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FUNDAMENTAL RELATION ON TERNARY HYPERSEMIRINGS

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Abstract In this paper, a new equivalence relation δ^* on a ternary hypersemiring S has been introduced. It has been observed that δ^* is the smallest strongly regular relation on S so that the quotient structure is a ternary semiring. The notion of a fundamental ternary semiring with respect to the fundamental relation δ^* on a ternary hypersemiring S has been introduced. Every ternary semiring with unital element is isomorphic to a fundamental ternary semiring has been established. After that, the fundamental relation δ^* in terms of a typical kind of subsets, called ternary strong $\mathcal{C}$-set of ternary hypersemiring S has been studied.

1 Introduction

The concept of a ternary algebraic system found its origins in the work of D. H. Lehmer [15]. In 1971, W. G. Lister [14] expanded upon this foundation by introducing the concept of the ternary ring, alongside presenting various representations for this mathematical construct. To extend this idea further, Dutta and Kar [6] introduced the notion of a ternary semiring in their work, considering it as a generalization of the ternary ring. A non-empty set, denoted as T, equipped with a binary operation referred to as addition and a ternary operation denoted by juxtaposition, is called ternary semiring if T is an additive commutative semigroup satisfying (i) $(uvw)xy = u(vwx)y = uv(wxy)$, (ii) $(u + v)xy = uxy + vxy$, (iii) $u(v + x)y = uvy + uxy$, (iv) $uv(x + y) = uvx + uvy$ for all $u, v, w, x, y \in T$.

The field of algebraic hyperstructures, is a firmly established branch within classical algebraic theory. It was originally introduced by the French mathematician F. Marty [11] in 1934. Hyperrings, a subclass of hyperstructures, were introduced through various approaches by different researchers. M. Krasner [12] introduced hyperrings, characterized by addition as a hyperoperation and multiplication as a binary operation. On the other hand, M. D. Salvo [19] introduced hyperrings where both addition and multiplication are hyperoperations. Rota [17] delved into the study of hyperrings, where addition is considered a binary operation and multiplication is treated as a hyperoperation. These particular hyperrings are referred to as multiplicative hyperrings.

The exploration of hypersemirings began with Ameri and Hedayati [1] in their work published in 2012, where they focused solely on addition as a hyperoperation. Another class of hyperstructures, known as multiplicative ternary hyperrings, was introduced by T.K. Dutta et al. [18] in 2015. In this context, addition serves as a binary operation, while multiplication is defined as a ternary hyperoperation. Subsequently, in 2018, N. Tamang and M. Mandal [20] defined and conducted a comprehensive study of ternary hypersemirings. These structures represent a generalization encompassing both multiplicative ternary hyperrings and ternary semirings. When we have a ternary semiring denoted as $(S, +, \circ)$, we can regard $(S, +, \circ)$ as a strongly distributive ternary hypersemiring by defining $a \circ b \circ c = \{abc\}$ for all a, b, $c \in S$. This particular ternary hypersemiring is referred to as “trivial ternary hypersemiring". For more details about ternary hypersemiring, we refer to [20], [16].

Fundamental relations represent a pivotal and intriguing concept within the realm of algebraic hyperstructures. They play a vital role in the derivation of ordinary algebraic structures from these hyperstructures. In the domain of hypergroups, the inaugural fundamental relation in hypergroups known as the β^*-relation, was introduced by Koskas [13] in 1970. Building upon this foundation, Freni [8] later presented the γ-relation for hypergroups, as a generalization of the initial β relation. Within the class of hyperrings, a variety of fundamental relations have been defined over time, each playing a crucial role in transforming a given hyperstructure into an equivalent structure. The primary among these is the γ^*-relation, innovatively introduced by Vougiouklis [21] on a hyperring R, where both addition and multiplication are hyperoperations. This relation leads to a quotient structure that aligns with the classical ring framework. Subsequently, fundamental relations have garnered attention from a multitude of researchers, including B. Davvaz [5] and V. Leoreanu [3], Ameneh Asadi et al. [2], S. Mirvakili and B. Davvaz [10], and Leoreanu-Fotea [4], many others.

The structure of the paper is organised as follows: In Section 2, we recall some definitions and results for their use in the sequel. In Section 3, we introduce the smallest strongly regular relation δ^* on ternary hypersemiring S, so that quotient S/δ^* is a ternary semiring and we called it fundamental relation on ternary hypersemiring S. Then we introduce the concept of a fundamental ternary semiring, that is a ternary semiring which is isomorphic to the ternary semiring of a nontrivial ternary hypersemiring. Then we observe that every ternary semiring with unital element is a fundamental ternary semiring. Finally, in Section 4, we establish the fundamental relation δ^* in terms of a typical kind of subsets, called ternary strong $\mathcal{C}$-set of ternary hypersemiring S.

Now we will summarise some key definitions and results that will be relevant to our subsequent discussions.

Definition 1.1. [7] Given an equivalence relation δ on a non-empty set S, two relations $\bar{\delta}$ and $\bar{\bar{\delta}}$ on the power set $P^*(S)$ are defined as follows:

(i) $U\bar{\delta}V$ if and only if for each element u in set U, there exists an element v in set V such that $u\delta v$ holds, and for each v' in set V, there exists an u' in set U such that $u'\delta v'$ holds.

(ii) $U\bar{\bar{\delta}}V$ is true if and only if for all $u \in U$ and $v \in V$, $u\delta v$ holds.

