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On the Hybrid Leonardo p-numbers

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Abstract *In this paper, we introduced hybrid Leonardo p-numbers and some algebraic properties of hybrid Leonardo p-numbers in terms of hybrid Fibonacci p-numbers.*

1 Introduction

The Leonardo numbers, Le_n [3] are defined by the non-homogeneous recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \text{ for } n \geq 2$$

with initial terms

$$Le_0 = Le_1 = 1.$$

Kuhapatanakul and Chobsorn [11] defined a generalization of Leonardo numbers, $Le_k(n)$ by the non-homogeneous recurrence relation

$$Le_k(n) = Le_k(n-1) + Le_k(n-2) + k \text{ for } n \geq 2$$

with initial terms

$$Le_k(0) = Le_k(1) = 1.$$

The Leonardo p-numbers, $Le_p(n)$ [7] are defined by the non-homogeneous recurrence relation

$$Le_p(n) = Le_p(n-1) + Le_p(n-p-1) + p \tag{1.1}$$

with $n > p$, for a given integer $p = 1, 2, 3, 4, \cdots$ and initial terms

$$Le_p(0) = Le_p(1) = \cdots = Le_p(p) = 1.$$

The non-homogeneous recurrence relation of Leonardo p-numbers, $Le_p(n)$ can be converted to the following homogeneous recurrence relation

$$Le_p(n) = Le_(n-1) + Le_p(n-p) - Le_p(n-2p-1) \text{ for } n > 2p$$

The Fibonacci p-numbers [6] are defined by the recurrence relation

$$F_p(n) = F_p(n-1) + F_p(n-p-1)$$

with $n > p+1$, for a given integer $p = 0, 1, 2, 3, \cdots$ and initial terms

$$F_p(1) = F_p(2) = \cdots = F_p(p) = F_p(p+1) = 1.$$

The Fibonacci p-numbers, $F_p(n)$ coincide with classical Fibonacci numbers, F_n [12, 13] for $p = 1$. e.g. $F_1(n) = F_n$.
The Lucas p-numbers [6] are given by the recurrence relation

$$L_p(n) = L_p(n-1) + L_p(n-p-1)$$

with $n > p+1$ and initial terms

$$L_p(1) = L_p(2) = \cdots = L_p(p) = 1, L_p(p+1) = p+2$$

where $p = 0, 1, 2, \cdots$.
For $p = 1$, $L_1(n) = L_n$ are known as classical Lucas numbers [14].
There is a relation among Leonardo p-numbers, $Le_p(n)$, Fibonacci p-numbers, $F_p(n)$ and Lucas p-numbers, $L_p(n)$ [7].

$$L_p(n) = F_p(n+1) + pF_p(n-p) \tag{1.2}$$

$$Le_p(n) = (p+1)F_p(n+1) - p \tag{1.3}$$

$$Le_p(n) = L_p(n+p+1) - F_p(n+p+1) - p \tag{1.4}$$

$$F_p(k+n) = F_p(k)F_p(n+1) + \sum_{r=1}^{p} F_p(k-r)F_p(n-p+r) \tag{1.5}$$

where k and n are positive integers.
Ozdemir [5] defined the set of hybrid numbers denoted by K which contains complex, dual and hyperbolic numbers. The set of the hybrid numbers, K is defined as follows:
$K = \{a + bi + c\varepsilon + dh : a, b, c, d \in \mathbf{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i\}$.
We can give operations and properties with the hybrid numbers. If we take two hybrid numbers $Z_1 = a_1 + b_1 i + c_1\varepsilon + d_1 h$ and $Z_2 = a_2 + b_2 i + c_2\varepsilon + d_2 h$, we get
$Z_1 = Z_2 \Leftrightarrow a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$ (Equality)
$Z_1 + Z_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)\varepsilon + (d_1 + d_2)h$ (Addition)
$Z_1 - Z_2 = (a_1 - a_2) + (b_1 - b_2)i + (c_1 - c_2)\varepsilon + (d_1 - d_2)h$ (Subtraction)
$sZ_1 = sa_1 + sb_1 i + sc_1\varepsilon + sd_1 h$ (Multiplication by scalar $s \in \mathbf{R}$)
Addition operation in K is both commutative and associative.
$0 = 0 + 0i + 0\varepsilon + 0h \in K$ is the additive identity element. The inverse of $Z = a + bi + c\varepsilon + dh$ is $-Z = -a - bi - c\varepsilon - dh$. Thus ($K$, +) is an Abellian group. The hybrid product is obtained by distributing the terms to the right, preserving the order of multiplication of the units and then writing the values of the following substituting each product of units by the equalities $i^2 = -1$, $\varepsilon^2 = 0$, $h^2 = 1$, $ih = -hi = \varepsilon + i$. From these equalities, we can obtain the product of any two hybrid units. The multiplication table of the units of hybrid numbers are as follows:

Table 1: Multiplication scheme of hybrid numbers

x	1	i	ε	h
1	1	i	ε	h
i	i	-1	1-h	$\varepsilon + i$
ε	ε	1+h	0	-ε
h	h	-$\varepsilon - i$	ε	1

Considering this table, it can be seen that the multiplication operation of the hybrid numbers has the property of associative but not to property of commutative. $1 = 1 + 0i + 0\varepsilon + 0h \in K$ is the multiplicative identity element. Therefore, (K, +, .) is a noncommutative ring with identity.
Recently, many researchers have studied related to hybrid numbers. In 2021, Y. Alp and E. G. Kocer [2] introduced the hybrid Leonardo numbers and their algebraic properties. Szynal-Liana and Wloch [4] defined the Fibonacci hybrid numbers and obtained the properties of these numbers. In 2022, E. G. Kocer and H. Alsan [1] introduced the generalized hybrid Fibonacci and Lucas p-numbers.
Hybrid Fibonacci p-numbers, $HF_p(n)$ and hybrid Lucas p-numbers, $HL_p(n)$ [1] are given by the following recurrence relations

$$HF_p(n) = F_p(n) + iF_p(n+1) + \varepsilon F_p(n+2) + hF_p(n+3) \tag{1.6}$$

and

$$HL_p(n) = L_p(n) + iL_p(n+1) + \varepsilon L_p(n+2) + hL_p(n+3) \tag{1.7}$$

where $p(> 0)$ is integer, $n > p$ and the basis $\{1, i, \varepsilon, h\}$ satisfies the condition
$i^2 = -1$, $\varepsilon^2 = 0$, $h^2 = 1$, $ih = -hi = \varepsilon + i$.
The hybrid Leonardo numbers, HLe_n [3] are defined by the non-homogeneous recurrence relation

$$HLe_n = HLe_{n-1} + HLe_{n-2} + A \text{ for } n \geq 2$$

where $A = 1 + i + \varepsilon + h$ with initial terms

$$HLe_0 = 1 + i + 3\varepsilon + 5h, Le_1 = 1 + 3i + 5\varepsilon + 9h.$$

Motivated by [3], we introduce a new hybrid numbers with Leonardo p-numbers. Hybrid Leonardo p-numbers associated with hybrid Fibonacci p-numbers and hybrid Lucas p-numbers. In this paper, we give the recurrence relation, Binet's formula, generating function, summation formula, Cassini identity, Catalan's identity, d'ocagne type identity, Honsberger identity for Hybrid Leonardo p-numbers.

2 Hybrid Leonardo p-numbers

Definition 2.1 Hybrid Leonardo p-numbers are given by the following recurrence relations

$$HLe_p(n) = Le_p(n) + iLe_p(n+1) + \varepsilon Le_p(n+2) + hLe_p(n+3) \tag{2.1}$$

where $p(> 0)$ is integer, $n > p$ and the basis $\{1, i, \varepsilon, h\}$ satisfies the condition
$i^2 = -1$, $\varepsilon^2 = 0$, $h^2 = 1$, $ih = -hi = \varepsilon + i$.
From the recurrence relations (1) and (8) , we get
$HLe_p(n) = Le_p(n) + iLe_p(n+1) + \varepsilon Le_p(n+2) + hLe_p(n+3)$
$= (Le_p(n-1) + Le_p(n-p-1) + p) + i(Le_p(n) + Le_p(n-p) + p) + \varepsilon(Le_p(n+1) + Le_p(n-p+1) + p) + h(Le_p(n+2) + Le_p(n-p+2) + p)$
$= HLe_p(n-1) + HLe_p(n-p-1) + p(1+i+\varepsilon+h) = HLe_p(n-1) + HLe_p(n-p-1) + pA$ where $A = 1+i+\varepsilon+h$.
Now, we calculate the values of $HLe_p(0)$, $HLe_p(1)$, $HLe_p(2)$, $\cdots$,HLe$_p(p-2)$, $HLe_p(p-1)$, $HLe_p(p)$ with the help

of Leonardo p-numbers
$HLe_p(0) = Le_p(0) + iLe_p(1) + \varepsilon Le_p(2) + hLe_p(3) = 1 + i + \varepsilon + h$
$HLe_p(1) = Le_p(1) + iLe_p(2) + \varepsilon Le_p(3) + hLe_p(4) = 1 + i + \varepsilon + h$
$HLe_p(2) = Le_p(2) + iLe_p(3) + \varepsilon Le_p(4) + hLe_p(5) = 1 + i + \varepsilon + h$
$HLe_p(p-3) = Le_p(p-3) + iLe_p(p-2) + \varepsilon Le_p(p-1) + hLe_p(p) = 1 + i + \varepsilon + h$
$HLe_p(p-2) = Le_p(p-2) + iLe_p(p-1) + \varepsilon Le_p(p) + hLe_p(p+1) = 1 + i + \varepsilon + h(p+2)$
$HLe_p(p-1) = Le_p(p-1) + iLe_p(p) + \varepsilon Le_p(p+1) + hLe_p(p+2) = 1 + i + \varepsilon(p+2) + h(2p+3)$
$HLe_p(p) = Le_p(p) + iLe_p(p+1) + \varepsilon Le_p(p+2) + hLe_p(p+3) = 1 + i(p+2) + \varepsilon(2p+3) + h(3p+4)$
$HLe_p(0)$, $HLe_p(1)$, $HLe_p(2)$, $\cdots$,HLe$_p(p-2)$, $HLe_p(p-1)$, $HLe_p(p)$ are initial conditions.
$HLe_p(n+1) = Le_p(n+1) + iLe_p(n+2) + \varepsilon Le_p(n+3) + hLe_p(n+4)$
$= (Le_p(n) + Le_p(n+1-p) - Le_p(n-2p)) + i(Le_p(n+1) + Le_p(n+2-p) - Le_p(n-2p+1)) + \varepsilon(Le_p(n+2) + Le_p(n+3-p) - Le_p(n-2p+2)) + h(Le_p(n+3) + Le_p(n+4-p) - Le_p(n-2p+3))$
$= (Le_p(n) + iLe_p(n+1) + \varepsilon Le_p(n+2) + hLe_p(n+3)) + (Le_p(n+1-p) + iLe_p(n+2-p) + \varepsilon Le_p(n+3-p) + hLe_p(n+4-p)) - (Le_p(n-2p) + iLe_p(n-2p+1) + \varepsilon Le_p(n-2p+2) + hLe_p(n-2p+3))$
$= HLe_p(n) + HLe_p(n+1-p) - HLe_p(n-2p)$.

Theorem 2.2 For $n \geq 0$, $HLe_p(n) = (p+1)HF_p(n+1) - p(1+i+\varepsilon+h)$.
Proof:
$HLe_p(n) = Le_p(n) + iLe_p(n+1) + \varepsilon Le_p(n+2) + hLe_p(n+3)$
$= ((p+1)F_p(n+1) - p) + i((p+1)F_p(n+2) - p) + \varepsilon((p+1)F_p(n+3) - p) + h((p+1)F_p(n+4) - p)$
$= (p+1)HF_p(n+1) - p(1+i+\varepsilon+h)$.
Theorem 2.3 For $n \geq 0$, $HLe_p(n) = HL_p(n+p+1) - HF_p(n+p+1) - p(1+i+\varepsilon+h)$.
Proof:
$HLe_p(n) = Le_p(n) + iLe_p(n+1) + \varepsilon Le_p(n+2) + hLe_p(n+3)$
$= (L_p(n+p+1) - F_p(n+p+1) - p) + i(L_p(n+p+2) - F_p(n+p+2) - p) + \varepsilon(L_p(n+p+3) - F_p(n+p+3) - p) + h(L_p(n+p+4) - F_p(n+p+4) - p)$
$= HL_p(n+p+1) - HF_p(n+p+1) - p(1+i+\varepsilon+h)$.
Theorem 2.4 Let $F_p(n)$ and $HF_p(n)$ be n^{th} Fibonacci p-numbers and hybrid Fibonacci p-numbers respectively and $HLe_p(n)$ be n^{th} hybrid Leonardo p-numbers. Then $HLe_p(k+n) = (p+1)(F_p(k)HF_p(n+2) + \sum_{r=1}^{p} F_p(k-r)HF_p(n+r+1-p)) - p(1+i+\varepsilon+h)$ where k and n are positive integers.
Proof:
$HLe_p(k+n) = (p+1)HF_p(k+n+1) - p(1+i+\varepsilon+h)$
$= (p+1)(F_p(k)HF_p(n+2) + \sum_{r=1}^{p} F_p(k-r)HF_p(n+r+1-p)) - p(1+i+\varepsilon+h)$.
Theorem 2.5 For $n \geq 0$, $\sum_{k=0}^{n} HLe_p(k) = (p+1)\sum_{k=1}^{n+1} HF_p(k) - (n+1)p(1+i+\varepsilon+h)$. **Proof:**
$\sum_{k=0}^{n} HLe_p(k) = \sum_{k=0}^{n}[(p+1)HF_p(k+1) - p(1+i+\varepsilon+h)]$
$= (p+1)\sum_{k=0}^{n} HF_p(k+1) - \sum_{k=0}^{n} p(1+i+\varepsilon+h)$
$= (p+1)\sum_{k=1}^{n+1} HF_p(k) - (n+1)p(1+i+\varepsilon+h)$.
Theorem 2.6 For $n \geq 0$, $\sum_{k=0}^{n} HLe_p(k) = \sum_{k=1}^{n+1} HL_p(k+p) - \sum_{k=1}^{n+1} HF_p(k+p) - (n+1)p(1+i+\varepsilon+h)$.
Proof:
$\sum_{k=0}^{n} HLe_p(k) = \sum_{k=0}^{n}[HL_p(k+p+1) - HF_p(k+p+1) - p(1+i+\varepsilon+h)]$
$= \sum_{k=0}^{n} HL_p(k+p+1) - \sum_{k=0}^{n} HF_p(k+p+1) - (n+1)p(1+i+\varepsilon+h)$
$= \sum_{k=1}^{n+1} HL_p(k+p) - \sum_{k=1}^{n+1} HF_p(k+p) - (n+1)p(1+i+\varepsilon+h)$.

Theorem 2.7 The generating function for the hybrid Leonardo p-numbers is $g(t) = (p+1)\sum_{n=0}^{\infty} HF_p(n+1)t^n - p(1+i+\varepsilon+h)\sum_{n=0}^{\infty} t^n$.
Proof:
$g(t) = \sum_{n=0}^{\infty} HLe_p(n)t^n$
$= \sum_{n=0}^{\infty}[(p+1)HF_p(n+1) - p(1+i+\varepsilon+h)]t^n$
$= (p+1)\sum_{n=0}^{\infty} HF_p(n+1)t^n - p(1+i+\varepsilon+h)\sum_{n=0}^{\infty} t^n$.
Theorem 2.8 The Binet formula for the hybrid Leonardo p-numbers is $HLe_p(n) = (p+1)[\sum_{k=1}^{p+1} \frac{\alpha_k^{n+1}}{(p+1)\alpha_k - p} + i\sum_{k=1}^{p+1} \frac{\alpha_k^{n+2}}{(p+1)\alpha_k - p} + \varepsilon\sum_{k=1}^{p+1} \frac{\alpha_k^{n+3}}{(p+1)\alpha_k - p} + h\sum_{k=1}^{p+1} \frac{\alpha_k^{n+4}}{(p+1)\alpha_k - p} - p(1+i+\varepsilon+h)$
where α_k are the distinct roots of the polynomial $x^{p+1} - x^p - 1$.
Proof:
$HLe_p(n) = (p+1)HF_p(n+1) - p(1+i+\varepsilon+h)$
$= (p+1)(F_p(n+1) + iF_p(n+2) + \varepsilon F_p(n+3) + hF_p(n+4)) - p(1+i+\varepsilon+h)$
$= (p+1)[\sum_{k=1}^{p+1} \frac{\alpha_k^{n+1}}{(p+1)\alpha_k - p} + i\sum_{k=1}^{p+1} \frac{\alpha_k^{n+2}}{(p+1)\alpha_k - p} + \varepsilon\sum_{k=1}^{p+1} \frac{\alpha_k^{n+3}}{(p+1)\alpha_k - p} + h\sum_{k=1}^{p+1} \frac{\alpha_k^{n+4}}{(p+1)\alpha_k - p} - p(1+i+\varepsilon+h)$
where α_k are the distinct roots of the polynomial $x^{p+1} - x^p - 1$.
Theorem 2.9 Let $n, m \geq 0$ the Honsberger identity for the hybrid Leonardo p-numbers is given by $HLe_p(n)HLe_p(m) + HLe_p(n+1)HLe_p(m+1) = (p+1)^2(HF_p(n+1)HF_p(m+1) + HF_p(n+2)HF_p(m+2)) - p(p+1)A(HF_p(n+1) + HF_p(m+1) + HF_p(n+2) + HF_p(m+2)) + 2p^2A^2$ where $A = 1+i+\varepsilon+h$.
Proof:
$HLe_p(n)HLe_p(m) + HLe_p(n+1)HLe_p(m+1) = [(p+1)HF_p(n+1) - p(1+i+\varepsilon+h)][(p+1)HF_p(m+1) - p(1+i+\varepsilon+h)] + [(p+1)HF_p(n+2) - p(1+i+\varepsilon+h)][(p+1)HF_p(m+2) - p(1+i+\varepsilon+h)]$
$= [(p+1)HF_p(n+1) - pA][(p+1)HF_p(m+1) - pA] + [(p+1)HF_p(n+2) - pA][(p+1)HF_p(m+2) - pA]$
$= (p+1)^2(HF_p(n+1)HF_p(m+1) + HF_p(n+2)HF_p(m+2)) - p(p+1)A(HF_p(n+1) + HF_p(m+1) + HF_p(n+2) + HF_p(m+2)) + 2p^2A^2$
where $A = 1+i+\varepsilon+h$.

Theorem 2.10 Let $m, n \geq 0$ the D'Ocagne's identity for the hybrid Leonardo p-numbers is given by $HLe_p(m)HLe_p(n+1) - HLe_p(m+1)HLe_p(n) = (p+1)^2(HF_p(m+1)HF_p(n+2) - HF_p(m+2)HF_p(n+1)) + p(p+1)A(HF_p(m+2) + HF_p(n+1) - HF_p(m+1) - HF_p(n+2))$ where $A = 1 + i + \varepsilon + h$.
Proof:
$HLe_p(m)HLe_p(n+1) - HLe_p(m+1)HLe_p(n) = [(p+1)HF_p(m+1) - p(1+i+\varepsilon+h)][(p+1)HF_p(n+2) - p(1+i+\varepsilon+h)] + [(p+1)HF_p(m+2) - p(1+i+\varepsilon+h)][(p+1)HF_p(n+1) - p(1+i+\varepsilon+h)]$
$= [(p+1)HF_p(m+1) - pA][(p+1)HF_p(n+2) - pA] + [(p+1)HF_p(m+2) - pA][(p+1)HF_p(n+1) - pA]$
$= (p+1)^2(HF_p(m+1)HF_p(n+2) - HF_p(m+2)HF_p(n+1)) + p(p+1)A(HF_p(m+2) + HF_p(n+1) - HF_p(m+1) - HF_p(n+2))$
where $A = 1 + i + \varepsilon + h$.
Theorem 2.11 For $n \geq 1$, Cassini's identity for the hybrid Leonardo p-numbers is given by $HLe_p^2(n) - HLe_p(n-1)HLe_p(n+1) = (p+1)^2(HF_p^2(n+1) - HF_p(n)HF_p(n+2)) + p(p+1)A(HF_p(n) + HF_p(n+2) - 2HF_p(n+1))$ where $A = 1 + i + \varepsilon + h$.
Proof:
$HLe_p^2(n) - HLe_p(n-1)HLe_p(n+1) = [(p+1)HF_p(n+1) - p(1+i+\varepsilon+h)]^2 - [(p+1)HF_p(n) - p(1+i+\varepsilon+h)][(p+1)HF_p(n+2) - p(1+i+\varepsilon+h)]$
$= [(p+1)HF_p(n+1) - pA]^2 - [(p+1)HF_p(n) - pA][(p+1)HF_p(n+2) - pA]$
$= (p+1)^2(HF_p^2(n+1) - HF_p(n)HF_p(n+2)) + p(p+1)A(HF_p(n) + HF_p(n+2) - 2HF_p(n+1))$
where $A = 1 + i + \varepsilon + h$.
Theorem 2.12 For $n \geq 1$, Catalan's identity for the hybrid Leonardo p-numbers is given by $HLe_p^2(n) - HLe_p(n-r)HLe_p(n+r) = (p+1)^2(HF_p^2(n+1) - HF_p(n-r+1)HF_p(n+r+1)) + p(p+1)A(HF_p(n-r+1) + HF_p(n+r+1) - 2HF_p(n+1))$ where $A = 1 + i + \varepsilon + h$.
Proof:
$HLe_p^2(n) - HLe_p(n-r)HLe_p(n+r) = [(p+1)HF_p(n+1) - p(1+i+\varepsilon+h)]^2 - [(p+1)HF_p(n-r+1) - p(1+i+\varepsilon+h)][(p+1)HF_p(n+r+1) - p(1+i+\varepsilon+h)]$
$= [(p+1)HF_p(n+1) - pA]^2 - [(p+1)HF_p(n-r+1) - pA][(p+1)HF_p(n+r+1) - pA]$
$= (p+1)^2(HF_p^2(n+1) - HF_p(n-r+1)HF_p(n+r+1)) + p(p+1)A(HF_p(n-r+1) + HF_p(n+r+1) - 2HF_p(n+1))$
where $A = 1 + i + \varepsilon + h$.

3 Conclusion

In this paper, we introduced hybrid Leonardo p-numbers. We established the properties of hybrid Leonardo p-numbers in terms of hybrid Fibonacci p-numbers. We also established the generating function, Binet's formula, Cassini identity, Catalan's identity, d'ocagne type identity, Honsberger identity for the hybrid Leonardo p-numbers. I hope that these results will be useful in number theory, Kinematics and differential equations like dual complex k-Fibonacci numbers [9], dual complex Fibonacci p-numbers [10], dual-complex numbers and their holomorphic functions [8].

References

[1] E. G. Kocer, H. Alsan, *Generalized Hybrid Fibonacci and Lucas p-numbers*, Indian journal of Pure and Applied Mathematics, **53** , 948–955, (2022).

[2] Y. Alp, E. G. Kocer, *Some properties of Leonardo numbers*, Konuralp J Math, **9(1)**, 183–189,(2021).

[3] Y. Alp, E. G. Kocer, *Hybrid Leonardo numbers*. Chaos, Solitons and Fractals, **150**, 111128, (2021).

[4] A. Szynal-Lina, I. Wloch, *Introduction to Fibonacci and Lucas hybrinomials*, Complex var. and Elliptic Equ., **65(10)** , 1736–1747, (2020).

[5] M. Ozdemir, *Introduction to hybrid numbers*, Adv. Appl. Clifford Algebras, **28(11)**, (2018).

[6] E. Kocer EG et al, *On the m extension of the Fibonacci and Lacus p-numbers*, Chaos, Solitons and Fractals, **40(4)** , 1890–1906,(2009).

[7] E. Tan, H. Leung, *On Leonardo p-numbers,* **23**, *#A7, (2023).*

[8] F. Messelmi, Dual complex numbers and their holomorphic functions*,* **Working paper or preprint: URL https:// hal.archives-ouvertes.fr/hal-01114178***, (2015).*

[9] F. T. Aydin, Dual Complex k-Fibonacci numbers,*Chaos, Solitons and Fractals,* **115***, 1–6, (2018).*

[10] B. Prasad, Dual Complex Fibonacci p-numbers*, Chaos, Solitons and Fractals,* **145***, 109922, (2021).*

[11] K. Kuhapatanakul, J. Chobsorn, On the generalized Leonardo numbers*, Integers,* **22***,* #A48*,(2022).*

[12] B. D. Bitim, N. Topal, Binomial Sum Formulas From the Exponential Generating Functions of (p, q)-Fibonacci and (p, q)-Lucas Quaternions*, Palestine Journal of Mathematics,* **10(1)***, 279–289 , (2021).*

[13] E. Mehdi-Nezhad, A. M. Rahimi, A Note on Fibonacci Numbers and the Golden Ratio of Order k*, Palestine Journal of Mathematics,* **13(2)***, 51–59 , (2024).*

[14] E. Kilic, N. Omur, I. Akkus, Y. T. Ulutas, VARIOUS SUMS INCLUDING THE GENERALIZED FIBONACCI AND LUCAS NUMBERS*, Palestine Journal of Mathematics,* **4(2)** *, 319–326, (2015).*

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