A q-DIFFERENCE EQUATION APPROACH FOR GENERALIZED AL-SALAM-CARLITZ POLYNOMIALS

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Abstract. In this paper, we focus on using the q-difference equation approach to derive generating function, another generating function, an extension of the generating function, the Rogers formula and its extension, Mehler's formula, and a Srivastava-Agarwal type generating function for the polynomials $h_n(x, y, a, b|q)$. In addition, we establish connections between q-difference equations and a transformation formula. By assigning specific values to parameters in the polynomials identities for $h_n(x, y, a, b|q)$, another generating function, an extension of the generating function, the Rogers formula and its extension, Mehler's formula, and a Srivastava-Agarwal type generating function for the Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ are determined. Additionally, connections between q-difference equations and a transformational identity for the polynomials $U_n(x, y, a; q)$ are established.

1 Introduction

In this paper, we employ the same concepts and terminology as in [12], and we assume that $|q| < 1$. The q-shifted factorial is defined as [10, 12]

$$(a; q)_k = \begin{cases} 1, & \text{if } k = 0, \\ (1 - a)(1 - aq) \cdots (1 - aq^{k-1}), & \text{if } k = 1, 2, 3, \cdots. \end{cases}$$

We also define

$$(a; q)_\infty = \prod_{k=0}^\infty (1 - aq^k).$$

The following notation will be used for the multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n.$$
$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

The generalized basic hypergeometric series $_r\phi_s$ is defined by [1, 5, 22]

$$_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x) = {}_r\phi_s \left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; q, x \right)$$

$$= \sum_{k=0}^\infty \frac{(a_1; q)_k \cdots (a_r; q)_k}{(q; q)_k (b_1; q)_k \cdots (b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} x^k,$$

where $q \neq 0$ when $r > s + 1$. Note that

$$_{r+1}\phi_r \left(\begin{array}{c} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{array} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} x^n, \quad |x| < 1.$$

The q-binomial coefficients are given by [18]

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

The Cauchy identity is given by [12, 26]

$$\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} x^k = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |x| < 1.$$

The Cauchy polynomials $P_n(x, y)$ are defined by [20, 25, 27]

$$P_n(x, y) = \left\{ \begin{array}{ll} (x - y)(x - qy)(x - q^2 y) \cdots (x - q^{n-1} y), & \text{if } n > 0; \\ 1, & \text{if } n = 0, \end{array} \right.$$

which has the generating function [2, 3, 16, 19]:

$$\sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(xt; q)_\infty}, \quad |xt| < 1. \tag{1.1}$$

In 1983, Goulden and Jackson gave the following identity [4, 13]:

$$P_n(x, y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} y^k x^{n-k}. \tag{1.2}$$

In 1965, Al-Salam and Carlitz [6] defined the following polynomials:

$$u_n^{(a)}(x; q) = (-a)^n q^{\binom{n}{2}} \, {}_2\phi_1 \left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array} ; q, qx/a \right).$$

The polynomials $u_n^{(a)}(x; q)$ can be rewritten as

$$u_n^{(a)}(x; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} a^k P_{n-k}(x, 1).$$

In 2003, Chen *et al.* [8] proposed the homogeneous q-difference operator D_{xy}, which is appropriate for studying Cauchy polynomials as follows:

$$D_{xy}\{f(x, y)\} = \frac{f(x, q^{-1}y) - f(qx, y)}{x - q^{-1}y}.$$

The homogeneous q-shift operator was constructed by Chen *et al.* [8] based on the homogeneous q-difference operator D_{xy} as follows:

$$\mathbb{E}(D_{xy}) = \sum_{n=0}^{\infty} \frac{D_{xy}^n}{(q; q)_n}.$$

In 2010, the definition of Al-Salam-Carlitz polynomials was expanded by Chen *et al.* [11] as follows:

$$U_n(x, y, a; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} a^k P_{n-k}(x, y). \tag{1.3}$$

Based on the operator D_{xy}, Chen *et al.* [11] construct the following homogeneous q-shift operator:

$$\mathbb{F}(aD_{xy}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(aD_{xy})^n}{(q;q)_n},$$

and they gave the following identities for $U_n(x, y, a; q)$:

Theorem 1.1. [11]. *Let $U_n(x, y, a; q)$ be defined as in* (1.3), *then*

- *The generating function for $U_n(x, y, a; q)$ is*

$$\sum_{n=0}^{\infty} U_n(x, y, a; q) \frac{t^n}{(q;q)_n} = \frac{(at, yt; q)_\infty}{(xt; q)_\infty}, \quad |xt| < 1. \tag{1.4}$$

- *The Rogers-type formula for $U_n(x, y, a; q)$ is*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} U_{n+m}(x, y, a; q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}$$

$$= \frac{(as, ys; q)_\infty}{(xs; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(xs; q)_k (at)^k}{(q;q)_k (as, ys; q)_k} \, {}_2\phi_1\left(\begin{array}{c} y/x, 0 \\ ysq^k \end{array} ; q, xt \right), \tag{1.5}$$

provided that $\max\{|xs|, |xt|\} < 1$.

- *The Mehler's formula for $U_n(x, y, a; q)$ is*

$$\sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} U_n(x, y, a; q) U_n(u, v, b; q) \frac{t^n}{(q;q)_n}$$

$$= \frac{(abt, ybt, avt; q)_\infty}{(xbt, aut; q)_\infty} \, {}_3\phi_2\left(\begin{array}{c} y/x, v/u, q/abt \\ q/aut, q/xbt \end{array} ; q, q \right), \tag{1.6}$$

where $y/x = q^{-r}$ or $v/u = q^{-r}$ for a non-negative integer r and $\max\{|xtbq^{-r}|, |autq^{-r}|\} < 1$.

The q-differential operator is defined as [9, 21]:

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}.$$

In 2013, Saad and Sukhi [28] defined the q-exponential operator $R(bD_q)$ as follows:

$$R(bD_q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k}(bD_q)^k.$$

They also proposed an operator technique to deriving Mehler's formula for Cauchy polynomials $P_n(x, y)$.

$$\sum_{n=0}^{\infty} P_n(x, y) P_n(z, w) \frac{t^n}{(q;q)_n} = \frac{(xwt; q)_\infty}{(xzt; q)_\infty} \, {}_1\phi_1\left(\begin{array}{c} w|z \\ xwt \end{array} ; q, yzt \right), \quad |xzt| < 1. \tag{1.7}$$

In 2020, Arjika [7] introduced the homogeneous q-shift operator $\widetilde{E}(a, b; D_q)$ as follows:

$$\widetilde{E}(a, b; D_q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a; q)_k}{(q;q)_k}(bD_q)^k.$$

Arjika [7] discovered the solution to the q-difference equation and expressed it in q-operator form.

Theorem 1.2. [7]. *Let $f(a, x, y)$ be a three-variable analytic function in a neighbourhood of $(a, x, y) = (0, 0, 0) \in \mathbb{C}^3$. If $f(a, x, y)$ satisfies the q-difference equation*

$$x[f(a, x, y) - f(a, x, qy)] = y[f(a, qx, qy) - f(a, x, qy)]$$
$$- ay[f(a, qx, q^2y) - f(a, x, q^2y)].$$

Then

$$f(a, x, y) = \widetilde{E}(a, y; D_q)\{f(a, x, 0)\}.$$

In 2024, Saad and Arif [23] constructed the homogeneous q-shift operator $\widetilde{\mathbb{F}}(a, b; D_{xy})$ as follows:

$$\widetilde{\mathbb{F}}(a, b; D_{xy}) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a; q)_k}{(q; q)_k}(bD_{xy})^k. \tag{1.8}$$

Also, they defined a generalized Al-Salam-Carlitz polynomials $h_n(x, y, a, b|q)$ by

$$h_n(x, y, a, b|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}}(a; q)_k b^k P_{n-k}(x, y). \tag{1.9}$$

The operator $\widetilde{\mathbb{F}}(a, b; D_{xy})$ demonstrates suitability for responding with the polynomials $h_n(x, y, a, b|q)$:

$$\widetilde{\mathbb{F}}(a, b; D_{xy})\{P_n(x, y)\} = h_n(x, y, a, b|q). \tag{1.10}$$

- Setting $a = 0$ and then $b = a$, the polynomials $h_n(x, y, a, b|q)$ reduce to the Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$, which were investigated earlier by Chen *et al.* [11].

The paper is organized as follows: Two theorems regarding q-difference equations, which will be utilized in subsequent sections, are presented and proved in Section 2. In section 3, we employ the q-difference equations technique to prove generating function, another generating function, an extended generating function, the Rogers formula and its extension, and Mehler's formula for the polynomials $h_n(x, y, a, b|q)$. In Section 4, the q-difference equation approach is utilized to derive the Srivastava-Agarwal type generating function. Finally, in Section 5, using the q-difference equation technique, we establish a transformational identity involving the generating function for the polynomials.

2 q-Difference equations for the q-shift operator $\widetilde{\mathbb{F}}(a, b; D_{xy})$

In this section, we propose and prove two theorems concerning q-difference equations for the homogeneous q-shift operator $\widetilde{\mathbb{F}}(a, b; D_{xy})$ and for the generalized Al-Salam-Carlitz polynomials $h_n(x, y, a, b|q)$. These two theorems are used to prove some identities for the polynomials $h_n(x, y, a, b|q)$.

Lemma 2.1. [14]. *If a complex-valued function is holomorphic (analytic) in each variable separately in an open domain $D \subset \mathbb{C}^n$, then it is holomorphic (analytic) in D.*

Lemma 2.2. [17]. *If $f(x_1, x_2, ..., x_k)$ is analytic at the origin $(0, 0, ..., 0) \in \mathbb{C}^k$, then, f can be expanded in an absolutely convergent power series*

$$f(x_1, x_2, ..., x_k) = \sum_{n_1, n_2, ..., n_k=0}^{\infty} \alpha_{n_1, n_2, ..., n_k} x_1^{n_1} x_2^{n_2} ... x_k^{n_k}.$$

Theorem 2.3. *Let $f(x, y, a, b)$ be a four-variable analytic function in a neighbourhood of $(x, y, a, b) = (0, 0, 0, 0) \in \mathbb{C}^4$. If $f(x, y, a, b)$ satisfies the q-difference equation*

$$(x - q^{-1}y)[f(x, y, a, b) - f(x, y, a, qb)] = b[f(qx, y, a, qb) - f(x, q^{-1}y, a, qb)]$$

$$+ ab[f(x, q^{-1}y, a, q^2b) - f(qx, y, a, q^2b)]. \tag{2.1}$$

Then

$$f(x, y, a, b) = \widetilde{\mathbb{F}}(a, b; D_{xy})\{f(x, y, a, 0)\}. \tag{2.2}$$

Proof. From Lemmas 2.1 and 2.2, we assume that

$$f(x, y, a, b) = \sum_{k=0}^{\infty} A_k(x, y, a)b^k. \tag{2.3}$$

Substituting (2.3) into (2.1) yields

$$(x - q^{-1}y)\left[\sum_{k=0}^{\infty} A_k(x, y, a)b^k - \sum_{k=0}^{\infty} A_k(x, y, a)(qb)^k\right]$$

$$= b\left[\sum_{k=0}^{\infty} A_k(qx, y, a)(qb)^k - \sum_{k=0}^{\infty} A_k(x, q^{-1}y, a)(qb)^k\right]$$

$$+ ab\left[\sum_{k=0}^{\infty} A_k(x, q^{-1}y, a)(q^2b)^k - \sum_{k=0}^{\infty} A_k(qx, y, a)(q^2b)^k\right].$$

$$(x - q^{-1}y)\sum_{k=0}^{\infty}(1 - q^k)A_k(x, y, a)b^k$$

$$= b\left[\sum_{k=0}^{\infty}[A_k(qx, y, a) - A_k(x, q^{-1}y, a)](qb)^k\right]$$

$$- ab\left[\sum_{k=0}^{\infty}[A_k(qx, y, a) - A_k(x, q^{-1}y, a)](q^2b)^k\right]$$

$$= b\left[\sum_{k=0}^{\infty}(q^k - aq^{2k})\left[A_k(qx, y, a) - A_k(x, q^{-1}y, a)\right]b^k\right]$$

$$= b\left[\sum_{k=0}^{\infty}-(q^k - aq^{2k})\left[A_k(x, q^{-1}y, a) - A_k(qx, y, a)\right]b^k\right]$$

$$= \sum_{k=0}^{\infty}-q^k(1 - aq^k)[A_k(x, q^{-1}y, a) - A_k(qx, y, a)]b^{k+1}.$$

Equating the coefficients of b^k for both sides, we get

$$(x - q^{-1}y)(1 - q^k)A_k(x, y, a)$$
$$= -q^{k-1}(1 - q^{k-1}a)[A_{k-1}(x, q^{-1}y, a) - A_{k-1}(qx, y, a)].$$

Hence

$$A_k(x, y, a) = \frac{-q^{k-1}(1 - q^{k-1}a)}{(1 - q^k)}\frac{[A_{k-1}(x, q^{-1}y, a) - A_{k-1}(qx, y, a)]}{(x - q^{-1}y)}$$

$$= \frac{-q^{k-1}(1 - q^{k-1}a)}{(1 - q^k)}\ D_{xy}\{A_{k-1}(x, y, a)\}.$$

By iteration, we gain

$$A_k(x, y, a) = \frac{(-1)^k q^{\binom{k}{2}}(a; q)_k}{(q; q)_k}\ D_{xy}^k\{A_0(x, y, a)\}.$$

Note that $f(x, y, a, 0) = A_0(x, y, a)$. Hence

$$A_k(x, y, a) = \frac{(-1)^k q^{\binom{k}{2}}(a; q)_k}{(q; q)_k} \; D_{xy}^k \{f(x, y, a, 0)\}. \tag{2.4}$$

Substituting (2.4) into (2.3), we get

$$\begin{aligned}
f(x, y, a, b) &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a; q)_k}{(q; q)_k} \; D_{xy}^k \{f(x, y, a, 0)\} b^k \\
&= \widetilde{\mathbb{F}}(a, b; D_{xy}) \{f(x, y, a, 0)\}. \quad \text{(by using (1.8))}
\end{aligned}$$

$\square$

- Setting $a = 0$ in Theorem 2.3, we get the following result:

    Corollary 2.4. *Let* $f(x, y, b)$ *be a three-variable analytic function in a neighbourhood of* $(x, y, b) = (0, 0, 0) \in \mathbb{C}^3$. *If* $f(x, y, b)$ *satisfies the q-difference equation*

    $$(x - q^{-1}y)[f(x, y, b) - f(x, y, qb)] = b[f(qx, y, qb) - f(x, q^{-1}y, qb)].$$

    Then

    $$f(x, y, b) = \mathbb{F}(bD_{xy}) \{f(x, y, 0)\}.$$

Theorem 2.5. *In the neighbourhood of* $(0, 0, 0, 0) \in \mathbb{C}^4$, *if* $f(x, y, a, b)$ *is a four-variable analytic function verifying* (2.1) *and if* $f(x, y, a, 0)$ *possess the following expression:*

$$f(x, y, a, 0) = \sum_{n=0}^{\infty} \mu_n P_n(x, y), \tag{2.5}$$

where μ_n *is independent of* x *and* y. *Then*

$$f(x, y, a, b) = \sum_{n=0}^{\infty} \mu_n h_n(x, y, a, b|q). \tag{2.6}$$

Proof. Using equation (2.2) and substituting the expansion from (2.5), we obtain

$$\begin{aligned}
f(x, y, a, b) &= \widetilde{\mathbb{F}}(a, b; D_{xy}) \{f(x, y, a, 0)\} \\
&= \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \sum_{n=0}^{\infty} \mu_n P_n(x, y) \right\} \\
&= \sum_{n=0}^{\infty} \mu_n \widetilde{\mathbb{F}}(a, b; D_{xy}) \{P_n(x, y)\} \\
&= \sum_{n=0}^{\infty} \mu_n h_n(x, y, a, b|q). \quad \text{(by using (1.10))}
\end{aligned}$$

$\square$

3 The q-difference equations and their applications for $h_n(x, y, a, b|q)$

This section focuses on using the q-difference equations technique to verify some identities for the polynomials $h_n(x, y, a, b|q)$.

3.1 Generating functions for $h_n(x, y, a, b|q)$

In this subsection, we will illustrate that the generating function, another generating function and the extended generating function for $h_n(x, y, a, b|q)$ are a direct application of Theorem 2.3.

Theorem 3.1 (Generating function for $h_n(x, y, a, b|q)$). *For $|xs| < 1$, we have*

$$\sum_{n=0}^{\infty} h_n(x, y, a, b|q) \frac{s^n}{(q;q)_n} = \frac{(ys;q)_\infty}{(xs;q)_\infty} \, {}_1\phi_1 \left(\begin{array}{c} a \\ 0 \end{array} ; q, bs \right). \tag{3.1}$$

Proof. Assume that

$$f(x, y, a, b) = \text{the right-hand side of (3.1)}$$

$$= \frac{(ys;q)_\infty}{(xs;q)_\infty} \, {}_1\phi_1 \left(\begin{array}{c} a \\ 0 \end{array} ; q, bs \right).$$

Now, we explain that $f(x, y, a, b)$ satisfies equation (2.1).

$$(x - q^{-1}y)[f(x, y, a, b) - f(x, y, a, qb)]$$

$$= (x - q^{-1}y) \left[\frac{(ys;q)_\infty}{(xs;q)_\infty} \, {}_1\phi_1 \left(\begin{array}{c} a \\ 0 \end{array} ; q, bs \right) - \frac{(ys;q)_\infty}{(xs;q)_\infty} \, {}_1\phi_1 \left(\begin{array}{c} a \\ 0 \end{array} ; q, qbs \right) \right]$$

$$= (x - q^{-1}y) \frac{(ys;q)_\infty}{(xs;q)_\infty} \left[\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q)_k}{(q;q)_k}(bs)^k - \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q)_k}{(q;q)_k}(qbs)^k \right]$$

$$= (x - q^{-1}y) \frac{(ys;q)_\infty}{(xs;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q)_k}{(q;q)_k}(bs)^k(1 - q^k)$$

$$= (-bs)(x - q^{-1}y) \frac{(ys;q)_\infty}{(xs;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q)_k(bs)^k}{(q;q)_k}(q^k - aq^{2k})$$

$$= b \left[\frac{(ys;q)_\infty}{(qxs;q)_\infty} - \frac{(q^{-1}ys;q)_\infty}{(xs;q)_\infty} \right] \left[\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q)_k(qbs)^k}{(q;q)_k} - a\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q)_k(q^2bs)^k}{(q;q)_k} \right]$$

$$= b \left[\frac{(ys;q)_\infty}{(qxs;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q)_k}{(q;q)_k}(qbs)^k - \frac{(q^{-1}ys;q)_\infty}{(xs;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q)_k}{(q;q)_k}(qbs)^k \right]$$

$$+ ab \left[\frac{(q^{-1}ys;q)_\infty}{(xs;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q)_k}{(q;q)_k}(q^2bs)^k - \frac{(ys;q)_\infty}{(qxs;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}(a;q)_k}{(q;q)_k}(q^2bs)^k \right]$$

$$= b[f(qx, y, a, qb) - f(x, q^{-1}y, a, qb)] + ab[f(x, q^{-1}y, a, q^2b) - f(qx, y, a, q^2b)].$$

Hence equation (2.1) is satisfied. Using equation (2.2), we have

$$f(x, y, a, b) = \widetilde{\mathbb{F}}(a, b; D_{xy})\{f(x, y, a, 0)\}$$

$$= \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \frac{(ys;q)_\infty}{(xs;q)_\infty} \right\}$$

$$= \sum_{n=0}^{\infty} \widetilde{\mathbb{F}}(a, b; D_{xy})\{P_n(x, y)\} \frac{s^n}{(q;q)_n} \quad \text{(by using (1.1))}$$

$$= \sum_{n=0}^{\infty} h_n(x, y, a, b|q) \frac{s^n}{(q;q)_n}. \quad \text{(by using (1.10))}$$

Hence the proof of the Theorem 3.1 is completed. □

Theorem 3.2 (Another generating function for $h_n(x, y, a, b|q)$). *For $|xs| < 1$, we have*

$$\sum_{n=0}^{\infty} h_n(x, y, a, b|q)(r/s; q)_n \frac{s^n}{(q; q)_n}$$

$$= \frac{(ys; q)_\infty}{(xs; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} r^k P_k(x, y)}{(q, ys; q)_k} \sum_{i=0}^{\infty} \frac{q^{2\binom{i}{2}+ki}(xs, a; q)_i (br)^i}{(q, q^k ys; q)_i} {}_1\phi_1 \left(\begin{array}{c} aq^i \\ 0 \end{array} ; q, bsq^i \right). \quad (3.2)$$

Proof. Assume that

$$f(x, y, a, b) = \text{the right-hand side of (3.2)}$$

$$= \frac{(ys; q)_\infty}{(xs; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} r^k P_k(x, y)}{(q, ys; q)_k}$$

$$\times \sum_{i=0}^{\infty} \frac{q^{2\binom{i}{2}+ki}(xs, a; q)_i (br)^i}{(q, q^k ys; q)_i} {}_1\phi_1 \left(\begin{array}{c} aq^i \\ 0 \end{array} ; q, bsq^i \right).$$

By the same technique used in Theorem 3.1, we can check that $f(x, y, a, b)$ satisfies equation (2.1). Using equation (2.2), w get

$$f(x, y, a, b) = \widetilde{\mathbb{F}}(a, b; D_{xy}) \{f(x, y, a, 0)\}$$

$$= \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \frac{(ys; q)_\infty}{(xs; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} r^k P_k(x, y)}{(q, ys; q)_k} \right\}$$

$$= \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} r^k P_k(x, y)}{(q; q)_k} \frac{(q^k ys; q)_\infty}{(xs; q)_\infty} \right\}$$

$$= \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} r^k P_k(x, y)}{(q; q)_k} \sum_{n=0}^{\infty} P_n(x, q^k y) \frac{s^n}{(q; q)_n} \right\} \quad \text{(by using (1.1))}$$

$$= \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} r^k s^n}{(q; q)_k (q; q)_n} P_{n+k}(x, y) \right\}$$

$$= \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} r^k s^{n-k}}{(q; q)_k (q; q)_{n-k}} P_n(x, y) \right\}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k}{2}} r^k s^{n-k}}{(q; q)_n} \widetilde{\mathbb{F}}(a, b; D_{xy}) \{P_n(x, y)\}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k}{2}} r^k s^{n-k}}{(q; q)_n} h_n(x, y, a, b|q) \quad \text{(by using (1.10))}$$

$$= \sum_{n=0}^{\infty} \frac{h_n(x, y, a, b|q)}{(q; q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} r^k s^{n-k}$$

$$= \sum_{n=0}^{\infty} h_n(x, y, a, b|q) \frac{P_n(s, r)}{(q; q)_n} \quad \text{(by using (1.2))}$$

$$= \sum_{n=0}^{\infty} h_n(x, y, a, b|q)(r/s; q)_n \frac{s^n}{(q; q)_n}.$$

Hence the proof of the Theorem 3.2 is completed.   □

- Setting $r = 0$ in equation (3.2), we get the generating function for $h_n(x, y, a, b|q)$ (equation (3.1)).

- Setting $a = 0$ and then $b = a$ in equation (3.2), we get the following result:

Corollary 3.3. *For $|xs| < 1$, we have:*

$$\sum_{n=0}^{\infty} U_n(x,y,a;q)(r/s;q)_n \frac{s^n}{(q;q)_n} = \frac{(ys,as;q)_\infty}{(xs;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} r^k P_k(x,y)}{(ys,q;q)_k}$$

$$\times \sum_{i=0}^{\infty} \frac{q^{2\binom{i}{2}+ki}(xs;q)_i (ar)^i}{(q^k ys, as, q;q)_i}. \tag{3.3}$$

- When $r = 0$ in equation (3.3), we get the generating function for $U_n(x,y,a|q)$ (equation (1.4)).

Theorem 3.4 (Extended generating function for $h_n(x,y,a,b|q)$). *For $|xt| < 1$, we have*

$$\sum_{n=0}^{\infty} h_{n+k}(x,y,a,b|q) \frac{t^n}{(q;q)_n} = \frac{(ytq^k;q)_\infty}{(xt;q)_\infty} \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} (-1)^i q^{\binom{i}{2}} (xt;q)_i (a;q)_i b^i P_{k-i}(x,y)$$

$$\times \; {}_1\phi_1 \left(\begin{array}{c} aq^i \\ 0 \end{array} ; q, btq^i \right). \tag{3.4}$$

Proof. Denoting the right-hand side of equation (3.4) by $f(x,y,a,b)$, we have:

$$f(x,y,a,b)$$

$$= \frac{(ytq^k;q)_\infty}{(xt;q)_\infty} \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} (-1)^i q^{\binom{i}{2}} (xt;q)_i (a;q)_i b^i P_{k-i}(x,y) \; {}_1\phi_1 \left(\begin{array}{c} aq^i \\ 0 \end{array} ; q, btq^i \right).$$

By the same way used in Theorem 3.1, we can check that $f(x,y,a,b)$ satisfies equation (2.1). Using equation (2.2), w find

$$f(x,y,a,b) = \widetilde{\mathbb{F}}(a,b;D_{xy})\{f(x,y,a,0)\}$$

$$= \widetilde{\mathbb{F}}(a,b;D_{xy}) \left\{ \frac{(ytq^k;q)_\infty}{(xt;q)_\infty} P_k(x,y) \right\}$$

$$= \widetilde{\mathbb{F}}(a,b;D_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x,q^k y) \frac{t^n}{(q;q)_n} P_k(x,y) \right\} \quad \text{(by using (1.1))}$$

$$= \widetilde{\mathbb{F}}(a,b;D_{xy}) \left\{ \sum_{n=0}^{\infty} P_{n+k}(x,y) \frac{t^n}{(q;q)_n} \right\}$$

$$= \sum_{n=0}^{\infty} \widetilde{\mathbb{F}}(a,b;D_{xy}) \left\{ P_{n+k}(x,y) \right\} \frac{t^n}{(q;q)_n} \quad \text{(by using (1.10))}$$

$$= \sum_{n=0}^{\infty} h_{n+k}(x,y,a,b|q) \frac{t^n}{(q;q)_n},$$

which is the left-hand side (3.4). $\qquad \square$

- Setting $a = 0$ and then $b = a$ in equation (3.4), we get the following extension of the generating function for the Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$.

Corollary 3.5.

$$\sum_{n=0}^{\infty} U_{n+k}(x,y,a;q) \frac{t^n}{(q;q)_n} = \frac{(ytq^k, at;q)_\infty}{(xt;q)_\infty} \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} (-1)^i q^{\binom{i}{2}} \frac{(xt;q)_i}{(at;q)_i} a^i P_{k-i}(x,y).$$

3.2 The Rogers formula for $h_n(x, y, a, b|q)$

The Rogers formula and its extension for the generalized Al-Salam-Carlitz polynomials $h_n(x, y, a, b|q)$ are obtained in this section, and their proofs are given using homogeneous q-difference equations.

Theorem 3.6 (Rogers formula for $h_n(x, y, a, b|q)$). *For* $\max\{|xs|, |xt|\} < 1$, *we have*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x, y, a, b|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}$$

$$= \frac{(ys;q)_\infty}{(xs;q)_\infty} \sum_{i=0}^{\infty} \frac{(-1)^i q^{\binom{i}{2}} (a, xs;q)_i (bt)^i}{(q, ys;q)_i} \, {}_1\phi_1 \left(\begin{array}{c} aq^i \\ 0 \end{array} ; q, bsq^i \right) {}_2\phi_1 \left(\begin{array}{c} y/x, 0 \\ ysq^i \end{array} ; q, xt \right). \quad (3.5)$$

Proof. Let $R(x, y, s, a, b) =$ the right-hand side of equation (3.5).

$$R(x, y, s, a, b) = \frac{(ys;q)_\infty}{(xs;q)_\infty} \sum_{i=0}^{\infty} \frac{(-1)^i q^{\binom{i}{2}} (a, xs;q)_i (bt)^i}{(q, ys;q)_i}$$

$$\times \, {}_1\phi_1 \left(\begin{array}{c} aq^i \\ 0 \end{array} ; q, bsq^i \right) {}_2\phi_1 \left(\begin{array}{c} y/x, 0 \\ ysq^i \end{array} ; q, xt \right).$$

We can see that $R(x, y, s, a, b)$ satisfies (2.1), we have

$$R(x, y, s, a, b) = \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ R(x, y, s, a, 0) \right\}$$

$$= \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \frac{(ys;q)_\infty}{(xs;q)_\infty} \, {}_2\phi_1 \left(\begin{array}{c} y/x, 0 \\ ys \end{array} ; q, xt \right) \right\}$$

$$= \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q;q)_n} \frac{(ysq^n;q)_\infty}{(xs;q)_\infty} \right\}$$

$$= \sum_{n=0}^{\infty} \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ P_n(x, y) \frac{(ysq^n;q)_\infty}{(xs;q)_\infty} \right\} \frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ P_n(x, y) \sum_{m=0}^{\infty} P_m(x, q^n y) \frac{s^m}{(q;q)_m} \right\} \frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \sum_{m=0}^{\infty} P_{n+m}(x, y) \frac{s^m}{(q;q)_m} \right\} \frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ P_{n+m}(x, y) \right\} \frac{t^n s^m}{(q;q)_n (q;q)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x, y, a, b|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}$$

$$= \text{left-hand side of (3.5)}.$$

$$\square$$

- Setting $a = 0$ and then $b = a$ in equation (3.5), we get Rogers formula for the polynomials $U_n(x, y, a; q)$ (equation (1.5)).

Theorem 3.7 (Extended Rogers formula for $h_n(x, y, a, b|q)$). *For* $|xw| < 1$, *we have:*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} h_{n+m+k}(x, y, a, b|q) \frac{t^n s^m w^k}{(q;q)_n (q;q)_m (q;q)_k}$$

$$= \frac{(yw;q)_\infty}{(xw;q)_\infty} \sum_{i=0}^\infty \frac{(-1)^i q^{\binom{i}{2}}(a,xw;q)_i (bt)^i}{(q,yw;q)_i} \sum_{j=0}^\infty \frac{(-1)^j q^{\binom{j}{2}}(aq^i,xwq^i;q)_j (bsq^i)^j}{(q,ywq^i;q)_j}$$

$$\times \sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k}{2}}(aq^{i+j};q)_k (bwq^{i+j})^k}{(q;q)_k} \sum_{n=0}^\infty P_n(x,y) \frac{t^n}{(q,ywq^{i+j};q)_n}$$

$$\times \sum_{m=0}^\infty P_m(x,q^n y) \frac{(sq^i)^m}{(q,ywq^{i+j+n};q)_m}. \tag{3.6}$$

Proof. Let

$E(x,y,a,b) =$ the right-hand side of equation (3.6)

$$= \frac{(yw;q)_\infty}{(xw;q)_\infty} \sum_{i=0}^\infty \frac{(-1)^i q^{\binom{i}{2}}(a,xw;q)_i (bt)^i}{(q,yw;q)_i} \sum_{j=0}^\infty \frac{(-1)^j q^{\binom{j}{2}}(aq^i,xwq^i;q)_j (bsq^i)^j}{(q,ywq^i;q)_j}$$

$$\times \sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k}{2}}(aq^{i+j};q)_k (bwq^{i+j})^k}{(q;q)_k} \sum_{n=0}^\infty P_n(x,y) \frac{t^n}{(q,ywq^{i+j};q)_n}$$

$$\times \sum_{m=0}^\infty P_m(x,q^n y) \frac{(sq^i)^m}{(q,ywq^{i+j+n};q)_m}.$$

We can check that $E(x,y,a,b)$ satisfies equation (2.1), we have

$$E(x,y,a,b) = \widetilde{\mathbb{F}}(a,b;D_{xy})\{E(x,y,a,0)\}$$

$$= \widetilde{\mathbb{F}}(a,b;D_{xy}) \left\{ \frac{(yw;q)_\infty}{(xw;q)_\infty} \sum_{n=0}^\infty \sum_{m=0}^\infty P_{n+m}(x,y) \frac{t^n s^m}{(q;q)_n (q;q)_m (yw;q)_{n+m}} \right\}$$

$$= \sum_{n=0}^\infty \sum_{m=0}^\infty \widetilde{\mathbb{F}}(a,b;D_{xy}) \left\{ P_{n+m}(x,y) \frac{(ywq^{n+m};q)_\infty}{(xw;q)_\infty} \right\} \frac{t^n s^m}{(q;q)_n (q;q)_m}$$

$$= \sum_{n=0}^\infty \sum_{m=0}^\infty \widetilde{\mathbb{F}}(a,b;D_{xy}) \left\{ P_{n+m}(x,y) \sum_{k=0}^\infty P_k(x,q^{n+m}y) \frac{w^k}{(q;q)_k} \right\} \frac{t^n s^m}{(q;q)_n (q;q)_m}$$

$$= \sum_{n=0}^\infty \sum_{m=0}^\infty \widetilde{\mathbb{F}}(a,b;D_{xy}) \left\{ \sum_{k=0}^\infty P_{n+m+k}(x,y) \frac{w^k}{(q;q)_k} \right\} \frac{t^n s^m}{(q;q)_n (q;q)_m}$$

$$= \sum_{n=0}^\infty \sum_{m=0}^\infty \sum_{k=0}^\infty \widetilde{\mathbb{F}}(a,b;D_{xy}) \{P_{n+m+k}(x,y)\} \frac{t^n s^m w^k}{(q;q)_n (q;q)_m (q;q)_k}$$

$$= \sum_{n=0}^\infty \sum_{m=0}^\infty \sum_{k=0}^\infty h_{n+m+k}(x,y,a,b|q) \frac{t^n s^m w^k}{(q;q)_n (q;q)_m (q;q)_k}$$

$= $ left-hand side of (3.6).

- Setting $s = 0$ and then $w = s$ in equation (3.6), we get equation (3.5).
- Setting $a = 0$ and then $b = a$ in equation (3.6), we get the following result:

Corollary 3.8. *For $|xw| < 1$, we have:*

$$\sum_{n=0}^\infty \sum_{m=0}^\infty \sum_{k=0}^\infty U_{n+m+k}(x,y,a;q) \frac{t^n s^m w^k}{(q;q)_n (q;q)_m (q;q)_k}$$

$$= \frac{(aw,yw;q)_\infty}{(xw;q)_\infty} \sum_{i=0}^\infty \frac{(-1)^i q^{\binom{i}{2}}(xw;q)_i (at)^i}{(q,yw,aw;q)_i} \sum_{j=0}^\infty \frac{(-1)^j q^{\binom{j}{2}}(xwq^i;q)_j (asq^i)^j}{(q,ywq^i,awq^i;q)_j}$$

$$\times \sum_{n=0}^\infty P_n(x,y) \frac{t^n}{(q,ywq^{i+j};q)_n} \sum_{m=0}^\infty P_m(x,q^n y) \frac{(sq^i)^m}{(q,ywq^{i+j+n};q)_m}.$$

$\square$

3.3   Mehler's formula for $h_n(x, y, a, b|q)$

The goal of this section is to illustrate how Theorem 2.3 and Theorem 2.5 work together to yield Mehler's formula for $h_n(x, y, a, b|q)$.

Theorem 3.9 (Mehler's formula for $h_n(x, y, a, b|q)$). *Let the polynomials $h_n(x, y, a, b|q)$ be defined as in* (1.9)*, then*

$$\sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} h_n(x, y, a, b|q) h_n(u, v, c, d|q) \frac{t^n}{(q;q)_n}$$

$$= \frac{(ydt;q)_\infty}{(xdt;q)_\infty} \sum_{n=0}^{\infty} \frac{(y/x, v/u;q)_n (uq/d)^n}{(q, q/xdt;q)_n} \sum_{i=0}^{\infty} P_i(u, vq^n) \frac{(a;q)_i (tbq^{-n})^i}{(q;q)_i}$$

$$\times \sum_{k=0}^{\infty} \frac{(xdtq^{-n}, aq^i;q)_k (bcdtq^{-n})^k q^{k^2-k}}{(q, ydt;q)_k} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (yq^n/x;q)_j (cxdtq^{-n+k})^j}{(q, ydtq^k;q)_j}$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (aq^{i+k};q)_m (bdtq^{-n+k})^m}{(q;q)_m}, \tag{3.7}$$

where $y/x = q^{-r}$ or $v/u = q^{-r}$ for a non-negative integer r and $|xdtq^{-r}| < 1$.

Proof. Let

$f(x, y, a, b) = $ the right-hand side of equation (3.7)

$$= \frac{(ydt;q)_\infty}{(xdt;q)_\infty} \sum_{n=0}^{\infty} \frac{(y/x, v/u;q)_n (uq/d)^n}{(q, q/xdt;q)_n} \sum_{i=0}^{\infty} P_i(u, vq^n) \frac{(a;q)_i (tbq^{-n})^i}{(q;q)_i}$$

$$\times \sum_{k=0}^{\infty} \frac{(xdtq^{-n}, aq^i;q)_k (bcdtq^{-n})^k q^{k^2-k}}{(q, ydt;q)_k} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (yq^n/x;q)_j (cxdtq^{-n+k})^j}{(q, ydtq^k;q)_j}$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (aq^{i+k};q)_m (bdtq^{-n+k})^m}{(q;q)_m}.$$

We can check that $f(x, y, a, b)$ satisfies equation (2.1), by the same method used in Theorem 3.1. Using equation (2.2), w get

$$f(x, y, a, 0) = \frac{(ydt;q)_\infty}{(xdt;q)_\infty} \sum_{n=0}^{\infty} \frac{(y/x, v/u;q)_n (uq/d)^n}{(q, q/xdt;q)_n} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (yq^n/x;q)_j (cxdtq^{-n})^j}{(q, ydt;q)_j}$$

$$= \frac{(ydt;q)_\infty}{(xdt;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} (y/x, v/u;q)_n (xut)^n (-q/xdt)^n q^{\binom{n}{2}}}{(q, q/xdt;q)_n}$$

$$\times \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (yq^n/x;q)_j (cxdtq^{-n})^j}{(q, ydt;q)_j}$$

$$= \frac{(ydt;q)_\infty}{(xdt;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} (y/x, v/u;q)_n (xut)^n (xdt;q)_{-n}}{(q;q)_n}$$

$$\times \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (yq^n/x;q)_j (cxdtq^{-n})^j}{(q, ydt;q)_j}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} P_n(x, y) P_n(u, v) t^n}{(q;q)_n} \frac{(ydt;q)_\infty}{(q^{-n}xdt;q)_\infty}$$

$$\times \ _1\phi_1 \left(\begin{array}{c} q^n y/x \\ ydt \end{array} ; q, cxdtq^{-n} \right). \tag{3.8}$$

If the condition $v/u = q^{-r}$ or $y/x = q^{-r}$ holds, it means that the sum in (3.8) is finite. Using equation (1.7), equation (3.8) is equal

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} P_n(x,y) P_n(u,v) t^n}{(q;q)_n} \sum_{k=0}^{\infty} P_k(1,c) P_k(x,q^n y) \frac{(dtq^{-n})^k}{(q;q)_n} \tag{3.9}$$

We point out that when n goes to infinity, the second sum in (3.9) does not converge. We can focus on the situation where $v/u = q^{-r}$ or $y/x = q^{-r}$, where r is a non-negative integer, to avoid this problem. Hence

$$f(x,y,a,0) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{n+k} q^{-\binom{n+k}{2}} P_{n+k}(x,y) t^{n+k}}{(q;q)_k (q;q)_n} (-1)^k q^{\binom{k}{2}} (c;q)_k d^k P_n(u,v)$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} P_n(x,y) t^n}{(q;q)_k (q;q)_{n-k}} (-1)^k q^{\binom{k}{2}} (c;q)_k d^k P_{n-k}(u,v)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} P_n(x,y) t^n}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} (c;q)_k d^k P_{n-k}(u,v)$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} h_n(u,v,c,d|q) \frac{t^n}{(q;q)_n} P_n(x,y). \quad \text{(by using (1.9))}$$

From (2.5), we get

$$\mu_n = (-1)^n q^{-\binom{n}{2}} h_n(u,v,c,d|q) \frac{t^n}{(q;q)_n}.$$

From (2.6), we get

$$f(x,y,a,b) = \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} h_n(x,y,a,b|q) h_n(u,v,c,d|q) \frac{t^n}{(q;q)_n}$$

$$= \text{the left-hand side of equation (3.7)}.$$

$\square$

- Setting $a = c = 0$ and then $b = a$ and $d = b$ in equation (3.7), we get Mehler's formula for Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$ (equation (1.6)).

4 Srivastava-Agarwal type generating functions

In this section, we give Srivastava-Agarwal type generating function for the generalized Al-Salam-Carlitz polynomials $h_n(x,y,a,b|q)$ by the homogeneous q-difference equation technique. By giving specific values to parameters in the Srivastava-Agarwal type generating function for the polynomials $h_n(x,y,a,b|q)$, we get the Srivastava-Agarwal type generating function for Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$ and the Cauchy polynomials $P_n(x,y)$.

The Hahn polynomials [15] are given by

$$\phi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k x^k,$$

which have the following generating function [6, 24]

$$\sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q) \frac{t^n}{(q;q)_n} = \frac{(\alpha x t; q)_\infty}{(t, xt; q)_\infty}, \quad \max\{|t|, |xt|\} < 1.$$

The following is the Srivastava-Agarwal type generating function for $\phi_n^{(\alpha)}(x|q)$ [29]:

$$\sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q)(\lambda;q)_n \frac{t^n}{(q;q)_n} = \frac{(\lambda t;q)_\infty}{(t;q)_\infty} \, {}_2\phi_1\left(\begin{array}{c} \alpha, \lambda \\ \lambda t \end{array} ; q, xt\right), \quad \max\{|t|,|xt|\} < 1.$$

Theorem 4.1. *For $|\alpha x t| < 1$, we have*

$$\sum_{n=0}^{\infty} P_n(\alpha,\lambda) h_n(x,y,a,b|q) \frac{t^n}{(q;q)_n}$$

$$= \frac{(\alpha yt;q)_\infty}{(\alpha xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} P_k(x,y)(\lambda t)^k}{(q,\alpha yt;q)_k} \sum_{i=0}^{\infty} \frac{(-1)^i q^{2\binom{i}{2}+ki}(\alpha xt,a;q)_i}{(q,q^k y\alpha t;q)_i}(b\lambda t)^i$$

$$\times \, {}_1\phi_1\left(\begin{array}{c} aq^i \\ 0 \end{array} ; q, b\alpha tq^i\right). \tag{4.1}$$

Proof. Denoting the right-hand side of Equation (4.1) by $S(x,y,a,b)$, then we have

$$S(x,y,a,b) = \frac{(\alpha yt;q)_\infty}{(\alpha xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} P_k(x,y)(\lambda t)^k}{(q,\alpha yt;q)_k} \sum_{i=0}^{\infty} \frac{(-1)^i q^{2\binom{i}{2}+ki}(\alpha xt,a;q)_i}{(q,q^k y\alpha t;q)_i}(b\lambda t)^i$$

$$\times \, {}_1\phi_1\left(\begin{array}{c} aq^i \\ 0 \end{array} ; q, b\alpha tq^i\right). \tag{4.2}$$

We can check that equation (4.2) satisfies (2.1), we have

$$S(x,y,a,b) = \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{S(x,y,a,0)\right\}$$

$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\frac{(\alpha yt;q)_\infty}{(\alpha xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} P_k(x,y)(\lambda t)^k}{(q,\alpha yt;q)_k}\right\}$$

$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} P_k(x,y)(\lambda t)^k}{(q;q)_k} \frac{(q^k \alpha yt;q)_\infty}{(\alpha xt;q)_\infty}\right\}$$

$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} P_k(x,y)(\lambda t)^k}{(q;q)_k} \sum_{n=0}^{\infty} P_n(x,q^k y)\frac{(\alpha t)^n}{(q;q)_n}\right\} \quad \text{(by using (1.1))}$$

$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} P_{n+k}(x,y)\frac{(-1)^k q^{\binom{k}{2}} \lambda^k \alpha^n t^{n+k}}{(q;q)_k(q;q)_n}\right\}$$

$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{k=0}^{\infty}\sum_{n=k}^{\infty} P_n(x,y)\frac{(-1)^k q^{\binom{k}{2}} \lambda^k \alpha^{n-k} t^n}{(q;q)_k(q;q)_{n-k}}\right\}$$

$$= \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{\sum_{n=0}^{\infty}\sum_{k=0}^{n} P_n(x,y)\frac{(-1)^k q^{\binom{k}{2}} \lambda^k \alpha^{n-k} t^n}{(q;q)_k(q;q)_{n-k}}\right\}$$

$$= \sum_{n=0}^{\infty} \widetilde{\mathbb{F}}(a,b;D_{xy})\left\{P_n(x,y) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}} \lambda^k \alpha^{n-k} \frac{t^n}{(q;q)_n}\right\}$$

$$= \sum_{n=0}^{\infty} P_n(\alpha,\lambda)\widetilde{\mathbb{F}}(a,b;D_{xy})\left\{P_n(x,y)\right\} \frac{t^n}{(q;q)_n} \quad \text{(by using (1.2))}$$

$$= \sum_{n=0}^{\infty} P_n(\alpha,\lambda) h_n(x,y,a,b|q) \frac{t^n}{(q;q)_n}, \quad \text{(by using (1.10))}$$

which is the left-hand side of (4.1). The proof is complete. $\square$

- Setting $\alpha = 1$ in equation (4.1), we get the following result:

Corollary 4.2 (The Srivastava-Agarwal type generating function for $h_n(x, y, a, b|q)$). *For* $|xt| < 1$, *we have*

$$\sum_{n=0}^{\infty} h_n(x, y, a, b|q)(\lambda; q)_n \frac{t^n}{(q; q)_n}$$

$$= \frac{(yt; q)_\infty}{(xt; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} P_k(x, y)(\lambda t)^k}{(q, yt; q)_k} \sum_{i=0}^{\infty} \frac{(-1)^i q^{2\binom{i}{2}+ki}(xt, a; q)_i}{(q, q^k yt; q)_i}(b\lambda t)^i$$

$$\times \; {}_1\phi_1 \left(\begin{array}{c} aq^i \\ 0 \end{array} ; q, btq^i \right).$$ (4.3)

- Setting $a = 0$ and then $b = a$ in equation (4.3), we get the following result:

Corollary 4.3 (The Srivastava-Agarwal type generating function for $U_n(x, y, a|q)$). *For* $|xt| < 1$, *we have*

$$\sum_{n=0}^{\infty} U_n(x, y, a|q)(\lambda; q)_n \frac{t^n}{(q; q)_n}$$

$$= \frac{(at, yt; q)_\infty}{(xt; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} P_k(x, y)(\lambda t)^k}{(q, yt; q)_k} \sum_{i=0}^{\infty} \frac{(-1)^i q^{2\binom{i}{2}+ki}(xt; q)_i}{(q, at, q^k yt; q)_i}(a\lambda t)^i.$$

- For $b = 0$ in equation (4.3), we get the following result:

Corollary 4.4 (The Srivastava-Agarwal type generating function for $P_n(x, y)$). *For* $|xt| < 1$, *we have*

$$\sum_{n=0}^{\infty} P_n(x, y)(\lambda; q)_n \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(xt; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} P_k(x, y)}{(q, yt; q)_k}(\lambda t)^k.$$

5 A Transformational identity involving generating function for $h_n(x, y, a, b|q)$

This section will focus on determining a transformational identity using the homogeneous *q*-difference equation technique, which includes the generating functions for $h_n(x, y, a, b|q)$.

Theorem 5.1. *If*

$$\sum_{k=0}^{\infty} A(k)P_k(x, y) = \sum_{k=0}^{\infty} B(k)\frac{(ytq^k; q)_\infty}{(xtq^k; q)_\infty}$$ (5.1)

is convergent, then

$$\sum_{k=0}^{\infty} A(k)h_k(x, y, a, b|q) = \sum_{k=0}^{\infty} B(k)\frac{(ytq^k; q)_\infty}{(xtq^k; q)_\infty} \; {}_1\phi_1 \left(\begin{array}{c} a \\ 0 \end{array} ; q, btq^k \right),$$ (5.2)

assuming (5.2) *is convergent.*

Proof. Let

$$f(x, y, a, b) = \text{the right-hand side of (5.2)}$$

$$= \sum_{k=0}^{\infty} B(k)\frac{(ytq^k; q)_\infty}{(xtq^k; q)_\infty} \; {}_1\phi_1 \left(\begin{array}{c} a \\ 0 \end{array} ; q, btq^k \right).$$

One can easily check that $f(x, y, a, b)$ satisfies (2.1) and by (2.2) we get

$$f(x, y, a, b) = \widetilde{\mathbb{F}}(a, b; D_{xy})\{f(x, y, a, 0)\}$$

$$= \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \sum_{k=0}^{\infty} B(k) \frac{(ytq^k; q)_{\infty}}{(xtq^k; q)_{\infty}} \right\}$$

$$= \widetilde{\mathbb{F}}(a, b; D_{xy}) \left\{ \sum_{k=0}^{\infty} A(k) P_k(x, y) \right\} \quad \text{(by using (5.1))}$$

$$= \sum_{k=0}^{\infty} A(k) \widetilde{\mathbb{F}}(a, b; D_{xy})\{P_k(x, y)\}$$

$$= \sum_{k=0}^{\infty} A(k) h_k(x, y, a, b|q). \quad \text{(by using (1.10))}$$

$$= \text{left-hand side of (5.2).}$$

$\square$

- Setting $a = 0$ and then $b = a$ in Theorem 5.1, we get the following result:

  Corollary 5.2. *If*

  $$\sum_{k=0}^{\infty} A(k) P_k(x, y) = \sum_{k=0}^{\infty} B(k) \frac{(ytq^k; q)_{\infty}}{(xtq^k; q)_{\infty}}$$

  is convergent, then

  $$\sum_{k=0}^{\infty} A(k) U_k(x, y, a; q) = \sum_{k=0}^{\infty} B(k) \frac{(ytq^k, atq^k; q)_{\infty}}{(xtq^k; q)_{\infty}}, \tag{5.3}$$

  assuming (5.3) *is convergent.*

6 Conclusion remarks

One useful way to prove q-identities is to use the q-difference equation technique. The identities of the Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ are extended to the generalized Al-Salam-Carlitz polynomials $h_n(x, y, a, b|q)$.

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