Definition 1.2. [7] An equivalence relation δ defined on a ternary hypersemiring $(S, +, \circ)$ is classified as follows:

(i) δ is considered regular if it is a congruence on the commutative semigroup $(S, +)$, which implies that for any $x,\ y,\ z \in S$, if $x\delta y$ holds, then $(x + z)\delta(y + z)$, and if $x\delta u$, $y\delta v$, $z\delta w$, then $(x \circ y \circ z)\bar{\delta}(u \circ v \circ w)$ for $x,\ y,\ z,\ u,\ v,\ w \in S$.

(ii) δ is termed strongly regular when it is congruence on the commutative semigroup $(S, +)$, this means that for any $x,\ y,\ z \in S$, if $x\delta y$, then $(x + z)\delta(y + z)$, and if $x\delta u$, $y\delta v$, $z\delta w$, then $(x \circ y \circ z)\bar{\bar{\delta}}(u \circ v \circ w)$ for any $x, y, x, u, v, w \in S$.

The second conditions mentioned in (i) and (ii) of this definition can be equivalently expressed as follows: For all $u,\ v,\ x,\ y$ in S, (i) In the regular case: $u\delta v$ implies $(u\circ x\circ y)\ \bar{\delta}\ (v\circ x\circ y)$, $(x \circ u \circ y)\ \bar{\delta}\ (x \circ v \circ y)$, and $(x \circ y \circ u)\ \bar{\delta}\ (x \circ y \circ v)$ (ii) In the strongly regular case: $u\delta v$ implies $(u \circ x \circ y)\ \bar{\bar{\delta}}\ (v \circ x \circ y)$, $(x \circ u \circ y)\ \bar{\bar{\delta}}\ (x \circ v \circ y)$, and $(x \circ y \circ u)\ \bar{\bar{\delta}}\ (x \circ y \circ v)$.

2 On Fundamental Relation and Fundamental Ternary Semiring

Let $(S, +, \circ)$ be a ternary hypersemiring. We define a binary relation δ on S as follows: $x\delta y$ if and only if there exist $n \in \mathbb{N}$, $m_i \in \mathbb{Z}_0^+$, $i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2m_i+1$ such that

$$x = y \text{ or } \{x, y\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}.$$

Clearly, δ is reflexive and symmetric. Consider δ^* be the transitive closure of the relation δ. Then, we have $x\delta^* y$ if and only if there exist elements $a_0, a_1, ..., a_m \in S$ with $x = a_0$ and $y = a_m$ such that $a_i\delta a_{i+1}$ for $i \in 0, 1, ..., m - 1$.

Theorem 2.1. *The relation δ^* is a strongly regular relation on S.*

Proof. Let $a, b \in S$ such that $a\delta^* b$ holds. If $a = b$, then it is obvious that $(a + c)\,\delta^*\,(b + c)$ and $(a \circ u \circ v)\,\bar{\bar{\delta^*}}\,(b \circ u \circ v)$ for any $c, u, v \in S$. Analogously, $(u \circ a \circ v)\,\bar{\bar{\delta^*}}\,(u \circ b \circ v)$ and $(u \circ v \circ a)\,\bar{\bar{\delta^*}}\,(u \circ v \circ b)$ for any $u, v \in S$. If $a \neq b$, then there exist $n \in \mathbb{N}, m_i \in \mathbb{Z}_0^+, i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2m_i + 1$ such that

$$\{a, b\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}.$$

Then, for any $c \in S$, $a + c \in \sum_{i=1}^{n} \prod_{j=1}^{k_i} a_{ij} + c$ and $b + c \in \sum_{i=1}^{n} \prod_{j=1}^{k_i} a_{ij} + c$. Now we set $m_{n+1} = 0$ and $a_{n+1,1} = c$, thus

$$\{a + c, b + c\} \subseteq \sum_{i=1}^{n+1} \prod_{j=1}^{2m_i+1} a_{ij}.$$

So, $(a + c)\,\bar{\bar{\delta^*}}\,(b + c)$ for any $c \in S$.
Again for arbitrary $u, v \in S$, $a \circ u \circ v \subseteq (\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}) \circ u \circ v \subseteq \sum_{i=1}^{n} (\prod_{j=1}^{2m_i+1} a_{ij}) \circ u \circ v$, also $b \circ u \circ v \subseteq \sum_{i=1}^{n} (\prod_{j=1}^{2m_i+1} a_{ij}) \circ u \circ v$. Here, we set $m_i' = m_i + 1$, $a_{i,2m_i+2} = u$, $a_{i,2m_i+3} = v$ for $i = 1, 2, ..., n$. Then

$$\{a \circ u \circ v, b \circ u \circ v\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i'+1} a_{ij}.$$

So for any $s \in a \circ u \circ v$ and $t \in b \circ u \circ v$, we have $s\delta t$. Thus $(a \circ u \circ v)\,\bar{\bar{\delta^*}}\,(b \circ u \circ v)$. Similarly, $(u \circ a \circ v)\,\bar{\bar{\delta^*}}\,(u \circ b \circ v)$ and $(u \circ v \circ a)\,\bar{\bar{\delta^*}}\,(u \circ v \circ b)$. Hence δ^* is a strongly regular relation on S. □

Theorem 2.2. *The relation δ^* is the smallest strongly regular relation on ternary hypersemiring S.*

Proof. If possible, let θ be the smallest strongly regular relation on S. We prove that $\theta = \delta^*$. It is clear that, $\theta \subseteq \delta^*$. If $x\delta y$, then there exist $n \in \mathbb{N}$, $m_i \in \mathbb{Z}_0^+$, $i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2m_i + 1$ such that

$$x = y \text{ or } \{x, y\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}.$$

Since θ is a strongly regular relation and $a\theta a$ for $a \in S$, we have $(a \circ b \circ c)\,\bar{\bar{\theta}}\,(a \circ b \circ c)$ for any $b, c \in S$. Also for any $z \in (a \circ b \circ c)$, we have $(z \circ d \circ e)\,\bar{\bar{\theta}}\,(z \circ d \circ e)$ for all $d, e \in S$. That implies $(a \circ b \circ c) \circ d \circ e\,\bar{\bar{\theta}}\,(a \circ b \circ c) \circ d \circ e$. Continuing this process, we get that any finite product of elements of S is θ related to itself. Thus, we obtain $(\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij})\ \bar{\bar{\theta}}\ (\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij})$. Hence $x\,\theta\,y$, that implies $\delta \subseteq \theta$. If $x\,\delta^*\,y$, then there exist $x = a_0, a_1, a_2, ..., a_m = y$ such that $a_i\,\delta\,a_{i+1}$ for $i = 0, 1, .., m - 1$. Since $\delta \subseteq \theta$ and θ is transitive, we have $\delta^* \subseteq \theta$. Therefore, δ^* is the smallest strongly regular relation on ternary hypersemiring S. □

Corollary 2.3. *By Theorem-*4.13 *[7], the quotient $(S/\delta^*, +, \circ)$ is a ternary semiring.*

Definition 2.4. The smallest strongly regular equivalence relation δ^* on S is called the fundamental relation on ternary hypersemiring S.

Proposition 2.5. *Let f be a homomorphism from a ternary hypersemiring $(S, +, \circ)$ to a ternary hypersemiring $(T, +', \circ')$. Then, the following are true:*

(1) $x\,\delta^*\,y$ implies $f(x)\,\delta^*\,f(y)$

(2) if f is an one-one homomorphism, then $a\delta b$ implies $x\delta y$ where $f(x) = a$ and $f(y) = b$ for some $x, y \in S$ and $a, b \in T$

(3) if f is an isomorphism, then $x\,\delta\,y$ if and only if $f(x)\,\delta\,f(y)$.

Proof. (1) Let $x\,\delta^*\,y$. Then there exists $a_1, a_2, .., a_n \in S$ with $a_1 = x$, $a_m = y$ such that $a_k\,\delta\,a_{k+1}$ for $1 \leq k \leq m-1$. Thus for $a_k\,\delta\,a_{k+1}$, there exist $n_k \in \mathbb{N}$, $m_{ik} \in \mathbb{Z}_0^+$ for $i = 1, 2, ..., n_k$ and $a_{ijk} \in S$, $1 \leq i \leq n_k$, $1 \leq j \leq 2m_ik + 1$ such that

$$\{a_k, a_{k+1}\} \subseteq \sum_{i=1}^{n_k} \prod_{j=1}^{2m_{ik}+1} a_{ijk}.$$

Since f is a homomorphism, we have $\{f(a_k), f(a_{k+1})\} = f(\{a_k, a_{k+1}\})$ $\subseteq f(\sum_{i=1}^{n_k} \prod_{j=1}^{2m_{ik}+1} a_{ijk}) = \sum_{i=1}^{n_k} f(\prod_{j=1}^{2m_{ik}+1} a_{ijk}) \subseteq \sum_{i=1}^{n_k} \prod_{j=1}^{2m_{ik}+1} f(a_{ijk})$ for all $k = 1, 2, .., m-1$. Therefore, $f(x)\,\delta^*\,f(y)$.

(2) Let $a = f(x)\,\delta f(y) = b$. Then there exist $n \in \mathbb{N}$, $2m_i + 1 \in \mathbb{N}$ where $m_i \in \mathbb{Z}_0^+$ for $i = 1, 2, ..., n$ and $c_{ij} \in T$ such that $\{f(x), f(y)\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} c_{ij}$. Since f is an one-one homomorphism, we have $\{x, y\} = f^{-1}\{f(x), f(y)\} \subseteq f^{-1}(\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} c_{ij}) = \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} f^{-1}(c_{ij})$. So, $x\,\delta\,y$.

(3) By (1) and (2), it is clear.

□

Theorem 2.6. *Let $f : S \longrightarrow T$ be an isomorphism from a ternary hypersemiring $(S, +, \circ)$ to a ternary hypersemiring $(T, +', \circ')$. Then the ternary semirings $(S/\delta^*, +, \cdot)$ and $(T/\delta^*, +', \cdot')$ are isomorphic.*

Proof. We define a mapping $h : S/\delta^* \longrightarrow T/\delta^*$ by $h(\delta^*(a)) = \delta^*(f(a))$ for all $a \in S$. Now $\delta^*(a) = \delta^*(b) \iff a\delta^*b \iff f(a)\,\delta^*\,f(b)$ (by Proposition-2.5) $\iff \delta^*(f(a)) = \delta^*(f(b)) \iff h(\delta^*(a)) = h(\delta^*(b))$. Thus h is well-defined and one-to-one. Clearly, h is also a surjective mapping. Now, we have the following equality $h(\delta^*(a) + \delta^*(b)) = h(\delta^*(a + b)) = \delta^*(f(a + b)) = \delta^*(f(a) +' f(b)) = \delta^*(f(a)) +' \delta^*(f(b)) = h(\delta^*(a)) +' h(\delta^*(b))$ and $h(\delta^*(a) \cdot \delta^*(b) \cdot \delta^*(c)) = h(\delta^*(x))$ (for some $x \in a \circ b \circ c$) $= \delta^*(f(x))$ (where $f(x) \in f(a) \circ' f(b) \circ' f(c)$)$=\delta^*(f(a)) \cdot' \delta^*(f(b)) \cdot' \delta^*(f(c)) = h(\delta^*(a)) \cdot' h(\delta^*(b)) \cdot' h(\delta^*(c))$. Hence h is a ternary homomorphism and so S/δ^* and T/δ^* are isomorphic. □

Definition 2.7. A ternary semiring $(S, +, \cdot)$ is called a fundamental ternary semiring if there exists a non-trivial ternary hypersemiring, say $(T, +, \circ)$, such that $(T/\delta^*, +', \cdot')$ is isomorphic to $(S, +, \cdot)$.

Theorem 2.8. *Let $(T, +, \cdot)$ be a ternary semiring. Then, for any semiring $(S, +', \cdot')$ there exist a binary operation $\oplus$ and a ternary hyperopration $\odot$ such that $(T \times S, \oplus, \odot)$ is a non-trivial ternary hypersemiring.*

Proof. Let $(S, +', \cdot')$ be a semiring. We define a binary operation $\oplus$ and a ternary hyperopration $\odot$ on $T \times S$ by $(r_1, s_1) \oplus (r_2, s_2) = (r_1 + r_2, s_1 +' s_2)$ and $(r_1, s_1) \odot (r_2, s_2) \odot (r_3, s_3) = \{(r_1{\cdot}r_2{\cdot}r_3, s_1), (r_1{\cdot}r_2{\cdot}r_3, s_2), (r_1{\cdot}r_2{\cdot}r_3, s_3), (r_1{\cdot}r_2{\cdot}r_3, 0)\}$ for $(r_1, s_1), (r_2, s_2), (r_3, s_3) \in R{\times}S$. Then it is easy to show that $(T \times S, \oplus)$ is a commutative semigroup and '$\odot$' is associative. Now, $((r_1, s_1){\oplus}(r_2, s_2)){\odot}(r_3, s_3){\odot}(r_4, s_4) = \{((r_1{+}r_2){\cdot}r_3{\cdot}r_4, s_1{+'}s_2), ((r_1{+}r_2){\cdot}r_3{\cdot}r_4, s_3), ((r_1{+}r_2){\cdot}r_3{\cdot}r_4, s_4), ((r_1{+}r_2){\cdot}r_3{\cdot}r_4, 0)\} \subseteq ((r_1, s_1){\odot}(r_3, s_3){\odot}(r_4, s_4)){\oplus}((r_2, s_2){\odot}(r_3, s_3){\odot}(r_4, s_4))$ for $(r_1, s_1), (r_2, s_2), (r_3, s_3), (r_4, s_4), (r_5, s_5) \in T \times S$. In similar way, the other inclusions hold for ternary hypersemiring. Therefore, $(T \times S, \oplus, \odot)$ is a ternary hypersemiring. □

The ternary hypersemiring $(T \times S, \oplus, \odot)$ is called associated ternary hypersemiring of T via S.

Theorem 2.9. *Let $(T, +, \cdot)$ and $(\tilde{T}, \tilde{+}, \tilde{\cdot})$ be two isomorphic ternary semirings. Then, for any semiring $(S, +', \cdot')$, the associated ternary hypersemirings $(T \times S, \oplus, \odot)$ of T via S and $(\tilde{T} \times S, \tilde{\oplus}, \tilde{\odot})$ of $\tilde{T}$ via S are isomorphic.*

Proof. Let $(S, +', \cdot')$ be a semiring and $f : T \longrightarrow \tilde{T}$ be an isomorphism. We define a mapping $h : T \times S \longrightarrow \tilde{T} \times S$ by $h((t,s)) = (f(t), s)$ for any $(t,s) \in T \times S$. Since f is an isomorphism, it is easy to check h is well-defined and a bijection mapping. Now, $h((t_1,s_1) \oplus (t_2,s_2)) = h((t_1+t_2, s_1+'s_2)) = (f(t_1+t_2), s_1+'s_2) = (f(t_1)\tilde{+}f(t_2), s_1+'s_2) = (f(t_1),s_1)\tilde{\oplus}(f(t_2),s_2) = h((t_1,s_1))\tilde{\oplus}h((t_2,s_2))$.
Also, $h((t_1,s_1) \odot (t_2,s_2) \odot (t_3,s_3))$
$= h\left(\{(t_1 \cdot t_2 \cdot t_3,\, s_1), (t_1 \cdot t_2 \cdot t_3,\, s_2), (t_1 \cdot t_2 \cdot t_3,\, s_3), (t_1 \cdot t_2 \cdot t_3,\, 0)\}\right)$
$= \{h((t_1 \cdot t_2 \cdot t_3,\, s_1)), h((t_1 \cdot t_2 \cdot t_3,\, s_2)), h((t_1 \cdot t_2 \cdot t_3,\, s_3)), h((t_1 \cdot t_2 \cdot t_3,\, 0))\}$
$= \{(f(t_1 \cdot t_2 \cdot t_3),\, s_1), (f(t_1 \cdot t_2 \cdot t_3),\, s_2), (f(t_1 \cdot t_2 \cdot t_3),\, s_3), (f(t_1 \cdot t_2 \cdot t_3),\, 0)\}$
$= \{(f(t_1)\tilde{\cdot}f(t_2)\tilde{\cdot}f(t_3),\, s_1), (f(t_1)\tilde{\cdot}f(t_2)\tilde{\cdot}f(t_3),\, s_2), (f(t_1)\tilde{\cdot}f(t_2)\tilde{\cdot}f(t_3),\, s_3),$
$(f(t_1)\tilde{\cdot}f(t_2)\tilde{\cdot}f(t_3),\, 0)\}$
$= (f(t_1),\, s_1)\ \tilde{\odot}\ (f(t_2),\, s_2)\ \tilde{\odot}\ (f(t_3),\, s_3) = h((t_1,\, s_1))\,\tilde{\odot}\,h((t_2,\, s_2))\,\tilde{\odot}\,h((t_3,\, s_3))$. Therefore, h is a good homomorphism and so the ternary hypersemirings $(T \times S, \oplus, \odot)$ and $(\tilde{T} \times S, \tilde{\oplus}, \tilde{\odot})$ are isomorphic. □

Theorem 2.10. *Every ternary semiring with a unital element is a fundamental ternary semiring.*

Proof. Let $(T, +, \cdot)$ be a ternary semiring with unital element 'e' and $(S, +', \cdot')$ be any semiring. Then, by Theorem 2.8, $(T \times S, \oplus, \odot)$ is a non-trivial ternary hypersemiring. Consequently, by Corollary 2.3, $(T \times S)/\delta^*$ is a ternary semiring, where δ^* is the fundamental relation on $T \times S$. Now we show that $\delta^*((t,s)) = \{(t,s') : s' \in S\}$ for $(t,s) \in T \times S$. Since $\{(t,s),(t,s')\} \subseteq (t,s) \odot (e,s') \odot (e,s)$ for any $(t,s') \in T \times S$. Thus we get $(t,s') \in \delta^*((t,s))$. On the other hand, if $(t'',s'')\, \delta^*\, (t,s)$. This obviously implies that, $t'' = t$. Therefore, $\delta^*((t,s)) = \{(t,s') : s' \in S\}$. Now we consider a map $f : (T \times S)/\delta^* \longrightarrow T$ by $f(\delta^*((t,s))) = t$ for all $\delta^*((t,s)) \in (T \times S)/\delta^*$. It is easy to check that the map is well-defined and one-to-one. Also, for any $t \in T$ we have $f(\delta^*((t,0))) = t$, thus f is an onto map. Now $f(\delta^*(t_1,s_1) + \delta^*(t_2,s_2)) = f(\delta^*((t_1+t_2, s_1+s_2)) = t_1+t_2 = f(\delta^*(t_1,s_1)) + f(\delta^*((t_2,s_2)))$ and $f(\delta^*((t_1,s_1)) \cdot \delta^*((t_2,s_2)) \cdot \delta^*((t_3,s_3))) = f(\delta^*((t_1t_2t_3, s_1))) = t_1t_2t_3 = f(\delta^*((t_1,s_1)))f(\delta^*((t_1,s_1)))f(\delta^*((t_1,s_1)))$. Thus, f is a homeomorphism and so $(T \times S)/\delta^*$ and T isomorphic. Therefore, $(T, +, \cdot)$ is a fundamental ternary semiring. □

Theorem 2.11. *Let $(T, +, \circ)$ be a ternary hypersemiring. Then, there exist a ternary semiring S, binary operation '$\oplus$' and ternary hyperoperation '$\odot$' such that $(T, +, \circ)$ can be embedded in $(T \times S, \oplus, \odot)$.*

Proof. Let $(T, +, \circ)$ be a ternary hypersemiring. By Corollary-2.3, $(T/\delta^*, +', \circ')$ is a ternary semiring. We set $S = (T/\delta^*, +', \circ')$. Now, on the set $T \times S$, we define the binary operation $\oplus$ and ternary hyperoperation $\odot$ by $(t_1, \delta^*(s_1)) \oplus (t_2, \delta^*(s_2)) = (t_1+t_2, \delta^*(s_1+s_2))$ and $(t_1,\delta^*(s_1)) \odot (t_2,\delta^*(s_2)) \odot (t_3,\delta^*(s_3)) = (t_1 \circ t_2 \circ t_3, \delta^*(s_1 \circ s_2 \circ s_3))$. Let $(t_1,\delta^*(s_1)) = (t_1', \delta^*(s_1'))$, $(t_2,\delta^*(s_2)) = (t_2',\delta^*(s_2'))$, $(t_3,\delta^*(s_3)) = (t_3',\delta^*(s_3'))$. So $t_1 = t_1'$, $\delta^*(s_1) = \delta^*(s_1')$, $t_2 = t_2'$, $\delta^*(s_2) = \delta^*(s_2')$ and $t_3 = t_3'$, $\delta^*(s_3) = \delta^*(s_3')$. Then $t_1 \circ t_2 = t_1' \circ t_2'$ and $\delta^*(s_1+s_2) = \delta^*(s_1'+s_2')$ (since δ^* is a congruence on $(T,+)$). Also, $t_1 \circ t_2 \circ t_3 = t_1' \circ t_2' \circ t_3'$ and $\delta^*(s_1 \circ s_2 \circ s_3) = \delta^*(s_1' \circ s_2' \circ s_3')$, since δ^* is a strongly regular relation. Thus the operations $\oplus$ and $\odot$ are well-defined. Next, we will show $(T \times S, \oplus, \odot)$ is a ternary hypersemiring. Clearly, $(T \times S, \oplus)$ is a commutative semigroup. For associativity of '$\odot$', $((t_1,\delta^*(s_1)) \odot (t_2,\delta^*(s_2)) \odot (t_3,\delta^*(s_3))) \odot (t_4,\delta^*(s_4)) \odot (t_5,\delta^*(s_5)) = ((t_1 \circ t_2 \circ t_3) \circ t_4 \circ t_5, \delta^*((s_1 \circ s_2 \circ s_3) \circ s_4 \circ s_5)) = (t_1 \circ (t_2 \circ t_3 \circ t_4) \circ t_5, \delta^*(s_1 \circ (s_2 \circ s_3 \circ s_4) \circ s_5)) = (t_1,\delta^*(s_1)) \odot ((t_2,\delta^*(s_2)) \odot (t_3,\delta^*(s_3))) \odot (t_4,\delta^*(s_4))) \odot (t_5,\delta^*(s_5))$. In a similar way, $(t_1,\delta^*(s_1)) \odot ((t_2,\delta^*(s_2)) \odot (t_3,\delta^*(s_3)) \odot (t_4,\delta^*(s_4))) \odot (t_5,\delta^*(s_5)) = (t_1,\delta^*(s_1)) \odot (t_2,\delta^*(s_2)) \odot ((t_3,\delta^*(s_3)) \odot (t_4,\delta^*(s_4)) \odot (t_5,\delta^*(s_5)))$ for any $(t_i,\delta^*(s_i)) \in T \times S$, $i = 1,2,..,5$. Hence,'$\odot$' is associative. Here, $((t_1,\delta^*(s_1)) \oplus (t_2,\delta^*(s_2))) \odot (t_3,\delta^*(s_3)) \odot (t_4,\delta^*(s_4)) = ((t_1+t_2) \circ t_3 \circ t_4,\ \delta^*((s_1+s_2) \circ s_3 \circ s_4)) \subseteq ((t_1 \circ t_3 \circ t_4)+(t_2 \circ t_3 \circ t_4), \delta^*((s_1 \circ s_3 \circ s_4)+(s_2 \circ s_3 \circ s_4)) = ((t_1,\delta^*(s_1)) \odot (t_3,\delta^*(s_3)) \odot (t_4,\delta^*(s_4))) \oplus ((t_2,\delta^*(s_2)) \odot (t_3,\delta^*(s_3)) \odot (t_4,\delta^*(s_4)))$ for any $(t_i,\delta^*(s_i)) \in T \times S$, $i = 1,2,3,4$. Silimarly, other inclusions of ternary hypersemiring also hold. So, $(T \times S, \oplus, \odot)$ is a ternary hypersemirng. Now, consider a mapping $\phi : T \longrightarrow T \times S$ defined by $\phi(t) = (t, \delta^*(t))$ for all $t \in T$. Then, $t_1 = t_2 \iff (t_1,\delta^*(t_1)) = (t_2,\delta^*(t_2)) \iff \phi(t_1) = \phi(t_2)$ for $t_1,t_2 \in T$. So ϕ is well-defined and one-one. Let $t_1,t_2 \in T$. Then $\phi(t_1+t_2) = (t_1+t_2, \delta^*(t_1+t_2)) = (t_1,\delta^*(t_1)) \oplus (t_2,\delta^*(t_2)) = \phi(t_1) \oplus \phi(t_2)$. Further, we have $\phi(t_1 \circ t_2 \circ t_3) = (t_1 \circ t_2 \circ t_3, \delta^*(t_1 \circ t_2 \circ t_3)) = (t_1,\delta^*(t_1)) \odot (t_2,\delta^*(t_2)) \odot (t_3,\delta^*(t_3)) =$

$\phi(t_1) \odot \phi(t_2) \odot \phi(t_3)$. This shows that ϕ is a one-one good homomorphism. Therefore, $(T, +, \cdot)$ is embedded in $(T \times S, \oplus, \odot)$. □

3 Ternary Strong $\mathcal{C}$-set

In this section, we will first define the concept of ternary strong $\mathcal{C}$-set, then we will describe the fundamental relation δ^* in terms of ternary strong $\mathcal{C}$-set of ternary hypersemiring S.

Definition 3.1. Let $(S, +, \circ)$ be a ternary hypersemiring and $\Omega = \{\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} : n \in \mathbb{N}, m_i \in \mathbb{Z}_0^+, a_{ij} \in S \, for \, 1 \leq i \leq n, 1 \leq j \leq 2m_i + 1\}$. A non-empty subset I of S is called a ternary strong $\mathcal{C}$-set if for any $A \in \Omega$, $I \cap A \neq \phi$ implies $A \subseteq I$.

Proposition 3.2. *For any strongly regular equivalence relation $\wp$ on ternary hypersemiring S, the equivalence class $\wp(a)$, $a \in S$ is a ternary strong $\mathcal{C}$-set.*

Proof. Let $A \in \Omega = \{\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} : n \in \mathbb{N}, m_i \in \mathbb{Z}_0^+, a_{ij} \in S, for \; 1 \leq i \leq n, 1 \leq j \leq 2m_i + 1\}$ be such that $A \cap \wp(a) \neq \phi$. Then there exists $x \in S$ such that $x \in A$ and $x \in \wp(a)$. Thus for any $y \in A$, there exist $n \in \mathbb{N}$, $m_i \in \mathbb{Z}_0^+$, $a_{ij} \in S$ for $1 \leq i \leq n$ and $1 \leq j \leq 2m_i + 1$ such that

$$\{x, y\} \subseteq A \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}.$$

That implies $x\delta^* y$. Since δ^* being the smallest strongly regular equivalence relation and $\wp$ a strongly regular equivalence relation, we have $\delta^* \subseteq \wp$. Thus $x\wp y$, and so $y \in \wp(a)$. Hence $A \subseteq \wp(a)$. Therefore, $\wp(a)$ is a ternary strong $\mathcal{C}$-set of S. □

Theorem 3.3. *Let A be a non-empty subset of ternary hypersemiring S. Then the following are equivalent:*

(1) A is a ternary strong $\mathcal{C}$-set of S;

(2) for $x \in A$ and $x \, \delta \, y$ implies $y \in A$;

(3) for $x \in A$ and $x \, \delta^* \, y$ implies $y \in A$.

Proof. (1) $\Rightarrow$ (2) Suppose that A is a ternary strong $\mathcal{C}$-set of S and $x \in A$, $x \, \delta \, y$ for some $x, y \in S$. Then there exist $n \in \mathbb{N}$, $m_i \in \mathbb{Z}_0^+$, $i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2m_i + 1$ such that

$$x = y \text{ or } \{x, y\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}.$$

If $x = y$, then it is clear. Otherwise, $x \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \cap A$. Since A is a ternary strong $\mathcal{C}$-set, we have $\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \subseteq A$ which implies $y \in A$.

(2) $\Rightarrow$ (3) Let $x \in A$ and $x\delta^* y$ for some $x, y \in S$. So there exist $n \in \mathbb{N}$ and $x = y_0, y_1, y_2, ..., y_n = y \in S$ such that $y_i \, \delta \, y_{i+1}$ for all $i \in \{0, 1, ..., n-1\}$. Since $x = y_0 \in A$ we have $y \in A$, by applying (2) n-times.

(3) $\Rightarrow$ (1) Let $(\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}) \cap A \neq \phi$ for some $n \in \mathbb{N}$, $m_i \in \mathbb{Z}_0^+$, $a_{ij} \in S$ for $1 \leq i \leq n$ and $1 \leq j \leq m_i$. Suppose $x \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \cap A$. Now for any $y \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}$, we have $x\delta^* y$. So by (3), $y \in A$. Hence $\sum_{i=1}^{n} \prod_{j=1}^{k_i} a_{ij} \subseteq A$. Therefore, A is a ternary strong $\mathcal{C}$-set of S. □

The intersection of any arbitrary collection of ternary strong $\mathcal{C}$-set of a ternary hypersemiring S is again a ternary strong $\mathcal{C}$-set of S.

Definition 3.4. Let A be a non-empty subset of a ternary hypersemiring S. The intersection of ternary strong $\mathcal{C}$-sets of S containing A is called the ternary strong $\mathcal{C}$-closure of A and denote it by $\mathcal{C}^*(A)$.

Since, S is itself a ternary strong $\mathcal{C}$-set in the ternary hypersemiring S, so $\mathcal{C}^*(A)$ exists for any subsets A of S.

Now, for a non-empty subset A of a ternary hypersemiring S, we set

$$K_1(A) = A$$

$$K_{m+1}(A) = \left\{x \in S : \exists\, a_{ij} \in S; x \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \textit{ and } \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \cap K_m(A) \neq \phi\right\}$$

$$K(A) = \bigcup_{m\geq 1} K_m(A).$$

Lemma 3.5. *For any non-empty subset A of S, the set $K(A)$ is a ternary strong $\mathcal{C}$-set containing A.*

Proof. Let $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap K(A) \neq \phi$ for some $n \in \mathbb{N}$, $k_i \in \mathbb{Z}_0^+$, $a_{ij} \in S$ where $i = 1, 2, ..., n$ and $1 \leq j \leq 2k_i + 1$. Then, there exists $m \in \mathbb{N}$ such that $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap K_m(A) \neq \phi$, which implies that $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \subseteq K_{m+1}(A) \subseteq K(A)$. Also, $K_1(A) = A \subseteq K(A)$ and so the set $K(A)$ is a ternary strong $\mathcal{C}$-set containing A. □

Corollary 3.6. *$A \subseteq B$ implies $K(A) \subseteq K(B)$, for any two non-empty subsets $A, B \in S$.*

Lemma 3.7. *For every $m(\geq 2) \in \mathbb{N}$ and $x \in S$, we have $K_m(K_2(\{x\})) = K_{m+1}(\{x\})$.*

Proof. $K_2(K_2(\{x\})) = \{x \in S : \exists\, a_{ij} \in S; x \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}$ *and* $\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \cap K_2(\{x\})) \neq \phi\} = K_3(\{x\})$. So, our assertion is true for $m = 2$. Let the assertion be true for some $l(> 2) \in \mathbb{N}$, i.e., $K_l(K_2(\{x\})) = K_{l+1}(\{x\})$. Let $K_{l+1}(K_2(x)) = \{x \in S : \exists\, a_{ij} \in S; x \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}$ *and* $\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \cap K_l(K_2(\{x\})) \neq \phi\} = \{x \in S : \exists\, a_{ij} \in S; x \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}$ *and* $\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \cap K_{l+1}(\{x\}) \neq \phi\} = K_{l+2}(\{x\})$. Hence, by induction $K_m(K_2(\{x\})) = K_{m+1}(\{x\})$ for any $m(\geq 2) \in \mathbb{N}$. □

Theorem 3.8. *For any non-empty subset of a ternary hypersemiring S, $\mathcal{C}^*(A) = K(A)$.*

Proof. By Lemma 3.5, we have $\mathcal{C}^*(A) \subseteq K(A)$. Let A' be a ternary strong $\mathcal{C}$-set containing A. Clearly, $K_1(A) \subseteq A'$. Suppose $K_m(A) \subseteq A'$ for some $m \in \mathbb{N}$. Let $x \in K_{m+1}(A)$. Then there exist elements $a_{ij} \in S$ such that $x \in \sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}$ and $\phi \neq (\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}) \cap K_m(A) \subseteq (\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}) \cap A'$. Since A' is a ternary strong $\mathcal{C}$-set, we have $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \subseteq A'$. So $K_{m+1}(A) \subseteq A'$ and thus by induction $K_m \subseteq A'$ for all $m \in \mathbb{N}$. So $K(A) \subseteq A'$. Therefore, $K(A) \subseteq \mathcal{C}^*(A)$, this implies that $\mathcal{C}^*(A) = K(A)$. □

Theorem 3.9. *If B is a non-empty subset of a ternary hypersemiring S, then*

$$\bigcup_{b\in B} \mathcal{C}^*(\{b\}) = \mathcal{C}^*(B).$$

Proof. Clearly, for every $b \in B$, $\mathcal{C}^*(\{b\}) \subseteq \mathcal{C}^*(B)$, by Theorem 3.8 and Corollary 3.6. Hence, $\bigcup_{b\in B} \mathcal{C}^*(\{b\}) \subseteq \mathcal{C}^*(B)$.

Conversely, at first we show $K_m(B) \subseteq \bigcup_{b\in B} K_m(\{b\})$ for any $m \in \mathbb{N}$. For $m = 1$, $K_1(B) = B = \bigcup_{b\in B} K_1(\{b\})$. Now let the assertion be true for some $l \in \mathbb{N}$, that is $K_l(B) \subseteq \bigcup_{b\in B} K_l(\{b\})$. Consider, $x \in K_{l+1}(B)$, then there exist $n \in \mathbb{N}$, $k_i \in \mathbb{Z}_0^+$, $i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2k_i + 1$ such that $x \in \sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}$ and $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap K_l(B) \neq \phi$. Now from hypothesis induction, we have $(\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}) \cap (\bigcup_{b\in B} K_l(\{b\})) \neq \phi$. So, there exists $b_1 \in B$ such that $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap K_l(\{b_1\}) \neq \phi$. Therefore, $x \in K_{l+1}(\{b_1\})$ and so $K_{l+1}(B) \subseteq \bigcup_{b\in B} K_{l+1}(\{b\})$. Now by Theorem 3.8, $y \in \mathcal{C}^*(B)$ implies $y \in K(B) = \bigcup_{m\geq 1} K_m(B) \subseteq \bigcup_{m\geq 1} \bigcup_{b\in B} K_m(\{b\})$. Then for some $b \in B$ and $m \in \mathbb{N}$ we have $y \in K_m(\{b\}) \subseteq K(\{b\}) = \mathcal{C}^*(\{b\}) \subseteq \bigcup_{b\in B} \mathcal{C}^*(\{b\})$. Therefore, $\bigcup_{b\in B} \mathcal{C}^*(\{b\}) = \mathcal{C}^*(B)$. □

We define a binary relation 'μ' on a ternary hypersemiring S as follows: $x \mu y$ if and only if $x \in K(\{y\})$, for all $x, y \in S$.

Proposition 3.10. *The relation μ is an equivalence relation on ternary hypersemiring S.*

Proof. For any $x \in S$ we have $x \in K(\{x\})$, thus $x\mu x$ holds and so μ is reflexive. To show μ is symmetric, we first prove that $x \in K_m(\{y\}) \iff y \in K_m(\{x\})$ for all $m \in \mathbb{N}$ and we prove it by induction on 'm'. If $m = 2$, there exist $a_{ij} \in S$ such that $x \in \sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}$ and $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap \{y\} \neq \phi$ which implies $x = y$ and so $y \in K_2(\{x\})$. Let the assertion be true for some $l(> 2) \in \mathbb{N}$, i.e., $x \in K_l(\{y\})$ implies $y \in K_l(\{x\})$. Now for $l+1 \in \mathbb{N}$, $x \in K_{l+1}(\{y\})$ implies there exist $a_{ij} \in S$ such that $x \in \sum_{i=1}^{n'} \prod_{j=1}^{k'_i} a_{ij}$ and $\sum_{i=1}^{n'} \prod_{j=1}^{k'_i} a_{ij} \cap K_l(\{y\}) \neq \phi$. Suppose $u \in (\sum_{i=1}^{n'} \prod_{j=1}^{k'_i} a_{ij}) \cap K_l(\{y\})$. So $u, x \in \sum_{i=1}^{n'} \prod_{j=1}^{k'_i} a_{ij}$ and $u \in K_l(\{y\})$. Thus $u \in K_2(\{x\})$ and by hypothesis $y \in K_l(\{u\})$. So by Lemma 3.7, $y \in K_l(K_2(\{x\})) = K_{l+1}(\{x\})$. Hence by induction, $x \in K_m(\{y\})$ implies $y \in K_m(\{x\})$ for any $m \in \mathbb{N}$. Now let $x \in K(\{y\})$. Then there exists $m \in \mathbb{N}$ such that $x \in K_m(\{y\})$. Thus $y \in K_m(\{x\})$, this means that $y \in K(\{x\})$. Therefore, μ is symmetric. To show μ is transitive, let $x \mu y$ and $y \mu z$ hold. Then $x \in K(\{y\}) = \mathcal{C}^*(\{y\})$, $y \in K(\{z\}) = \mathcal{C}^*(\{z\})$. Let A be a ternary strong $\mathcal{C}$-set containing $\{z\}$. Then $y \in A$ and $x \in K(\{y\}) = \mathcal{C}^*(\{y\}) \subseteq A$. So, x in any ternary strong $\mathcal{C}$-set containing $\{z\}$. Thus, $x \in \mathcal{C}^*(\{z\}) = K(\{z\})$. Hence $x \mu z$. □

Theorem 3.11. *The equivalence relation μ on a ternary hypersemiring S coincides with the fundamental relation δ^* on S.*

Proof. Let $x \mu y$ for some $x, y \in S$. If $x = y$, then $x \delta^* y$. