $K(X,Y) = K(Y,X)$. We shall also write

$$K_X Y = K_Y X = K(X,Y). \tag{4.6}$$

From (4.2), we have

$$\eta(K(X,Y)) = \eta(\nabla_X Y - \tilde{\nabla}_X\ Y) = 0.$$

Then $K(X,Y) \in S^B(TM)$, where $S^B(TM)$ denotes the Blaschke screen. Finally, it is easy to see from (4.3) that

$$(K_X B)(Y,Z) = (\nabla_X B)(Y,Z) = L(X,Y,Z). \tag{4.7}$$

Remark 4.2. It is easy to verify:

(1) $B(K_X Y, Z) = B(Y, K_X Z)$, for all $X,Y,Z \in \Gamma(TM)$.

(2) We can relate the cubic L form to the difference tensor K by

$$L(X,Y,Z) = -2B(K_X Y, Z). \tag{4.8}$$

(3) The induced Blaschke connection ∇ and $\tilde{\nabla}$ coincide with each other if and only if the cubic form vanishes identically, this means $K = 0$.

Next, we say that the screen distribution $S(TM)$ is **totally umbilic** if there exists a smooth function λ such that

$$C(X,PY) = \lambda g(X,PY), \tag{4.9}$$

for all $X,Y \in \Gamma(TM)$ or equivalently,

$$A_N X = \lambda PY, \tag{4.10}$$

for all $X \in \Gamma(TM)$. A lightlike hypersurface of a semi-Riemannian manifold is said to be a **parallel lightlike hypersurface** if its second fundamental form $h = B \otimes N$ satisfies

$$(\nabla_X h)(Y,Z) = 0, \tag{4.11}$$

for all $X,Y,Z \in \Gamma(TM)$ that is

$$(\nabla_X B)(Y,Z) + \tau(Z)B(Y,Z) = L(X,Y,Z) + \tau(Z)B(Y,Z) = 0. \tag{4.12}$$

We have the following result.

Proposition 4.3. *Let $f\colon\ M \to \mathbb{R}^{n+2}_1$, $n \geq 2$, be a lightlike hypersurface with its Blaschke structure. If the cubic form L is identically zero, then*

(i) the Blaschke screen is totally umbilic,

(ii) M is parallel.

Proof. First, assume that A_N has the property that for each vectors field X with $B(X,X) \neq 0$, we have $A_N X = \lambda PX$, where λ is a certain scalar possibly depending on X. Now, consider a quasi-orthonormal basis $(\xi, X_1, \ldots, X_n)$ relative to B. Since $B(X_i, X_i) \neq 0$ and $PX_i = X_i$, by assumption there exists scalars λ, $1 \leq i \leq n$, such that $A_N X_i = \lambda_i X_i$, $1 \leq i \leq n$. Our aim is to prove that all λ_i are equal. Let $i \neq j$ and $Z = X_i + 4X_j$. We have $B(Z,Z) \neq 0$. By assumption, there exists a non-null scalar α_{ij} such that $A_N Z = \alpha_{ij} Z$. On the other hand, we have $A_N Z = A_N(X_i + 4X_j) = \alpha_i X_i + 4\alpha_j X_j = \alpha_{ij}(X_i + 4X_j)$. Then

$$(\alpha_i - \alpha_{ij})X_i + 4(\alpha_j - \alpha_{ij})X_j = 0.$$

Since, X_i and X_j are linear independent, we get $\alpha_{ij} = \alpha_i = \alpha_j$, which shows that all α_i are equal.

Now, from (a) in Proposition 3.2 [4] for $k = 0$, we have:

$$R(X,Y)Z = B(Y,Z)A_N X - B(X,Z)A_N Y, \tag{4.13}$$

for all $X, Y, Z \in \Gamma(TM)$. Assumption $L = \nabla B = 0$ means that

$$X \cdot B(Y, Z) = B(\nabla_X Y, Z) + B(Y, \nabla_X Z).$$

Thus,

$$\begin{aligned}
B(R(X,Y)W,Z) &= B(\nabla_X\nabla_Y W, Z) - B(\nabla_Y\nabla_X W, Z) - B(\nabla_{[X,Y]}W, Z) \\
&= X \cdot B(\nabla_Y W, Z) - B(\nabla_Y W, \nabla_X Z) - Y \cdot B(\nabla_X W, Z) \\
&+ B(\nabla_X W, \nabla_Y Z) - [X,Y]B(W,Z) + B(W, \nabla_{[X,Y]}Z) \\
&= X(YB(W,Z) - B(W,\nabla_Y Z)) - YB(W,\nabla_X Z) \\
&+ B(W, \nabla_Y\nabla_X Z) - Y(XB(W,Z) - B(W,\nabla_X Z)) \\
&+ XB(W,\nabla_Y Z) - B(W,\nabla_X\nabla_Y Z) - [X,Y]B(W,Z) \\
&+ B(W,\nabla_{[X,Y]}Z) \\
&= B(W,\nabla_Y\nabla_X Z) - B(W,\nabla_X\nabla_Y Z) + B(W,\nabla_{[X,Y]}Z) \\
&= -B(W, R(X,Y)Z),
\end{aligned}$$

then, for all $X, Y, Z, W \in \Gamma(TM)$ we have

$$B(R(X,Y)W,Z) + B(W,R(X,Y)Z) = 0. \tag{4.14}$$

In particular if $Y = W = Z$, equation (4.13) becomes:

$$B(R(X,Y)Y,Y) = 0. \tag{4.15}$$

By using (4.13), we have

$$B(Y,Y)B(A_N X,Y) = B(X,Y)B(A_N Y,Y). \tag{4.16}$$

Consider a quasi-orthonormal basis $(\xi, X_1, \ldots, X_n)$ relative to B such that $(X_1, \ldots, X_n)$ spans $S(TM)$. Then

$$B(X_j,X_j)B(A_N X_i,X_j) = B(X_i,X_j)B(A_N X_j,X_j) = 0.$$

Hence $B(A_N X_i, X_j) = 0$ for all $i \neq j$. Since $A_N X_i \in S^B(TM)$, for all $i = 1, \ldots, n$ and $\ker B = \{\xi\}$, it follows that there are scalars α_i such that $A_N X_i = \alpha_i X_i$, for $i = 1, \ldots, n$. $B(A_N X_i, X_i) = \alpha_i B(X_i, X_i) = \alpha_i$, then $\alpha_i \neq 0$. Hence, we note $B(X_i, X_i) \neq 0$ and $A_N X_i = \alpha_i X_i$, for $i = 1, \ldots, n$. then, it follows from the first part of the proof that all α_i are equal. We denote their common value by α. From equation (4.16), take $X = \xi$ and $Y = X_i$, we have $B(A_N\xi, X_i) = 0$ for $i = 1, \ldots, n$, which show that $A_N\xi = 0$. Let $X \in \Gamma(TM)$, $X = a^0\xi + a^i X_i$. Then

$$A_N X = a^i A_N X_i = \alpha a^i X_i = \alpha PX,$$

which proves (1).

(2) is a consequence of $\tau = 0$ in the Blaschke conditions. □

5 Symmetry properties of ∇L

The covariant differential ∇L of the cubic form L is denoted by

$$(\nabla L)(U,V,W;X) := (\nabla_X L)(U,V,W),$$

where $\nabla_X L$ is the covariant derivative of L defined by

$$\begin{aligned}
(\nabla_X L)(U,V,W) &= X \cdot L(U,V,W) - L(\nabla_X U, V, W) \\
&- L(U, \nabla_X V, W) - L(U, V, \nabla_X W),
\end{aligned}$$

where U, V, W are vectors field.

We say that ∇L is **totally symmetric** if $(\nabla_X L)(U,V,W)$ is symmetric relative to its four variables. Since ∇L is symmetric in U, V, W just like L, total symmetry follows if it is symmetric in X and U.

Consider the covariant differential $\widetilde{\nabla}$ defined in the Lemma 4.1. We have:

Lemma 5.1. *∇L is totally symmetric if and only if $\tilde{\nabla} L$ is totally symmetric.*

Proof. Since $L(U,V,W)$ is a C^∞ function on M and K_X acts on $L(U,V,W)$ as derivation, we have $K_X L(U,V,W) = \nabla_X L(U,V,W) - \tilde{\nabla}_X L(U,V,W) = XL(U,V,W) - XL(U,V,W) = 0$. Then by using (4.8) and (4.5) we get

$$
\begin{aligned}
(\nabla_X L)(U,V,W) &= (\tilde{\nabla}_X L)(U,V,W) + (K_X L)(U,V,W) \\
&= (\tilde{\nabla}_X L)(U,V,W) - L(K_X U,V,W) \\
&+ 2B(K_X V, K_U W) + 2B(K_U V, K_X W) \\
&= (\tilde{\nabla}_X L)(U,V,W) - L(K_X U,V,W) \\
&- L(X,V,K_U W) - L(X,W,K_U V) \\
&= (\tilde{\nabla}_X L)(U,V,W) - L(K_U X,V,W) \\
&- L(X,W,K_U V) - L(X,V,K_U W) \\
&= (\tilde{\nabla}_X L)(U,V,W) + (K_U L)(X,V,W) \\
&= (\tilde{\nabla}_X L)(U,V,W) + (\nabla_U L)(X,V,W) \\
&- (\tilde{\nabla}_U L)(X,V,W).
\end{aligned}
$$

Thus,

$$
(\nabla_X L)(U,V,W) - (\nabla_U L)(X,V,W) = (\tilde{\nabla}_X L)(U,V,W) - (\tilde{\nabla}_U L)(X,V,W),
$$

which prove the lemma. □

Lemma 5.2. *∇L is totally symmetric if and only if the Blaschke screen is totally umbilic.*

Proof. By straightforward calculation and the fact that ∇ is torsion-free, we get for any vector field W:

$$
\begin{aligned}
&(\nabla_X L)(Y,Z,W) - (\nabla_Y L)(X,Z,W) = \\
&-B\left(R(X,Y)Z,W\right) - B\left(Z,R(X,Y)W\right) \\
&\overset{(4.13)}{=} B(X,Z)B(A_N Y,W) - B(Y,Z)B(A_N X,W) \\
&-B(Y,W)B(A_N X,Z) + B(X,W)B(A_N Y,Z) \\
&= B\Big(B(X,Z)A_N Y - B(Y,Z)A_N X \\
&-B(A_N X,Z)Y + B(A_N Y,Z)X, W\Big).
\end{aligned} \tag{5.1}
$$

By formula (5.1), we see that ∇L is totally symmetric if and only if

$$
B\Big(B(X,Z)A_N Y - B(Y,Z)A_N X B(ANX,Z)Y + B(A_N Y,Z)X, W\Big) = 0, \tag{5.2}
$$

for any vector field W. Since $\ker B =<\xi>$, there exists $\alpha \in C^\infty(M)$ such that

$$
B(X,Z)A_N Y - B(Y,Z)A_N X - B(A_N X,Z)Y + B(A_N Y,Z)X = \alpha\xi. \tag{5.3}
$$

Then,

$$
\eta\Big(B(X,Z)A_N Y - B(Y,Z)A_N X - B(A_N X,Z)Y + B(A_N Y,Z)X\Big) = \eta(\alpha\xi).
$$

Since η is a 1-form and $\eta(A_N X) = \eta(A_N Y) = 0$, $\eta(\xi) = 1$, it follows that

$$
\alpha = -B(A_N X,Z)\eta(Y) + B(A_N Y,Z)\eta(X). \tag{5.4}
$$

So, (5.3) becomes

$$\begin{aligned} B(X,Z)A_NY &- B(Y,Z)A_NX - B(A_NX,Z)Y + B(A_NY,Z)X \\ &+ B(A_NX,Z)\eta(Y)\xi - B(A_NY,Z)\eta(X)\xi = 0. \end{aligned} \tag{5.5}$$

The trace of the linear mapping taking Y to the left-hand side of (5.5) is

$$B(X,Z)tr_BA_N - B(A_NX,Z) - nB(A_NX,Z) + B(A_NX,Z) = 0.$$

Then

$$B(X,Z)tr_BA_N - nB(A_NX,Z) = B\Big((tr_BA_N)X - nA_NX, Z\Big) = 0.$$

Since $X = PX + \eta(X)\xi$ and $B(\xi,.) = 0$, then

$$B\Big((tr_BA_N)PX - nA_NX, Z\Big) = 0,$$

for all Z. Hence

$$(tr_BA_N)PX - nA_NX = 0,$$

which implies that

$$A_N(X) = \frac{tr_BA_N}{n}PX, \tag{5.6}$$

and the Blaschke screen is totally umbilic. Conversely, if $A_N = \lambda P$, we have (5.2) by using (5.5) and so totally symmetriy of ∇L . □

Remark 5.3. By direct calculation, we have
$(\nabla L)(U,V,W;X) - (\nabla L)(U,V,W;X) = (R(X,U)\cdot B)(V,W)$, where $R(X,Y)$ acts on B as a derivation. Then, we see that the condition of the previous lemma is equivalent to $R(X,Y)B = 0$.

In the sequel, we need the following Lemma.

Lemma 5.4. *Let $f : M \to \mathbb{R}^{n+2}_1$ be a lightlike hypersurface with its Blaschke structure. If ∇L is totally symmetric, the operator $R(X,Y)$ is skew-symmetry with respect to B i.e. for all $X,Y,Z,W,T \in \Gamma(TM)$*

$$B(R(X,Y)Z,W) = -B(Z,R(X,Y)W).$$

Proof. For any $X,Y,U,V,W,T \in \Gamma(TM)$, we have

$$\begin{aligned} B(R(X,Y)U,V) &\overset{(4.13)}{=} B[B(Y,U)A_NX - B(X,U)A_NY, V] \\ &= B[\lambda B(Y,U)PX - \lambda B(X,U)PY, V] \\ &= B[B(\lambda PY,U)PX - B(\lambda PX,U)PY, V] \\ &= B[B(A_NY,U)PX - B(A_NX,U)PY, V] \\ &= B(A_NY,U)B(X,V) - B(A_NX,U)B(Y,V) \\ &= B[B(X,V)A_NY - B(Y,V)A_NY, U] \\ &= -B(R(X,Y)V,U). \end{aligned} \tag{5.7}$$

Then, $B(R(X,Y)U,V) = -B(R(X,Y)V,U)$. □

A lightlike submanifold M of a semi-Riemannian manifold is said to be **Ricci semi-symmetric** if the following condition is satisfied [11]

$$(R(X,Y)\cdot Ric)(U,V) = 0, \ \forall X,Y,U,V \in \Gamma(TM), \tag{5.8}$$

where $R(X,Y)$ and Ric are curvature operator and Ricci tensor on M. In general, the screen distribution is not integrable. In the case of totally symmetry of ∇L we have:

Theorem 5.5. *Let $f : M \to \mathbb{R}^{n+2}_1$ be a lightlike hypersurface with its Blaschke structure. If ∇L is totally symmetric, then*

(i) The Blaschke screen distribution is integrable,

(ii) M is Ricci semi-symmetric.

Proof. Combining Lemma 5.4 and (iii) of Theorem 3.2 in [6], we have the first assertion. From Corollary 6.1 in [3] and the Lemma 5.4, we infer that

$$Ric(X,Y) = B(X,Y)tr_g A_N - B(A_N X, Y) = (tr_g A_N - \lambda)B(X,Y), \tag{5.9}$$

where λ is such that $A_N = \lambda P$.

$$\begin{aligned} \Big(R(X,Y)\cdot Ric(U,V)\Big) &= -Ric\Big(R(X,Y)U,V\Big) - Ric\Big(U,R(X,Y)V\Big) \\ &\overset{(5.9)}{=} -(tr_g A_N - \lambda)\Big[B(R(X,Y)U,V) \\ &+ B(U,R(X,Y)V)\Big], \end{aligned} \tag{5.10}$$

then assertion 2 follows by using the previous Lemma. □

6 Conformal Blaschke structures

In this section, we study lightlike hypersurfaces with conformal Blaschke structure relative to the symmetry of ∇L. We recall the definition of a screen conformal lightlike hypersurface of a semi-Riemannian manifold M.

Definition 6.1. [3] A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold M is said to be locally screen (resp. globally) conformal if on any coordinate neighborhood $\mathcal{U}$ (resp. $\mathcal{U} = M$), the shape operators A_N and $\overset{*}{A}_\xi$ of M and its screen distribution $S(TM)$ are related by

$$A_N = \varphi \overset{*}{A}_\xi, \tag{6.1}$$

where φ is a non vanishing smooth function on $\mathcal{U}$ (resp. $\mathcal{U} = M$).

It is easy to see that (6.1) is equivalent to

$$C(Y, PZ) = \varphi B(Y, Z), \tag{6.2}$$

for all $X, Y \in \Gamma(TM|_\mathcal{U})$.

Throughout this section, we consider a lightlike hypersurface for which the Blaschke structure is conformal. At the end of the section, we will give two examples of lightlike hypersurfaces which Blaschke structure is conformal. Next, we say that the degenerate hypersurface M is totally umbilic if for any coordinate neighborhood $\mathcal{U} \subset M$, there exists a smooth function λ such that

$$B(X,Y) = \lambda g(X,Y), \tag{6.3}$$

for all $X, Y \in \Gamma(TM|_\mathcal{U})$, that is

$$\overset{*}{A}_\xi X = \lambda PX, \ \forall X \in \Gamma(TM|_\mathcal{U}).$$

Then, we prove the following.

Theorem 6.2. *Let (M, g, N) be a lightlike hypersurface of $\mathbb{R}^{n+2}_1$ with its Blaschke structure such that the Blaschke screen is conformal. Let L be the cubic form define on M. If ∇L is totally symmetric, then*

(i) M is totally umbilic;

(ii) (M, g) is an Einstein manifold;

(iii) if $n \geq 2$, the conformal factor φ is constant on any leaf of the Blaschke screen $S(TM)$ and satisfies the partial differential equation.

$$\xi \cdot \varphi - \varphi = 0. \tag{6.4}$$

Proof. (i) Let $X \in \Gamma(TM)$. Since L is totally symmetric, by using Lemma5.2 and equation (6.1) we obtain

$$\overset{*}{A}_\xi X = \frac{\lambda}{\varphi} PX, \tag{6.5}$$

where λ is given by (5.6), which shows that M is totally umbilic.

(ii) For all $X, Y \in \Gamma(TM)$, by equation (5.9) we get

$$\begin{aligned} Ric(X,Y) &= (tr_g A_N - \lambda)B(X,Y) \\ &= (tr_g A_N - \lambda)g(\overset{*}{A}_\xi X, Y) \\ &\overset{(6.5)}{=} \frac{\lambda}{\varphi}(tr_g A_N - \lambda)g(X,Y). \end{aligned}$$

(iii) From Theorem 6.1 in [3], we have

$$(\nabla_X C)(Y, PZ) = (\nabla_Y C)(X, PZ), \tag{6.6}$$

for all $X, Y \in \Gamma(TM)$, where we set

$$\begin{aligned} (\nabla_X C)(Y,PZ) &= X \cdot C(Y,PZ) - C(\nabla_X Y, PZ) \\ &- C(Y, \nabla_X PZ) \\ &\overset{(6.5)}{=} X \cdot (\varphi B(Y,Z)) - \varphi B(\varphi_X Y, Z) \\ &- \varphi B(Y, \nabla_X PZ). \end{aligned} \tag{6.7}$$

From (f) in Proposition3.1 of [4] and by using $\tau = 0$, we have

$$\nabla_X PZ = \nabla_X Z - X \cdot \eta(Z)\xi + \eta(Z) \overset{*}{A}_\xi X. \tag{6.8}$$

Then,

$$\begin{aligned} (\nabla_X C)(Y,PZ) &= X \cdot (\varphi B(Y,Z)) - \varphi B(\nabla_X Y, Z) \\ &- \varphi B(Y, \nabla_X Z) - \eta(Z) B(Y, \overset{*}{A}_\xi X) \\ &= (X \cdot \varphi)B(Y,Z) + \varphi L(X,Y,Z) \\ &- \eta(Z) B(Y, \overset{*}{A}_\xi X). \end{aligned} \tag{6.9}$$

By the same way,

$$\begin{aligned} (\nabla_Y C)(X,PZ) &= (Y \cdot \varphi)B(X,Z) + \varphi L(Y,X,Z) \\ &- \eta(Z) B(X, \overset{*}{A}_\xi Y). \end{aligned} \tag{6.10}$$

Now, combining relations (6.6), (6.9) and (6.10), we obtain

$$(X \cdot \varphi)B(Y,Z) = (Y \cdot \varphi)B(X,Z), \tag{6.11}$$

for all X, Y, Z in $\Gamma(TM)$ i.e

$$B\Big((X \cdot \varphi)Y - (Y \cdot \varphi)X, Z\Big) = 0,$$

for any $Z \in \Gamma(TM)$. It comes that

$$(X \cdot \varphi)Y - (Y \cdot \varphi)X = \beta\xi, \tag{6.12}$$

for all $X, Y \in \Gamma(TM)$, where $\beta = X\cdot(\varphi)\eta(Y) - Y\cdot(\varphi)\eta(X)$. Let M' be a leaf of $S(TM)$. It is readily verified that

$$(X\cdot\varphi)Y - (Y\cdot\varphi)X = 0. \tag{6.13}$$

for all X, Y in $\Gamma(TM')$. Since $rankS(TM) \geq 2$ ($n \geq 2$) and the screen distribution $S(TM)$ is smooth, we can choose X, Y to be linearly independent, then (6.13) implies $X\cdot\varphi = 0$, for all $X \in \Gamma(TM')$. Hence φ is constant on any leaf of $S(TM)$.

Now, we compute:

$$\begin{aligned}
(\nabla_X C)(Y, PZ) &= X\cdot C(Y, PZ) - C(\nabla_X Y, PZ)\\
&- C(Y, \overset{*}{\nabla}_X PZ)\\
&\overset{(4.10)}{=} (X\cdot\lambda)g(Y,Z) + \lambda X\cdot g(Y,Z)\\
&- \lambda g(\nabla_X Y, Z) - \lambda g(Y, \nabla_X PZ - C(X,PZ)\xi)\\
&= (X\cdot\lambda)g(Y,Z) + \lambda X\cdot g(Y,Z)\\
&- \lambda g(\nabla_X Y, Z) - \lambda g(Y, \nabla_X PZ)\\
&\overset{(6.8)}{=} (X\cdot\lambda)g(Y,Z) + \lambda X\cdot g(Y,Z)\\
&- \lambda g(\nabla_X Y, Z) - \lambda g(Y, \nabla_X Z) + \eta(Z)g(\overset{*}{A}_\xi X, Y)\\
&= (X\cdot\lambda)g(Y,Z) + \lambda(\nabla_X g)(Y,Z) + \eta(Z)g(\overset{*}{A}_\xi X, Y)\\
&\overset{(2.17)}{=} (X\cdot\lambda)g(Y,Z) + \lambda B(X,Y)\eta(Z)\\
&+ \lambda B(X,Z)\eta(Y) + \eta(Z)g(\overset{*}{A}_\xi X, Y),
\end{aligned}$$

then,

$$\begin{aligned}
(\nabla_X C)(Y, PZ) &= (X\cdot\lambda)g(Y,Z) + \lambda B(X,Y)\eta(Z)\\
&+ \lambda B(X,Z)\eta(Y) + \eta(Z)g(\overset{*}{A}_\xi X, Y).
\end{aligned} \tag{6.14}$$

Interchanging X into Y we get

$$\begin{aligned}
(\nabla_Y C)(X, PZ) &= (Y\cdot\lambda)g(X,Z) + \lambda B(X,Y)\eta(Z)\\
&+ \lambda B(Y,Z)\eta(X) + \eta(Z)g(\overset{*}{A}_\xi Y, X).
\end{aligned} \tag{6.15}$$

Now, substituting (6.14) and (6.15) in (6.6), by using $g(\overset{*}{A}_\xi X, Y) = g(\overset{*}{A}_\xi Y, X)$, we obtain

$$(Y\cdot\lambda)g(X,Z) + \lambda B(Y,Z)\eta(X) = (X\cdot\lambda)g(Y,Z) + \lambda B(X,Z)\eta(Y), \tag{6.16}$$

for all $X, Y, Z \in \Gamma(TM)$. In particular, if $X = \xi$, $Z = Y \in \Gamma(S(TM))$,

$$\begin{aligned}
(\xi\cdot\lambda)g(Y,Y) &= \lambda B(X,Y)\\
&\overset{(6.2)}{=} \frac{\lambda^2}{\varphi}.
\end{aligned}$$

Since $g(Y,Y) \neq 0$, we have $\xi\cdot\lambda - \frac{\lambda^2}{\varphi} = 0$. Hence,

$$\xi\cdot\lambda - \frac{\lambda^2}{\varphi} \overset{(5.6)}{=} (\xi\cdot\varphi)\frac{tr_B\,\overset{*}{A}_\xi}{n} + \varphi\frac{(\xi\cdot tr_B\,\overset{*}{A}_\xi)}{n} - \varphi\frac{(tr_B\,\overset{*}{A}_\xi)^2}{n^2} = 0.$$

That is

$$(\xi\cdot\varphi)\frac{tr_B\,\overset{*}{A}_\xi}{n} + \varphi\frac{(\xi\cdot tr_B\,\overset{*}{A}_\xi)}{n} - \varphi\frac{(tr_B\,\overset{*}{A}_\xi)^2}{n^2} = 0. \tag{6.17}$$

Let $(E_i)_{i=1,\ldots,n}$ be an orthonormal basis of $S(TM)$,

$$tr_g A_\xi = \sum_{i=1}^{n} g(\overset{*}{A}_\xi\, E_i, E_i) \overset{(6.5)}{=} \frac{\lambda}{\varphi}\sum_{i=1}^{n} g(E_i, E_i) = \frac{\lambda}{\varphi} n \overset{(5.6)}{=} tr_B\, \overset{*}{A}_\xi\,.$$

Hence by (3) in Theorem 5.1, we have

$$tr_B\, \overset{*}{A}_\xi = n. \tag{6.18}$$

Relations (70) and (71) imply (57) and the proof follows. □

Example 6.3. Lightcone Λ_0^3.
Let us consider the lightcone Λ_0^3 as the immersion

$$\begin{aligned} f:\; M = \mathbb{R}^3\backslash\{0\} &\longrightarrow \mathbb{R}_1^4 \\ (x,y,z) &\longmapsto [x,\ y,\ z,\ \epsilon(x^2+y^2+z^2)^{\frac{1}{2}}]\ ;\epsilon = \pm 1. \end{aligned}$$

Locally, Λ_0^3 is the graph $(x,y,z) \longmapsto t = \epsilon(x^2+y^2+z^2)^{\frac{1}{2}}\ \epsilon = \pm 1$. It is known that lightcon has the Blaschke structure ([3]) given by

$$\widetilde{N} = \frac{1}{\sqrt{2}t}(x\partial_x + y\partial_y + z\partial_z - t\partial_t)$$

and

$$\widetilde{\xi} = \frac{1}{\sqrt{2}t}(x\partial_x + y\partial_y + z\partial_z + t\partial_t).$$

It is readily checked that

$$\widetilde{N} = \widetilde{\xi} - \sqrt{2}\partial_t. \tag{6.19}$$

Given $X, Y \in \Gamma(S(T\Lambda_0^3))$, we have

$$C(X,Y) = \overline{g}(\nabla_X Y, \widetilde{N}) = \overline{g}(\overline{\nabla}_X Y, \widetilde{N}) = -\overline{g}(Y, \overline{\nabla}_X\ \widetilde{N}) \overset{(6.19)}{=} g(Y, \overset{*}{A}_{\widetilde{\xi}}\ X), \tag{6.20}$$

that is

$$g(A_{\widetilde{N}}X, Y) = g(Y, \overset{*}{A}_{\widetilde{\xi}}\ X),\ \ \forall\ \ X, Y \in \Gamma(S(T\Lambda_0^3)).$$

Therefore, we have

$$A_{\widetilde{N}}X = A_{\widetilde{\xi}}X,\quad \forall\ \ X \in \Gamma(S(T\Lambda_0^3)). \tag{6.21}$$

As $A_{\widetilde{N}}\widetilde{\xi} = A_{\widetilde{\xi}}\widetilde{\xi} = 0$, we infer the following relation

$$A_{\widetilde{N}}X = A_{\widetilde{\xi}}X \quad \forall\ \ X \in \Gamma(T\Lambda_0^3).$$

That is the blaschke structure on Λ_0^3 is coformal screen with conformal function $\varphi = 1$.

Example 6.4. de Monge Surfaces in $\mathbb{R}_1^3$
Consider the graph M of the function F, $\ x \longmapsto F(x,y)$ as the immersion

$$\begin{aligned} f\ :\Omega \subset \mathbb{R}^2 &\longrightarrow \mathbb{R}_1^3 \\ (y,z) &\longmapsto (F(y,z), y, z), \end{aligned}$$

$F \in C^\infty(\Omega)$. M is lightlike if and only if

$$(F_y')^2 + (F_z')^2 = 1. \tag{6.22}$$

It is known that M has the Blaschke structure ([3]) given by

$$\widetilde{N}= \frac{1}{2}[\Delta F]^{\frac{1}{3}}N - \frac{2}{3}J(F)(\Delta F)^{-\frac{5}{3}}W - \frac{1}{72}(\Delta F)^{-\dfrac{10}{3}}J(F)^2\xi,$$

and

$$\widetilde{\xi}= [\Delta F]^{-\frac{1}{3}}(\partial_x + F'_y\partial_y + F'_z\partial_z),$$

where $J(F) = F'_zF^{(3)}_{zzy} + F'_zF^{(3)}_{yyy} - F'_yF^{(3)}_{zzz} - F'_yF^{(3)}_{yyz}$.

By observing that $\widetilde{N}= \frac{1}{2}[\Delta F]^{\frac{2}{3}}\ \widetilde{\xi}\ -[\Delta F]^{\frac{1}{3}}\partial_t$ if $J(F) = 0$, and using the same process as in the previous example we have

$$A_{\widetilde{N}} = \frac{1}{2}[\Delta F]^{\frac{2}{3}}A_{\widetilde{\xi}},$$

which implies that M is screen conformal with $\varphi = \frac{1}{2}[\Delta F]^{\frac{2}{3}}$.

7 Minimal Blaschke lightlike hypersurfaces

Deformations of lightlike hypersurfaces and the corresponding variational formula are two main themes of this section.

Now, let M be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ and $f : M \to \bar{M}$ an isometric immersion.

An *isometric variation* (or a deformation) of M is a smooth map $F : I \times M \to \bar{M}$, $(t, x) \mapsto F(t, x) = F_t(x)$, $I = (-\epsilon, \epsilon)$ such that for each $t \in I$, $f_t = F(t, .) : M \to \bar{M}$ is an isometric immersion and $F(0, x) = f(x)$. Each immersion f_t induces a metric $g_t = (f_t)^*\bar{g}$ on M.

Definition 7.1. We said that an isometric variation F is a degenerate isometric variation of M if the induced metric g_t is degenerate for each $t \in I$.

Let us give an example of such degenerate isometric variations.

Example 7.2. Let $\mathbb{R}^4_2$ be a semi-Euclidean space with semi-Euclidean metric. Consider the lightlike hypersurface $M :\ x^3 = x^0 + \frac{1}{2}(x^1 + x^2)^2$.

We define a variation F of M as:

$$\begin{array}{rcl} F :\ I \times \mathbb{R}^3_1 & \longrightarrow & \mathbb{R}^4_2, \quad I = (-\frac{1}{4}, \frac{1}{4}) \\ (t, x^0, x^1, x^2) & \mapsto & (x^0, x^1, x^2, x^3 = x^0 + \frac{1}{2}(t^2 + 1)(x^1 + x^2)^2) \end{array}$$

It is easy to check that

$$\ker g_t = Spans\left\{\xi_t = \frac{\partial}{\partial x^0} + (t^2 + 1)(x^1 + x^2)\frac{\partial}{\partial x^1} - (t^2 + 1)(x^1 + x^2)\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\right\}.$$

This means that $M_t : x^3 = x^0 + \frac{1}{2}(t^2 + 1)(x^1 + x^2)^2$ is a lightlike hypersurface for all t. The radical distribution $Rad(TM_t) = TM_t^{\perp}$ is spanned by

$$\left\{\xi_t = \frac{\partial}{\partial x^0} + (t^2 + 1)(x^1 + x^2)\frac{\partial}{\partial x^1} - (t^2 + 1)(x^1 + x^2)\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\right\}.$$

Then the lightlike transversal vector bundle is given by

$$\begin{array}{rcl} tr(TM_t) & = & Spans\left\{N_t = -\dfrac{1}{2(1 + (1 + t^2)^4(x^1 + x^2)^2)^2}\left(\dfrac{\partial}{\partial x^0} + (t^2 + 1)(x^1 + x^2)\dfrac{\partial}{\partial x^1}\right.\right. \\ & & \left.\left. +(t^2 + 1)(x^1 + x^2)\dfrac{\partial}{\partial x^2} - \dfrac{\partial}{\partial x^3}\right)\right\}. \end{array}$$

The corresponding screen distribution $S(TM_t)$ is spanned by

$$\left\{V_1 = \frac{\partial}{\partial x^1} - (t^2 + 1)(x^1 + x^2)\frac{\partial}{\partial x^0}; V_2 = \frac{\partial}{\partial x^2} + (t^2 + 1)(x^1 + x^2)\frac{\partial}{\partial x^3}\right\}.$$

In the sequel, we are interesting to the variational formula of lightlike hypersurfaces with its Blaschke structure analogue to the variation in the classical affine differential geometry [10].

Let $f : M \to \mathbb{R}^{n+1}_1$ be a $1-$degenerate isometric. The volume of the immersion is the integral

$$v(f) = \int_M \omega_{B^N},$$

where ω_{B^N} is the metric volume form M relative to B^N.

In order to state the variational formula, let us fix the notion. Let M be an orientable manifold, possibly with smooth boundary ∂M. Let $f : M \to \mathbb{R}^{n+1}_1$ be a degenerate immersion. A *smooth degenerate variation* of M is a smooth map $F : I \times M \to \mathbb{R}^{n+1}_1$, $I = (-\epsilon, \epsilon)$ $(\epsilon > 0)$ with the following property: for $f_t(x) = F(t, x)$,

(i) f_t is a degenerate immersion,

(ii) $f_0 = f$,

(iii) supp F:= $\overline{\{x \in M \,:\, F(x,t) \neq x \text{ for som } t \in I = (-\epsilon, \epsilon)\}}$ being compact subset of M.

Remark 7.3. A smooth degenerate variation of M is a degenerate isometric variation F of M with support (supp F) being a compact subset of M.

Let $\{\xi_t, N_t\}$ be a pair of Blaschke normalization for f_t which depends smoothly on t. Set

$$\theta^N_t(X_1, \dots, X_n) = det[f_{t*}(X_1), \dots, f_{t*}(X_n), N_t], \tag{7.1}$$

such that $\xi_0 = \xi$, $N_0 = N$ and $\theta_0 = \theta$. Assume that the immersion f_t is Blaschke relative to $\{\xi_t, N_t\}$, the volume of f_t is then

$$V(t) = \int_M \theta_t.$$

Denote by $V = F_*(\frac{\partial}{\partial t})_{|t=0}$ the variation vector field which is decomposed as $V = V_0 + \psi N$, where V_0 is tangential. First of all we need the following algebraic result.

Lemma 7.4. *[9] Let $A(t) = (a_(ij))$, $|t| < \epsilon$ be a smooth family of $n \times n$ matrices satisfying $A(0) = I$ (the identity matrix). Then*

$$\frac{d}{dt}A(t)|_{t=0} = trA'(0).$$

With the notion above we have the following formula.

Theorem 7.5.

$$\frac{\partial V(t)}{\partial t}|_{t=0} = -\frac{n}{n+1}\int_M \psi tr_B A_N \theta. \tag{7.2}$$

Proof. By Gauss-Weingarten formulae, we have

$$\begin{aligned} D_X f_{t*}(Y) &= f_{t*}(\nabla^t_X Y) + B_t(X,Y)N_t, &(7.3)\\ D_X N_t &= f_{t*}(A^t_N(X)). &(7.4)\end{aligned}$$

Let $\{X_1 \dots, X_n\}$ be a unimodular basis of f: $\theta(X_1, \dots, X_n) = 1$. It has the property $\left|det[\overline{B}(\overline{X}_i, \overline{X}j)]\right| = 1$. We extend vector field X_i to $I \times M$ depending trivially on t. Then

$$\left[X_i, \frac{\partial}{\partial t}\right] = 0. \tag{7.5}$$

Set

$$\varphi(t) = \left|det[\overline{B_t}(\overline{X}_i, \overline{X}j)]\right|.$$

Determine the function λ on $I \times M$ so that

$$\theta_t(\lambda X_1, \dots, \lambda X_n) = 1.$$

Then by the assumption that $\{\xi_t, N_t\}$ is the pair of Blaschke normalisation for each t, we have $\left|det[\overline{B_t}(\overline{X}_i, \overline{X}j)]\right| = 1$. On the other hand, since both θ_t, and θ are volume forms on M , there existe a function ρ_t such that

$$\theta_t = \rho_t\theta.$$

It is easy to see that $\rho_t = \varphi^{\frac{1}{2}}$. Now we have

$$\frac{\partial}{\partial t}\theta_t = \frac{1}{2}\varphi^{-\frac{1}{2}}\frac{\partial\varphi}{\partial t}\theta,$$

hence at $t = 0$,

$$\frac{\partial}{\partial t}\theta_t|_{t=0} = \frac{1}{2}\varphi^{-\frac{1}{2}}\frac{\partial\varphi}{\partial t}|_{t=0}\theta. \tag{7.6}$$

From Lemma7.4, We have

$$\frac{\partial\varphi}{\partial t}|_{t=0} = \sum \overline{B}^{ij}\frac{\partial}{\partial t}\overline{B_t}(\overline{X}_i, \overline{Y}_j)|_{t=0}. \tag{7.7}$$

By differentiating of (7.1), we get another representation: Set

$$\frac{\partial}{\partial t}N_t|_{t=0} = f_*(Z) + aN,$$

where Z is the tangent vector to M. Then

$$\begin{aligned}
\frac{\partial}{\partial t}\theta_t(X_1, \ldots, X_n)|_{t=0} &= \sum det(X_1, \ldots, D_V f_{t*}(X_i), \ldots, N_t) \\
&\quad + \mathrm{det}(X_1, \ldots, \frac{\partial}{\partial t}N_t)_{t=0} \\
&\overset{(7.5)}{=} \sum det(X_1, \ldots, D_{X_i}V, \ldots, N) + \mathrm{det}(X_1, \ldots, aN) \\
&= \sum \mathrm{det}(X_i, \ldots, \nabla_{X_i}V_0 - \psi A_N X_i, \ldots, N) \\
&\quad + \mathrm{det}(X_1, \ldots, aN) \\
&= \sum \mathrm{det}(X_i, \ldots, \widetilde{\nabla}_X V_0 + K_X V_0, \ldots, N) \\
&\quad - \psi tr A_N\theta(X_1, \ldots, X_n) + \mathrm{det}(X_1, \ldots, aN) \\
&= (\mathrm{div}V_0 + trK_X)\theta(X_1, \ldots, X_n) - \psi tr A_N\theta(X_1, \ldots, X_n) \\
&\quad + a\theta(X_1, \ldots, X_n) \\
&= (\mathrm{div}V_0 - \psi tr A_N + a)\theta(X_1, \ldots, X_n).
\end{aligned} \tag{7.8}$$

This implies by using (7.6) and (7.8)

$$a = \frac{1}{2}\frac{\partial}{\partial t}\varphi|_{t=0} - \mathrm{div}V_0 + \psi tr A_N. \tag{7.9}$$

Now we differentiate (7.3) relative to t, the left hand side is

$$\begin{aligned}
D_{\frac{\partial}{\partial t}}D_X f_{t*}(Y) &= D_X D_{\frac{\partial}{\partial t}} f_{t*}(Y) \\
&= D_X D_Y V \\
&= D_X(\nabla_Y V_0 + B(V_0, Y)N - \psi A_N Y + (Y \cdot \psi)N) \\
&= \nabla_X\nabla_Y V_0 + B(V_0, Y)A_N X - X \cdot \psi A_N Y \\
&- \psi\nabla_X A_N Y - (Y \cdot \psi)A_N X + [B(X, \nabla_Y V_0) \\
&+ X \cdot B(V_0, Y) - \psi B(X, A_N Y) + X \cdot (Y \cdot \psi)]N.
\end{aligned} \tag{7.10}$$

The derivative at t = 0 of the right hand side of (7.3) is equal to

$$
\begin{aligned}
D_{\frac{\partial}{\partial t}}(\nabla^t_X Y)|_{t=0} \quad & +\frac{\partial}{\partial t}B_t(X,Y)|_{t=0}N + B(X,Y)(Z+aN) \\
&= D_{\nabla_X Y}(V_0+\psi N) + \frac{\partial}{\partial t}B_t(X,Y)|_{t=0}N + B(X,Y)(Z+aN) \\
&= [B(\nabla_X Y, V_0) + \nabla_X Y\cdot\psi + \frac{\partial}{\partial t}B_t(X,Y)|_{t=0} + aB(X,Y)]N \\
&= \nabla_{\nabla_X Y}V_0 - \psi A_N(\nabla_X Y) + B(X,Y)Z. \qquad (7.11)
\end{aligned}
$$

Therefore by comparing transversal parts of the two above equations, we get

$$
\begin{aligned}
\frac{\partial}{\partial t}B_t(X,Y)|_{t=0} + aB(X,Y) &= X\cdot B(V_0,Y) - B(\nabla_X Y, V_0) + B(\nabla_Y V_0, X) \\
& \quad -\nabla_X Y\cdot\psi - \psi B(X, A_N X) + X\cdot(Y\cdot\psi) \\
&= L(X,Y,V_0) + B(\nabla_X V_0, Y) + B(\nabla_Y V_0, X) \\
& \quad +\mathrm{Hess}_\psi(X,Y) - \psi B(X, A_N Y) \\
&= L(X,Y,V_0) + B(\widetilde{\nabla}_X\, V_0 + K_X V_0, Y) \\
& \quad +B(\widetilde{\nabla}_Y\, V_0 + K_Y V_0, X) + \mathrm{Hess}_\psi(X,Y) \\
& \quad -\psi B(X, A_N Y).
\end{aligned}
$$

Since $\mathrm{trace}_B((Y,Z)\mapsto L(X,Y,Z)) \overset{(31)}{=} \mathrm{trace}_B((Y,Z)\mapsto -2B(K_X Y, Z)) = -2\,\mathrm{trace}(K_X)$.
Then taking trace_B of the bilinear function represented by each term shows the identity

$$\frac{\partial\varphi}{\partial t}|_{t=0} + (n-1)a = 2\mathrm{div}V_0 + \Delta\psi - \psi tr A_N. \qquad (7.12)$$

Then by using (7.9) we have

$$\frac{1}{2}\frac{\partial\varphi}{\partial t}|_{t=0} = \frac{n+1}{n}\mathrm{div}V_O + \frac{1}{n}\Delta\psi - n\psi tr_B A_N. \qquad (7.13)$$

The boundary condition(3) in the definition of variation implies the theorem in view of Green's theorem. □

Definition 7.6. Let M be a lightlike hypersurface of $\mathbb{R}^{n+1}_1$ with its Blaschke structure. **The Blaschke mean curvature of the second fundamental form** H_B is defined by

$$H_B = \frac{n}{n+1}\mathrm{trace}_B A_N.$$

A Blaschke lightlike hypersurface is said to be B-minimal if H_B vanishes everywhere, that is, when the functional V above is critical at this immersion.

Remark 7.7. The terminology of B -minimal was introduced by Steven V. [12] for non-degenerate hypersurfaces of an Euclidean space. The terminology of B -minimal is just a definition, since a critical point of volume is not necessarily a minimum. The notion of mean curvature, which belongs to the extrinsic geometry of the hypersurface and can be characterized as a measure for the rate of area growth under deformations of the hypersurface, can be tailored to the geometry of the second fundamental form: a function that measures the rate at which the total area of the hypersurface, as surveyed in the geometry of its second fundamental form, changes under a deformation is called the mean curvature of the second fundamental form. This concept has been introduced by E. Glässner for surfaces in $\mathbb{R}^3$ and F. Manhart for hypersurfaces in $\mathbb{R}^{m+1}$ (see [12]).

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