**Palestine Journal of Mathematics**

**Vol 14(3)(2025)** , 359–373 © Palestine Polytechnic University-PPU 2025

# PLANARITY INDEX OF FUZZY MULTI-SEMIGRAPHS

Archana S. and Preethi Kuttipulackal

Communicated by Nasreen Kausar

MSC 2020 Classifications: Primary 05C72; Secondary 05C10.

Keywords and phrases: fuzzy semigraph, fuzzy multi-semigraph, intersection index, planarity index, strongly planar fuzzy semigraph, strong fuzzy semigraph.

*The authors thank the reviewers and editor for their pertinent remarks which improved the quality of the paper, and the funding agency, UGC, Government of India, for granting the financial assistance.*

Corresponding Author: Archana S.

**Abstract** The concept of a multi-semigraph is introduced as a natural extension of a multigraph, further extending to the notion of a fuzzy multi-semigraph and a study of its planarity. Also, this paper explores the concept of strong edges for the fuzzy multi-semigraph. Planarity refers to the crossing of edges at points other than the nodes. Since non-planarity causes disturbance in many situations, like signals and road networks, we associate a degree to the crossing, namely the intersection index at each crossing point. This leads to defining the planarity index of a fuzzy multi-semigraph. Further, it presents a study on the planarity index of a fuzzy multi-semigraph by examining the local rail and road networks.

## 1 Introduction

The mathematician Leonard Euler developed the novel approach to graph theory in 1736 to solve the celebrated "Königsberg Bridge Problem". This structure helps to model the bilateral connection between objects. Even though graph theory has widespread application in multiple fields, to give more importance to the role of edges E. Sampathkumar introduced a new concept in 2000, called semigraphs. Rather than describing pairwise relationships between objects, a semigraph models physical scenarios with sequential activities; where any two activities have at most one node in common. To make this concept a perfect tool for modelling, we introduced an advanced concept, called a multi-semigraph; where there is no restriction for the number of common nodes.

The traditional graph concept fails to illustrate the uncertainties existing in the real world. To overcome this limitation, L. A. Zadeh introduced objects with a grade of membership, in 1965. Allowing membership values to the nodes and edges the concept of fuzzy graph emerged as an extension of graph theory. A. Rosenfeld developed fuzzy equivalent concepts of several graph properties discussed in [10]. The mathematical development of fuzzy graph theory in $1975-2017$ was well discussed by S. C. Mathew, J. N. Mordeson and D. S. Malik in [6].

Working in multi-semigraphs, we identified the use and importance of assigning membership values to nodes and edges; consequently, we introduced fuzzy multi-semigraph, which will be an ideal tool for modelling contexts, where sequential activities play a role.

Planar graphs in the classification of graphs have a key significance because of their special characteristics and vast applications. After the development of fuzzy graphs, the concept of fuzzy planar graphs, by defining intersecting values in fuzzy multi-graphs, was introduced by S. Samanta, A. Pal, and M. Pal [12, 13]. Many complicated designs using fuzzy planar graphs will be less complex with fuzzy planar multi-semigraphs. There are many practical situations in which crossings of edges make disturbances—for example, the design problem of subway tunnels, circuits, oil/gas pipelines, etc. The concept of planarity addresses these problems. Various indices defined on graphs and fuzzy graphs are detailed in [5, 11, 13].

The primary objective of the research is to extend these concepts in a natural way to multi-semigraph and fuzzy multi-semigraph, a highly developed tool for modelling a real-world scenario with or without uncertainties. A permanent solution for all real-world problems is impractical but we can reduce the impact by best approximations. This paper demonstrates a network involving rail and road, scaled down to a region in the state of Kerala, India; and introduces the best approximate solution as an index which can be utilised when constructing a new bridge to reduce the rail and road traffic and accidents.

## 2 Preliminaries

Graph theory satisfies many properties based on the node set but not on the edge set. E. Sampathkumar improved such areas in graph theory and introduced the concept of semigraphs in 2000.

**Definition 2.1.** [14] A semigraph $H^*$ with the node set $N$ and edge set $E$ is the pair $(N, E)$, $N \neq \phi$ and $E$ is a collection of $r$-tuples, for $r \geq 2$, of different members of $N$ satisfying the following requirements:

(1) any pair of elements in $E$ share a maximum of one node,

(2) any two edges $(n_1, n_2, \ldots, n_r)$ and $(m_1, m_2, \ldots, m_s)$ are equal if, and only if, $r = s$ and at least one of the conditions listed below occurs for $j$ such that $1 \leq j \leq r$,

 (a) $n_j = m_j$,

 (b) $n_j = m_{r-j+1}$.

**Definition 2.2.** [14] A partial edge of an edge $e = (n_1, n_2, \ldots, n_r)$ in a semigraph $H^*$ is a sub sequence $(n_i, n_{i+1}, \ldots, n_j)$ for $1 \leq i < j \leq r$. The nodes $n_1$ and $n_r$ in the edge $e$ are end nodes of $e$ and the remaining nodes are all middle nodes.

There are different types of semigraphs.

**Definition 2.3.** [2] A uniform semigraph has the same cardinality (the number of nodes in an edge) for each edge in the semigraph.

A fuzzy multi-set allows an element of a non-empty set to appear multiple times with either the same or different membership values.

**Definition 2.4.** [7] Let $X$ be a non-empty set, then a fuzzy multi-set $A$ drawn from $X$ is defined by a count membership function of $A$, denoted by $CM_A$, such that $CM_A : X \to Q$, where $Q$ is the set of all crisp multi-sets drawn from the unit interval $[0, 1]$.

Connecting the notions of fuzzy graphs and semigraphs, K. Radha and P. Renganathan established a new concept called fuzzy semigraph.

**Definition 2.5.** [8] A fuzzy semigraph $H$ defined on a semigraph $H^* = (N, E)$ is described as $H = (N, \sigma, \mu, \eta)$ in which $\sigma : N \to [0, 1]$, $\mu : N \times N \to [0, 1]$ and $\eta : E \to [0, 1]$ satisfy the conditions:

(1) $\mu(n_i, n_j) \leq \sigma(n_i) \wedge \sigma(n_j)$ for all the elements $n_i, n_j \in N$,

(2) $\eta(e) = \mu(n_1, n_2) \wedge \mu(n_2, n_3) \wedge \cdots \wedge \mu(n_{r-1}, n_r) \leq \sigma(n_1) \wedge \sigma(n_r)$
 if $e = (n_1, n_2, \ldots, n_r) \in E$ and $r \geq 2$.

The thought of fuzzy semigraph was introduced in 2021. After that a study of analysis of this concept using the edge degree was discussed by the authors in [1]. As the regularity in a graph ensures certain symmetry, that can have executions in different network analyses, [1] discusses the regularity of fuzzy semigraph. Also, three types of isomorphisms in a fuzzy semigraph are introduced in [9].

**Definition 2.6.** [8] An edge $e = (n_1, n_2, \ldots, n_r)$ of a fuzzy semigraph $(N, \sigma, \mu, \eta)$ defined as an effective edge if it satisfy the conditions

(1) $\eta(e) = \sigma(n_1) \wedge \sigma(n_r)$ and

(2) $\mu(n_j, n_{j+1}) = \sigma(n_j) \wedge \sigma(n_{j+1})$ for $1 \leq j \leq r - 1$.

## 3 Results

Consider a real-world scenario in which sequential activities characterise the situations. Semigraph is a perfect tool for modelling such scenarios. However, the only disadvantage of such modelling using semigraphs defined by E. Sampathkumar is that not all the situations are created in such a way that any two sequential activities have at most one node in common. This prompted us to modify the definition of a semigraph by defining multi-semigraphs.

**Definition 3.1.** A multi-semigraph $H^*$ is a pair $(N, E)$ with a non-empty node set $N$ and a set of $r$-tuples, called edge set $E$, for $r \geq 2$, of distinct members of $N$.

An edge $(n_1, n_2, \ldots, n_{r-1}, n_r)$ may also be expressed as $(n_r, n_{r-1}, \ldots, n_2, n_1)$.

**Definition 3.2.** A multi-semigraph $H^* = (N, E)$ is referred to as a simple semigraph if it meets additional conditions that any two elements in $E$ can have at most one common node, and any two edges $(n_1, n_2, \ldots, n_r)$ and $(m_1, m_2, \ldots, m_s)$ are equal if, and only if, $r = s$ and at least one of the conditions listed below exists for $j$ such that $1 \leq j \leq r$,

(1) $n_j = m_j$,

(2) $n_j = m_{r-j+1}$.

Considering these terminologies, the concept of semigraph defined by E. Sampathkumar in [14] can be viewed as a simple semigraph. Note that the concept of multi-semigraph coincides with multi-graph when the $r$-tuple becomes an ordered pair.

**Definition 3.3.** The crisp semigraph of a fuzzy semigraph $(N, \sigma, \mu, \eta)$ is defined as $(N^*, E^*)$ in which $N^* = \{n \in N | \sigma(n) > 0\}$ and $E^* = \{e \in E | \eta(e) > 0\}$.

For convenience, we adopt a different diagrammatic representation for multi-semigraphs. The difference is in the representation of the middle nodes. [14] E. Sampathkumar has used $2d$-gons to represent a node which is the middle node for $d$ edges. Instead, we use small circles without filling for such nodes but edges with more than one common node are marked using various colors or designs.

**Example 3.4.** A network interaction can be represented using a multi-semigraph, where the nodes represent the devices (for example: smartphones, laptops, etc) and the edges are sequences of nodes in which the arrangement of nodes can represent the direction of flow of data in the interaction.

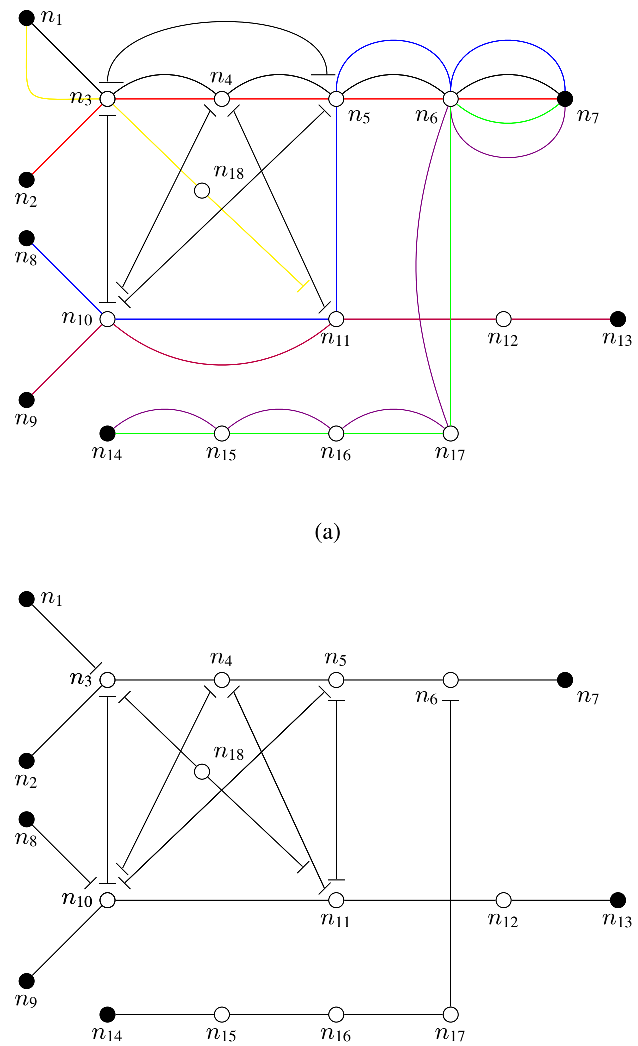
In general networks, different sequences representing flows may have common nodes, see Figure 1a. There may be situations where the intersection of two different sequences can be at a single node, see Figure 1b. [14] The semigraph defined by E. Sampathkumar with the property that any two edges can have at most one node in common is termed a simple semigraph when we extended the definition to include multiple edges in Definition 3.1.

The multi-semigraph in Figure 1a has the node set $N = \{n_1, n_2, n_3, \ldots, n_{18}\}$ and edge set $E = \{\ e_1 = (n_1, n_3, n_4, n_5, n_6, n_7)$, $e_2 = (n_1, n_3, n_{18}, n_{11})$, $e_3 = (n_2, n_3, n_4, n_5, n_6, n_7)$, $e_4 = (n_3, n_5)$, $e_5 = (n_3, n_{10})$, $e_6 = (n_4, n_{10})$, $e_7 = (n_4, n_{11})$, $e_8 = (n_8, n_{10}, n_{11}, n_5, n_6, n_7)$, $e_9 = (n_{14}, n_{15}, n_{16}, n_{17}, n_6, n_7)$, $e_{10} = (n_{14}, n_{15}, n_{16}, n_{17}, n_6, n_7)$, $e_{11} = (n_9, n_{10}, n_{11}, n_{12}, n_{13})$, $e_{12} = (n_5, n_{10}), \}$, where the edges $e_9$ and $e_{10}$ are multiple edges.

The concept of multiple edges, that is distinct edges associating the same $n$-tuples, with different membership values can be more suitably accommodated with the concept of multi-subset especially in fuzzy concepts.

**Definition 3.5.** A fuzzy multi-semigraph defined on a multi-semigraph $H^* = (N, E)$ is defined as $(N, \sigma, E_1, E_2)$ with a non-empty set $N$ and $\sigma : N \to [0, 1]$ is a function and $E_1 = \{((n, m), \mu_1(n, m), \mu_2(n, m), \ldots, \mu_{r_{nm}}(n, m))\ |(n, m) \in N \times N\}$ is a fuzzy multi-subset of $N \times N$, $\mu_i : N \times N \to [0, 1]$ such that

$$\mu_i(n, m) \leq \sigma(n) \wedge \sigma(m) \text{ for } n, m \in N$$




(a)

(b)

Figure 1: (a) A multi-semigraph, (b) A simple semigraph

in which $i = 1, 2, \ldots, r_{nm}$ and $r_{nm} = max\{i \mid \mu_i((n, m)) \neq 0, (n, m) \in N \times N\}$. The set $E_2 = \{(e_j, \eta_1(e_j), \eta_2(e_j), \ldots, \eta_{r_j}(e_j)) | e_j \in E\}$ is a fuzzy multi-subset of $E$, $\eta_i : E \to [0, 1]$ which satisfies

$$\eta_i(e_j) = \mu_i(n_1, n_2) \wedge \mu_i(n_2, n_3) \wedge \cdots \wedge \mu_i(n_{s-1}, n_s), \ \forall\, i = 1, 2, \ldots, r_j$$

if the edge $e_j$ is the $s$-tuple $(n_1, n_2, \ldots, n_{s-1}, n_s)$ and $r_j = max\{i \mid \eta_i(e_j) \neq 0, e_j \in E\}$.

If each edge $e_j$ in $E$ is 2-uniform, then the sets $E_1$ and $E_2$ will overlap completely. This makes the fuzzy multi-semigraph into a fuzzy multi-graph. Furthermore, if there exists exactly one fuzzy relation on $N \times N$, then the fuzzy multi-set $E_1$ became a fuzzy set. This yields a fuzzy graph.

A multi-semigraph can be transformed into a fuzzy multi-semigraph by defining appropriate functions $\sigma : N \to [0, 1]$ and $\mu_i : N \times N \to [0, 1]$.

Consider the multi-semigraph defined in Figure 1a. Since the node set $N$ is the collection of devices in the interaction we can define the functions $\sigma : N \to [0, 1]$, where $\sigma(n), n \in N$ as the normalized capacity of the device $n$ to store data and $\mu_i : N \times N \to [0, 1]$, where $\mu_i(n, m)$ as the normalized rate of flow of data between the devices $n$ and $m$ for $n, m \in N$.

In Figure 1a, assign $\sigma(n_1) = 0.52$, $\sigma(n_2) = 0.9$, $\sigma(n_3) = 0.73$, $\sigma(n_4) = 0.84$, $\sigma(n_5) = 0.49$, $\sigma(n_6) = 0.69$, $\sigma(n_7) = 0.57$, $\sigma(n_8) = 0.8$, $\sigma(n_9) = 0.32$, $\sigma(n_{10}) = 0.5$, $\sigma(n_{11}) = 0.45$, $\sigma(n_{12}) = 0.22$, $\sigma(n_{13}) = 0.9$, $\sigma(n_{14}) = 0.17$, $\sigma(n_{15}) = 0.33$, $\sigma(n_{16}) = 0.43$, $\sigma(n_{17}) = 0.65$, and $\sigma(n_{18}) = 0.69$, as the membership value of the nodes and $\mu_2(n_2, n_3) = 0.68$, $\mu_1(n_1, n_3) = 0.41$, $\mu_2(n_1, n_3) = 0.51$, $\mu_1(n_3, n_4) = 0.63$, $\mu_2(n_3, n_4) = 0.5$, $\mu_1(n_4, n_5) = 0.33$, $\mu_2(n_4, n_5) = 0.32$, $\mu_1(n_3, n_5) = 0.39$, $\mu_1(n_5, n_6) = 0.15$, $\mu_2(n_5, n_6) = 0.27$, $\mu_3(n_5, n_6) = 0.42$, $\mu_1(n_6, n_7) = 0.57$, $\mu_2(n_6, n_7) = 0.48$, $\mu_3(n_6, n_7) = 0.36$, $\mu_4(n_6, n_7) = 0.44$, $\mu_5(n_6, n_7) = 0.49$, $\mu_1(n_3, n_{10})$=0.35, $\mu_2(n_3, n_{18}) = 0.63$, $\mu_1(n_4, n_{10}) = 0.45$, $\mu_1(n_4, n_{11}) = 0.39$, $\mu_1(n_5, n_{10})$=0.21, $\mu_2(n_{18}, n_{11})$=0.4, $\mu_3(n_{11}, n_5) = 0.28$, $\mu_3(n_8, n_{10}) = 0.43$, $\mu_1(n_9, n_{10}) = 0.29$, $\mu_1(n_{10}, n_{11}) = 0.37$, $\mu_3(n_{10}, n_{11}) = 0.2$, $\mu_1(n_{11}, n_{12}) = 0.15$, $\mu_1(n_{12}, n_{13}) = 0.19$, $\mu_4(n_{14}, n_{15}) = 0.11$, $\mu_5(n_{14}, n_{15}) = 0.16$, $\mu_4(n_{15}, n_{16}) = 0.21$, $\mu_5(n_{15}, n_{16}) = 0.31$, $\mu_4(n_{16}, n_{17}) = 0.3$, $\mu_5(n_{16}, n_{17}) = 0.25$, $\mu_4(n_{17}, n_6) = 0.54$, $\mu_5(n_{17}, n_6) = 0.22$, and all other $\mu_i$ values are zero.

Hence $\eta_1(e_1) = 0.5$, $\eta_2(e_2) = 0.4$, $\eta_2(e_3) = 0.27$, $\eta_1(e_4) = 0.39$, $\eta_1(e_5) = 0.35$, $\eta_1(e_6) = 0.45$, $\eta_1(e_7) = 0.39$, $\eta_3(e_8) = 0.2$, $\eta_4(e_9) = 0.11$, $\eta_5(e_{10}) = 0.16$, $\eta_1(e_{11}) = 0.2$, $\eta_1(e_{12}) = 0.2$, and all other $\eta_i$ values are zero. This results in a fuzzy multi-semigraph from a multi-semigraph.

Note that $e_9$ and $e_{10}$ are the only multiple edges of the multi-semigraph in Figure 1. Thus these edges can be considered as a single edge (say $e_{multi}$) with different membership values in the fuzzy multi-semigraph. Thus

$$\begin{aligned} E_2 &= \{(e_1, 0.5), (e_2, 0, 0.4), (e_3, 0, 0.27), (e_4, 0.39), (e_5, 0.35), (e_6, 0.45), (e_7, 0.39), \\ &\quad (e_8, 0, 0, 0.2), (e_{multi}, 0, 0, 0, 0.11, 0.16), (e_{11}, 0.15), (e_{12}, 0.21)\}. \end{aligned}$$

We can define the crisp multi-semigraph of a fuzzy multi-semigraph analogous to the Definition 3.3.

### 3.1 Intersection Index in a Fuzzy Multi-Semigraph

The planarity of a graph is related to the crossing of edges at points other than nodes. In this section, we assign an index to the points that are not nodes where two or more edges in a fuzzy multi-semigraph intersect.

**Definition 3.6.** Consider a fuzzy multi-semigraph $(N, \sigma, E_1, E_2)$. Let $P$ be a point (not a node) where the edges $e_j$ with membership values $\eta_i(e_j)$ for $i = 1, 2, \dots r_j$ meet with end nodes $n_j$ and $m_j$ in between the consecutive adjacent nodes $x_j$ and $y_j$ of the edge $e_j$. Then define

$$I_{e_j}^i = \frac{1}{2}\left(\frac{\eta_i(e_j)}{\sigma(n_j) \wedge \sigma(m_j)} + \frac{\mu_i(x_j, y_j)}{\sigma(x_j) \wedge \sigma(y_j)}\right)$$

Then the intersection index $I_P$ at $P$ is

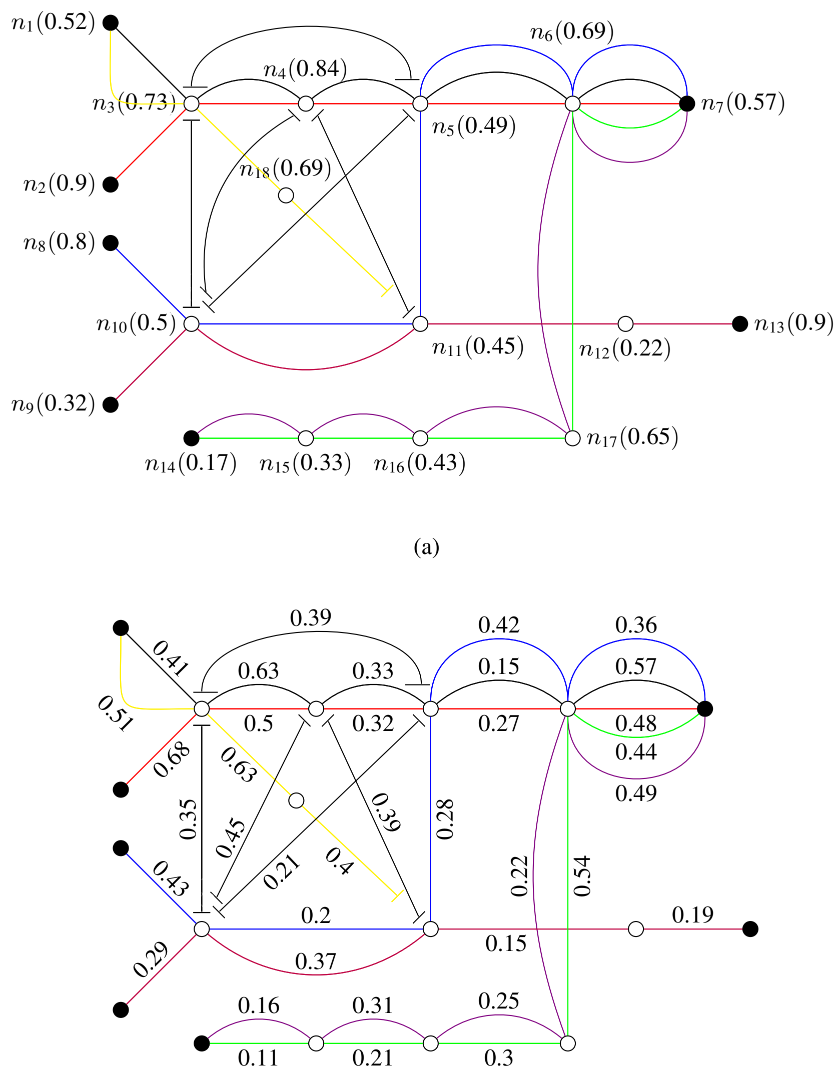
$$I_P = \frac{1}{k}\sum I_{e_j}^i \tag{3.1}$$

where the summation is taken over all pairs $(e_j, \eta_i(e_j))$, $i = 1, 2, \dots, r_j$ and $\eta_i \neq 0$ which meet at the point $P$ and $k$ is number of such pairs.

Suppose the membership values $\eta_i(e_j)$ for $i = 1, 2, \dots, r_j$ is non-zero for the edges $e_j$ in a fuzzy multi-semigraph then intersection index given in equation 3.1 became

$$I_P = \frac{1}{\sum_j r_j}\sum I_{e_j}^i$$

**Example 3.7.** Consider the points of intersections $P_1, P_2$, and $P_3$ marked in Figure 3. Here the point $P_1$ is in between the edges $e_2 = (n_1, n_3, n_{18}, n_{11})$ and $e_6 = (n_4, n_{10})$, the point $P_2$ is in between $e_2$ and $e_{12} = (n_5, n_{10})$, and the point $P_3$ in between $e_{12}$ and $e_7 = (n_4, n_{11})$. Also $P_1$ is in between the consecutive adjacent nodes $n_3$ and $n_{18}$). Here we want to consider the pairs $(e_2, \eta_2(e_2) = 0.4)$, $(e_6, \eta_1(e_6) = 0.45)$, $(e_{12}, \eta_1(e_{12}) = 0.21)$, and $(e_7, \eta_1(e_7) = 0.39)$. Then we have,




(b)

Figure 2: A fuzzy multi-semigraph of Figure 1a in which (a) $\sigma$ values are marked, (b) $\mu_i$ values are marked.

$$I^2_{e_2} = \frac{1}{2}\left(\frac{\eta_2(e_2)}{\sigma(n_1)\wedge\sigma(n_{11})} + \frac{\mu_2(n_3, n_{18})}{\sigma(n_3)\wedge\sigma(n_{18})}\right) = \frac{1}{2}\left(\frac{0.4}{0.45} + \frac{0.63}{0.69}\right) = 0.9009$$

$$I^1_{e_6} = \frac{1}{2}\left(\frac{\eta_1(e_6)}{\sigma(n_4)\wedge\sigma(n_{10})} + \frac{\mu_1(n_4, n_{10})}{\sigma(n_4)\wedge\sigma(n_{10})}\right) = \frac{1}{2}\left(\frac{0.45}{0.5} + \frac{0.45}{0.5}\right) = 0.9$$

$$I^1_{e_7} = \frac{1}{2}\left(\frac{\eta_1(e_7)}{\sigma(n_4)\wedge\sigma(n_{11})} + \frac{\mu_1(n_4, n_{11})}{\sigma(n_4)\wedge\sigma(n_{11})}\right) = \frac{1}{2}\left(\frac{0.39}{0.45} + \frac{0.39}{0.45}\right) = 0.8666.$$

$$I^1_{e_{12}} = \frac{1}{2}\left(\frac{\eta_1(e_{12})}{\sigma(n_5)\wedge\sigma(n_{10})} + \frac{\mu_1(n_5, n_{10})}{\sigma(n_5)\wedge\sigma(n_{10})}\right) = \frac{1}{2}\left(\frac{0.21}{0.49} + \frac{0.21}{0.49}\right) = 0.4285$$

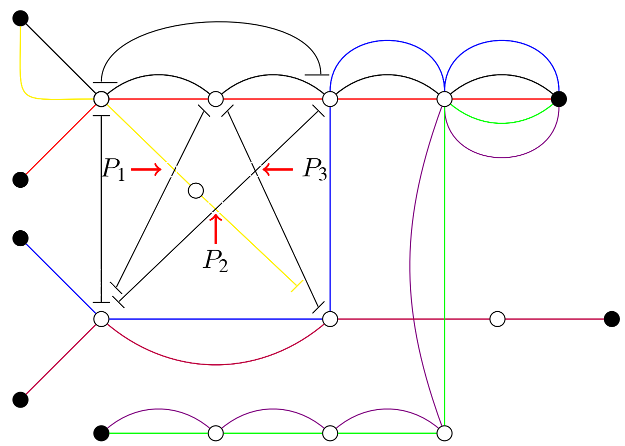



Figure 3: The multi-semigraph in Figure 1a with the points of intersections $P_1, P_2$, and $P_3$ are marked.

Thus the intersection index at the points $P_1, P_2$ and $P_3$ are

$$I_{P_1} = \frac{0.9009 + 0.9}{2} = 0.9004$$

$$I_{P_2} = \frac{0.9009 + 0.4285}{2} = 0.6647$$

$$I_{P_3} = \frac{0.4285 + 0.8666}{2} = 0.6475.$$

A similar computation will be performed if the intersections occur with multiple edges.

**Theorem 3.8.** *Let $(N, \sigma, E_1, E_2)$ be a fuzzy multi-semigraph. Then the intersection index at a point is maximum if the point of intersection occurs between effective edges of the fuzzy multi-semigraph.*

*Proof.* Assume that $P$ is a point where the effective edges $e_j$ for $j = 1, 2, 3, \dots r$ with membership values $\eta_i(e_j)$ for $i = 1, 2, \dots, s$ intersect and are in between the consecutive adjacent nodes $x_j$ and $y_j$ of the edge $e_j$ in the fuzzy multi-semigraph $(N, \sigma, E_1, E_2)$. Then

$$I^i_{e_j} = 1 \; \forall \; i, j$$

Thus the intersection index at the point $P$ is

$$I_P = \frac{1 + 1 + \cdots 1 \; (s \times r \text{ times})}{s \times r} = 1.$$

This gives the maximum value to the intersection index at the point. □

**Definition 3.9.** An edge $e = (n_1, n_2, \dots, n_r)$ in a fuzzy semigraph $(N, \sigma, \mu, \eta)$ is a strong edge if it meet the requirements:

$$\frac{1}{2}(\sigma(n_1) \wedge \sigma(n_r)) \leq \eta(e) \text{ and}$$

$$\frac{1}{2}(\sigma(n_j) \wedge \sigma(n_{j+1})) \leq \mu(n_j, n_{j+1}), \text{ for } 1 \leq j \leq r - 1.$$

Otherwise, called a weak edge.

**Definition 3.10.** The edge $e$ is said to be an $e$-strong edge if it satisfies the condition

$$\frac{1}{2}(\sigma(n_1) \wedge \sigma(n_r)) \leq \eta(e).$$

A strong edge is always an $e$-strong edge. But the converse is not true.

**Definition 3.11.** A fuzzy semigraph $(N, \sigma, \mu, \eta)$ is a strong fuzzy semigraph if each edges in $(N, \sigma, \mu, \eta)$ are strong edges.

The bounds of membership value of strong edges suggest a meagre chance of uncertainty. The strong edges make any scenario more stable compared to weak edges. Consider the context of network interaction explained in Example 3.4. Here strong edges ensure the effective data flow between the devices. Similarly, the networks which mandated to possess a membership value within some bounds, for example: electrical networks, can use the concept of strong edges if the lower bound falls as in Definition 3.9.

**Theorem 3.12.** *Let $(N, \sigma, E_1, E_2)$ be a fuzzy multi-semigraph. Then the intersection index at a point that occurs between strong edges is at most 0.5.*

*Proof.* Assume that $P$ be a point, where the strong edges $e_j$ for $j = 1, 2, 3, \ldots, r$ with membership values $\eta_i(e_j)$ intersect for $i = 1, 2, \ldots, s$ and in between the consecutive adjacent nodes $x_j$ and $y_j$ of the edge $e_j$ in the fuzzy multi-semigraph $(N, \sigma, E_1, E_2)$. Then

$$I^i_{e_j} \leq 0.5 \; \forall i, j$$

Hence the intersection index at the point $P$ is

$$I_P \leq \frac{0.5 + 0.5 + \cdots 0.5 \; (s \times r \text{ times})}{s \times r} = 0.5$$

□

**Remark 3.13.** Consider a fuzzy multi-semigraph $(N, \sigma, E_1, E_2)$ which is 2 - uniform, that is $(N, \sigma, E_1, E_2)$ is a fuzzy multi-graph. Let $e_j = (n_j, m_j)$ for $j = 1, 2, 3, \ldots r$ be edges with non-zero membership value $\eta_i(e_j)$ for $i = 1, 2, \ldots, s$ intersect at a point $P$. Since the $\eta$-value coincide with the $\mu$-value in a fuzzy multi-graph,

$$I^i_{e_j} = \frac{1}{2}\left(\frac{\eta_i(e_j)}{\sigma(n_j) \wedge \sigma(m_j)} + \frac{\mu_i(n_j, m_j)}{\sigma(n_j) \wedge \sigma(m_j)}\right) = \frac{\mu_i(n_j, m_j)}{\sigma(n_j) \wedge \sigma(m_j)}.$$

Thus the intersection index at the point $P$ is

$$I_P = \frac{1}{s \times r} \sum_{i=1}^{s} \sum_{j=1}^{r} \frac{\mu_i(n_j, m_j)}{\sigma(n_j) \wedge \sigma(m_j)}.$$

Suppose $\mu_l$ is the only non-zero membership value of the edge $e_k$, for fixed integers $l$ and $k$, intersecting at a point $P$. Then

$$I_P = \frac{\mu_l(n_k, m_k)}{\sigma(n_k) \wedge \sigma(m_k)}$$

which coincides with the concept of intersection value in a fuzzy multi-graph.

### 3.2 Planarity Index ($PI$) of a Fuzzy Multi-Semigraph

**Definition 3.14.** For a fuzzy multi-semigraph $(N, \sigma, E_1, E_2)$ the planarity index, denoted by $PI$, is defined as

$$PI = \frac{1}{1 + \sum_{j=1}^{t} I_{P_j}}$$

in which $P_1, P_2, \ldots, P_t$ are the intersecting points between the edges in a geometrical depiction of $(N, \sigma, E_1, E_2)$.

Every fuzzy multi-semigraph has a $PI$. Also note that the $PI$ in a fuzzy multi-semigraph increases as the sum of the intersection index in the fuzzy multi-semigraph decreases.

Note that if no intersections occur in a depiction of a fuzzy multi-semigraph $(N, \sigma, E_1, E_2)$ then $PI$ is one, which means the fuzzy multi-semigraph can be drawn in a plane without edge crossing as required for planar graphs. In other words $(N, \sigma, E_1, E_2)$ has a plane embedding with no edge crossings.

A flow chart representing the computation of $PI$, with a minimum number of edge crossing for the existence of an intersection is given in Figure 8. The same analysis is adopted if the intersecting points are caused by the crossing of three or more edges.

**Example 3.15.** The $PI$ of the fuzzy multi-semigraph in the Figure 2 is

$$PI = \frac{1}{1 + \sum_{1=i}^{3} I_{P_i}} = \frac{1}{1 + (0.9004 + 0.6647 + 0.6475)} = \frac{1}{3.2126} = 0.3113.$$

**Theorem 3.16.** *If each intersecting points occur in between effective edges, then $PI$ of the fuzzy multi-semigraph $(N, \sigma, E_1, E_2)$ is,*

$$PI = \frac{1}{1+t}$$

*where $t$ is the number of intersecting points.*

*Proof.* Assume that each intersecting points in the fuzzy multi-semigraph occur between the effective edges. Consider $P_1, P_2, \ldots, P_t$ are the intersecting points between these edges where $t$ is the number of intersections in the fuzzy multi-semigraph. Then by Theorem 3.8, the intersection index is one for each intersection. Hence the $PI$ is given by

$$PI = \frac{1}{1 + \sum_{i=1}^{t} 1} = \frac{1}{1+t}.$$

□

**Theorem 3.17.** *If the points of intersection in a fuzzy multi-semigraph occur between the strong edges, then $PI$ of a fuzzy multi-semigraph is at most $\frac{2}{2+t}$ where $t$ is the number of intersecting points.*

*Proof.* Assume that each intersecting point in the fuzzy multi-semigraph is in between the strong edges. Consider $P_1, P_2, \ldots, P_t$ are the intersecting points between these edges where $t$ is the number of intersections in the fuzzy multi-semigraph. Then by Theorem 3.12, the intersection index is at least $1/2$ for each intersection. Hence the $PI$ is given by

$$PI \leq \frac{1}{1 + \sum_{i=1}^{t} \frac{1}{2}} = \frac{2}{2+t}.$$

□

**Theorem 3.18.** *If the points of intersection in a fuzzy multi-semigraph occur between $e$-strong edges, then the $PI$ of a fuzzy multi-semigraph is at most $\frac{4}{4+t}$ where $t$ is the number of intersecting points.*

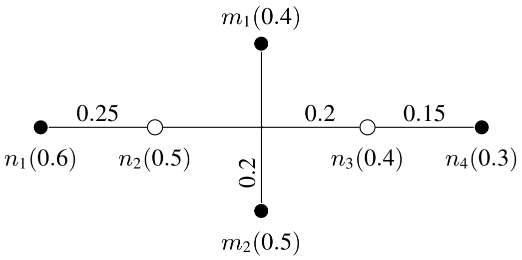
*Proof.* Assume that each intersecting points in the fuzzy multi-semigraph occur between the $e$-strong edges. Consider $P_1, P_2, \ldots, P_t$ are the intersecting points between these edges where $t$ is the number of intersections in the fuzzy multi-semigraph. Then the intersection index is at least $1/4$ for each intersection. Hence the $PI$ is given by

$$PI \leq \frac{1}{1 + \sum_{i=1}^{t} \frac{1}{4}} = \frac{4}{4+t}.$$

□

Since $\frac{2}{2+t} \leq \frac{4}{4+t}$ where t is a non-negative integer, then the strength of $PI$ of a fuzzy multi-semigraph with points of intersection occurs only between $e$-strong edges can be greater than the strength of the intersections occurs only between strong edges.

**Definition 3.19.** A fuzzy multi-semigraph with a planarity index $PI > 0.5$ is said to be a strongly planar fuzzy multi-semigraph. Otherwise, it is called a weakly planar fuzzy multi-semigraph.



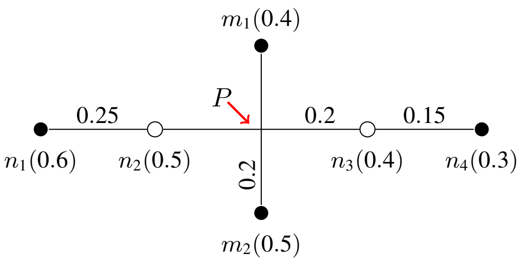




Figure 4: A fuzzy multi-semigraph with $PI = 2/3$

**Theorem 3.20.** *Consider a strongly planar fuzzy multi-semigraph. Then*

*(a) The set of intersecting points between effective edges is empty.*

*(b) The number of intersecting points between strong edges is no more than one.*

*Proof.* Assume a strongly planar fuzzy multi-semigraph $(N, \sigma, E_1, E_2)$.

(a) The $PI$ of $(N, \sigma, E_1, E_2)$ is greater than 0.5. If possible, let the strongly planar fuzzy multi-semigraph have an intersecting point $P$ between effective edges in $(N, \sigma, E_1, E_2)$. Then the intersection index at the point $P$ is 1. Hence the $PI$ of the fuzzy multi-semigraph is 0.5, which gives a contradiction. Thus no points of intersection occur between the effective edges of a strongly planar fuzzy multi-semigraph.

(b) If possible, let the fuzzy multi-semigraph have two intersecting points $P_1$ and $P_2$ occur between strong edges. Thus the intersection index at the points $P_i$ for $i = 1, 2$ are at least 0.5. Hence the $PI$ of the fuzzy multi-semigraph is at most 0.5, which gives a contradiction. Note that as the number of intersecting points increases the $PI$ reduces. Thus in a strongly planar fuzzy multi-semigraph, at most one intersecting point occurs between strong edges.

□

The Theorem 3.20 (a) says that a strongly planar fuzzy multi-semigraph is always planar if it is effective.

**Theorem 3.21.** *Consider a fuzzy multi-semigraph in which $PI > 2/3$. Then no intersecting points exist between strong edges.*

*Proof.* Assume a fuzzy multi-semigraph $(N, \sigma, E_1, E_2)$ with $PI > 2/3$. Choose a point of intersection $P$ between strong edges. Then the intersection index at the point $P$ is at least 0.5. Thus the $PI$ is not more than $2/3$, which shows a contradiction. □

Hence a fuzzy multi-semigraph with $PI > 2/3$ is planar if it is a strong fuzzy multi-semigraph.

**Remark 3.22.** The Theorem 3.21 does not hold if $PI$ equals $2/3$.

In the Figure 4 the edge $e = (n_1, n_2, n_3, n_4)$ with membership value 0.15 intersect at a point $P$ with the edge $f = (m_1, m_2)$ with the membership value 0.2 in between the consecutive adjacent nodes $n_2, n_3$ of the edge $e$ and $m_1, m_2$ of the edge $f$. Both the edges are strong. Here

$$I_P = \frac{1}{2}$$

Thus the intersection index $I_P$ at the point $P$ is $1/2$ and the $PI$ at the point $P$ is

$$PI = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}.$$

Here the fuzzy semigraph has $PI = 2/3$ and has edge crossing between its strong edges.

### 3.3 An Application

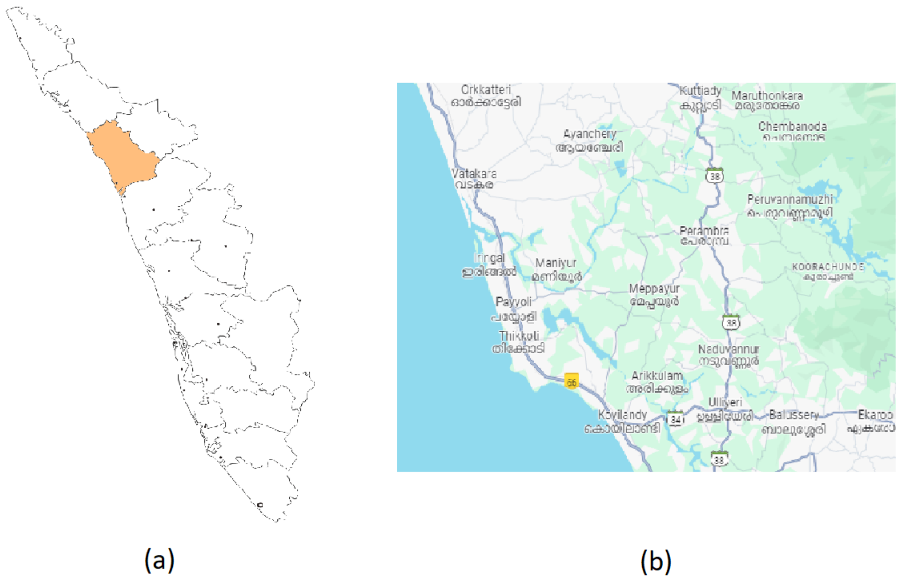
There are approximately 428 level crossings in the state of Kerala. These crossings are a major cause of rail or road accidents, and they also create hazards. A permanent solution to this real-world problem is to build a road over-bridge or an underpass. But it is impracticable to construct such bridges over or under all the level crossings.

Here we find the intersection index at each level crossing, which gives a measure of priority from which we can observe the most suitable (within construction constraints) level crossings where an over or under bridge is necessary to build.

For a study, we scaled down the road/railway system of the state Kerala in between Vadakara and Koyilandy, two major cities in the district Kozhikode. We construct a semigraph by putting major cities and railway stations between Vadakara and Koyilandy as the collection of nodes and an edge passing through these nodes as a road/railway track connecting these cities. Define the membership value to the nodes and pair of nodes so that the semigraph becomes a fuzzy semigraph. The membership values are assigned based on the capacity of that city. That measures how congested the city is. For that, we have taken into account several behavioural properties of that location, such as historical significance, traffic facilities, the average number of vehicles visiting that place, tourism, schools/colleges, central/state government offices, hospitals/dispensaries, merchant shops/houses, huge private firms, etc. So the value of $\sigma$ is inversely proportional to how the city is congested and the value of $\mu$ is directly proportional.

Here note that the edge representing the railway track is an effective edge. An over bridge already exists at the point of intersection $U$. Now we check the intersection index at the points of intersection $P, Q, R, S, T, V$ and $W$.

$$\begin{aligned}
I_P &= \frac{\frac{0.32}{0.35} + 1}{2} = 0.9571 \\
I_Q &= \frac{\frac{\frac{0.2}{0.4} + \frac{0.29}{0.39}}{2} + 1}{2} = 0.8109 \\
I_R &= \frac{\frac{0.3}{0.39} + 1}{2} = 0.8846 \\
I_S &= \frac{\frac{\frac{0.13}{0.5} + \frac{0.13}{0.4}}{2} + 1}{2} = 0.7979 \\
I_T &= \frac{\frac{0.1}{0.4} + 1}{2} = 0.625 \\
I_V &= \frac{\frac{\frac{0.1}{0.4} + \frac{0.25}{0.5}}{2} + 1}{2} = 0.6875 \\
I_W &= \frac{\frac{0.19}{0.2} + 1}{2} = 0.975
\end{aligned}$$




Note: (a) From " Kozhikode district, Kerala.png-Wikimedia Commons," by Crawford88, 2016, Original Creation by Uploader(https://commons.wikimedia.org/wiki/File:Kozhikode_district,_Kerala.png). CC BY-SA 4.0. (b) Adapted from "Google Maps," (https://www.google.com/maps/@14.275067,74.4407821,7z/ data=!5m1!1e2entry=ttu&g_ep=EgoyMDI0MTAwNS4yIKXMDSoASAFQAw%3D%3D)

Figure 5: (a)Kerala State where Kozhikode district is marked [3], (b) The location map of the area described in the Subsection 3.3[4]

Thus we found

$$I_W > I_P > I_R > I_Q > I_V > I_S > I_T$$

According to the study, we see that the preference should be given in the increasing order for construction of over/underpasses.

The planarity index $PI$ of the fuzzy semigraph in the Figure 7 is,

$$PI = \frac{1}{1 + I_P + I_Q + I_R + I_S + I_T + I_V + I_W} = \frac{1}{1 + 5.6901} = 0.1494.$$

Suppose we have constructed an over/under bridge at the point of intersection $W$, the point where the intersection index is maximum. Then by counting $PI$ of such a fuzzy semigraph, we will not count $I_W$, because $W$ is no longer a point of intersection. Thus

$$PI_W = \frac{1}{1 + I_P + I_Q + I_R + I_S + I_T + I_V +} = \frac{1}{1 + 4.7151} = 0.1749.$$

If we done similar process by putting the point $T$ and $Q$ separately instead of $W$, we have $PI$ of the respective fuzzy semigraph as follows,

$$PI_T = \frac{1}{1 + I_P + I_Q + I_R + I_S + I_V + I_W} = \frac{1}{1 + 5.0651} = 0.1648$$

$$PI_Q = \frac{1}{1 + I_P + I_T + I_R + I_S + I_V + I_W} = \frac{1}{1 + 4.8792} = 0.1701$$

Note that $PI_W$ represents the $PI$ of the fuzzy semigraphs where an over/under bridge is constructed at the point of intersection $W$. A similar notation is used for the same in other points of intersection.

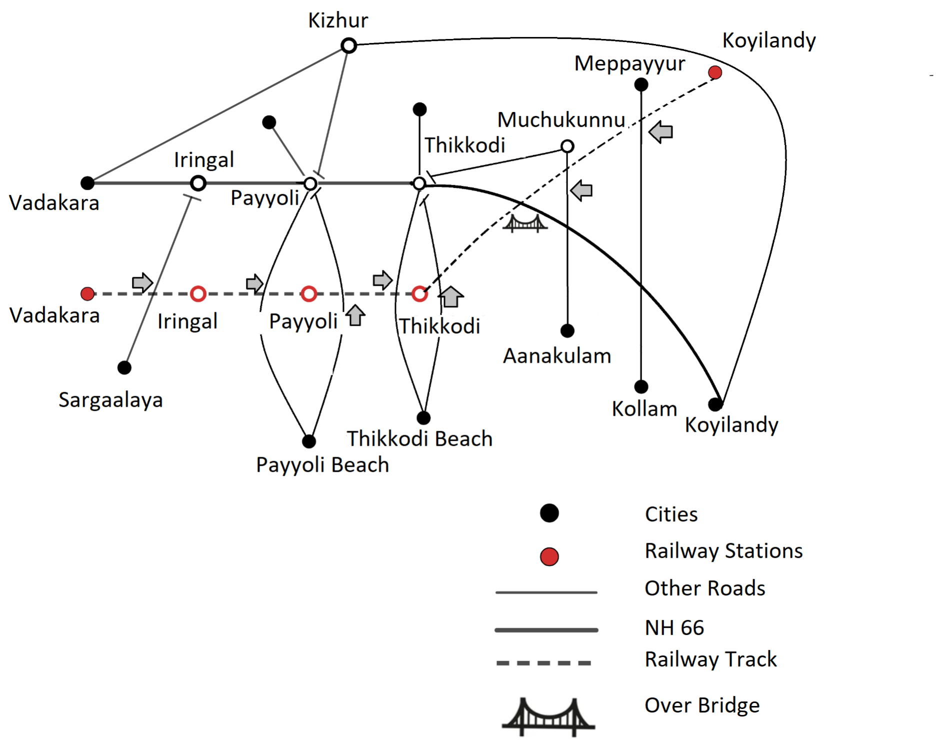



Figure 6: semigraph representation of rail/road system from Vadakara to Koyilandy

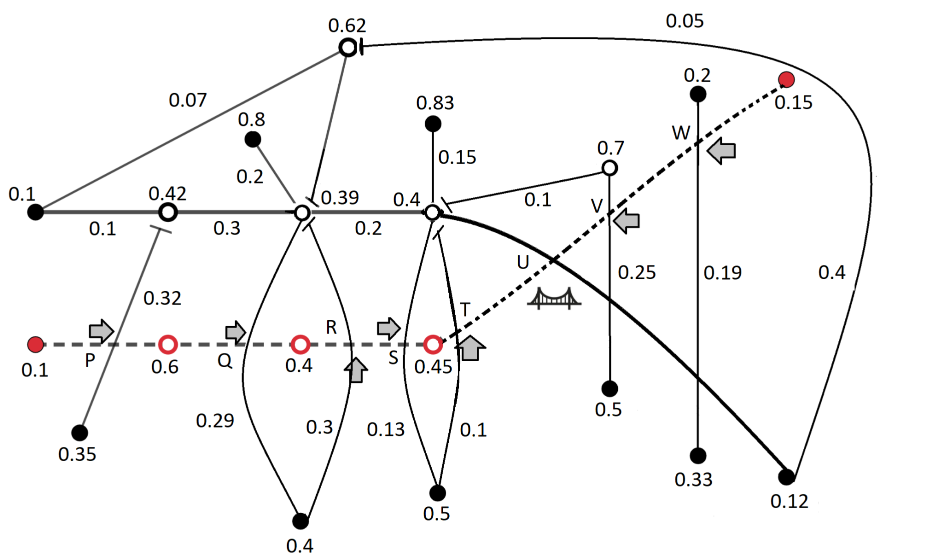



Figure 7: fuzzy semigraph representation of rail/road system from Vadakara to Koyilandy

We have

$$PI_W > PI_P > PI_R > PI_Q > PI_V > PI_S > PI_T > PI.$$

The construction of a new over/under bridge at $W$ increases the planarity index (or something closer to planar) of the network compared to other crossings. So preference should be given to $W$.

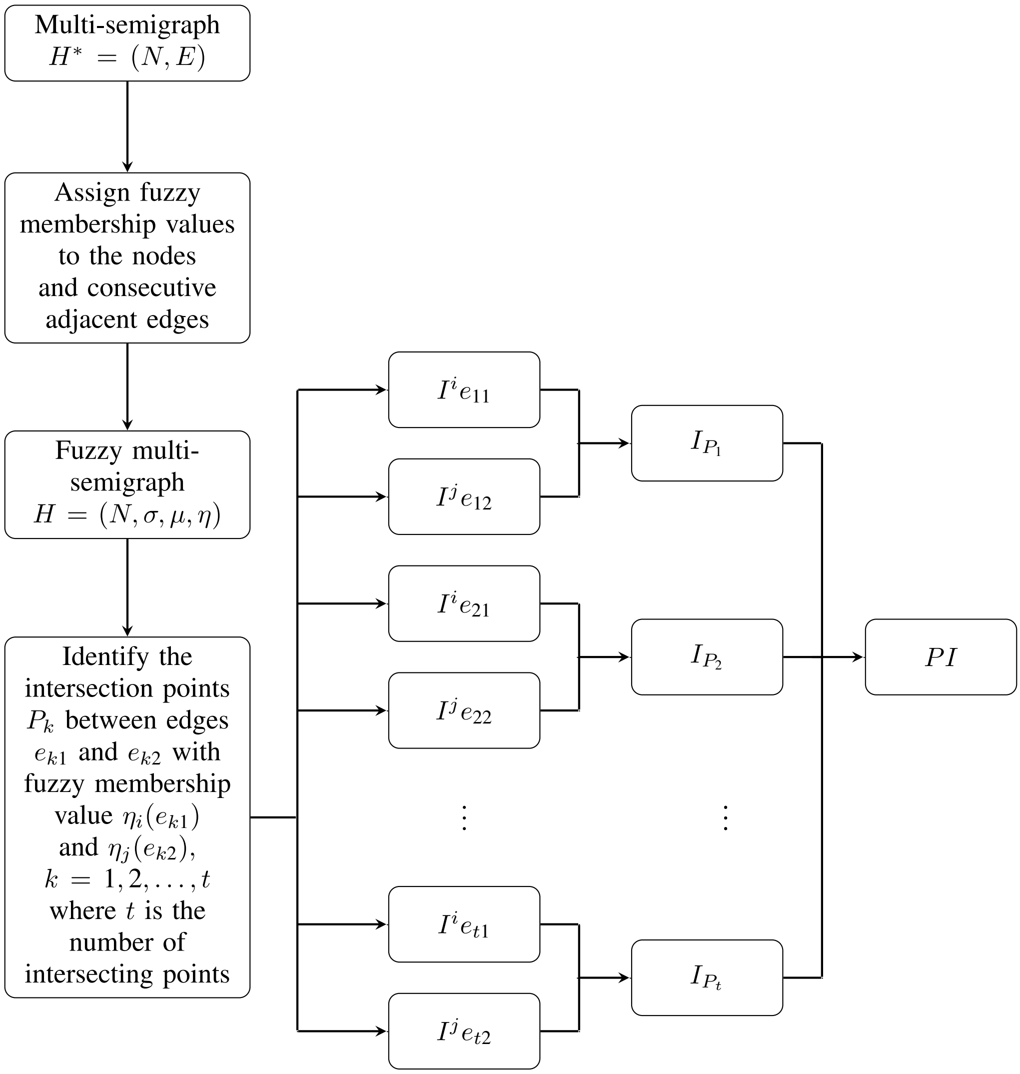



Figure 8: A flow chart showing the computation of $PI$

## 4 Conclusion

In this research, we introduced the concept of multi-semigraph and Fuzzy multi-semigraph by advancing the semigraph theory. Our observations demonstrate that the concept of fuzzy multi-semigraph is a perfect tool for modelling any practical problems which can be expressed using several parameters. With these tools, we have effectively modelled a rail and road network interaction. The intersection and planarity indices detailed in this work are relevant, because crossing of edges causes practical difficulties in many occasions, even though maintaining the connections are important. Multi-semigraphs and fuzzy multi-semigraphs can be used to model the functioning of various firms and systems where each function is a sequence of activities, also various functions of the human body like a neural network.

Even though the study provides valuable observations, there are some limitations too. While addressing neural networks in biological fields, direction holds a significant role.

Further scope of this study spreads over many areas and we have started working on n-planarity, the concept of level sets, and further extension of multi-semigraphs through multi-subsets of N. The concept of strong edges is really interesting and applicable in certain cases like electrical networks.

## References


[1] S. Archana, and P. Kuttipulackal, *Line regular fuzzy semigraphs*, Baghdad Sci. J., **20(1(SI))**, 0288–0293, (2023).

[2] B. Y. Bam, *On some problems of graph theory in semigraphs*, PhD thesis, Savitribai Phule Pune University, (2005).

[3] Crawford88. (2016). Kozhikode district, Kerala.png-Wikimedia Commons, commons.wikimedia, https://commons.wikimedia.org/wiki/File:Kozhikode_district,_Kerala.png

[4] Google Maps. https://www.google.com/maps/@11.53436,75.6958497,9z/data=!5m1!1e2?entry=ttu&g_ep=EgoyMDI0MTAwMi4xIKXMDSoASAFQAw%3D%3D

[5] A. Khalid, N. Kausar, M. Munir, H. Aydi, S. Kousar, and Y. U. Gaba, *Topological Indices for Two Special Families of Graphs of Diameter Three*, Advances in Mathematical Physics, **2021(1)**, 4051026, (2021).

[6] S. Mathew, J. N. Mordeson, and D. S. Malik, *Fuzzy Graph Theory*, Springer International Publishing, Cham, Switzerland, (2019).

[7] J. Moshahary, *Topological structure on fuzzy multisets*, Int. J. Sci. Res. Manag., **5(7)**, 6461–6465, (2017).

[8] K. Radha, and P. Renganathan, *Effective fuzzy semigraphs*, Adv Appl. Math. Sci., **20(5)**, 895–904, (2021).

[9] K. Radha, and P.Renganathan, *Isomorphic Properties of Fuzzy Semigraphs*, Adv Appl. Math. Sci., **21(6)**, 3371–3382, (2022).

[10] A. Rosenfeld, *Fuzzy graphs*, Fuzzy sets and their applications to cognitive and decision processes, 77–95, (1975).

[11] G. R. Roshini, S. B. Chandrakala, M. V. Kumar, and B. Sooryanarayana, *Non-neighbor topological indices of honeycomb networks*, Palestine Journal of Mathematics, **10(SI 1)**, 52–58, (2021).

[12] S. Samanta, and M. Pal, *Fuzzy planar graphs*, IEEE Transactions on Fuzzy Systems, **23(6)**, 1936–1942, (2015).

[13] S. Samanta, M. Pal, and A. Pal, *New concepts of fuzzy planar graph*, International Journal of Advanced Research in Artificial Intelligence, **3(1)**, 52–59, (2014).

[14] E. Sampathkumar, C. M. Deshpande, B. Y. Bam, L. Pushpalatha, and V. Swaminathan, *Semigraphs and their applications*, Report on the DST Project, (2000).


### Author information


Archana S., Research Scholar, Department of Mathematics, University of Calicut, India.
E-mail: archanas@uoc.ac.in

Preethi Kuttipulackal, Associate Professor, Department of Mathematics, University of Calicut, India.
E-mail: preethikp@uoc.ac.in



Received: 2024-08-29
Accepted: 2024-10-15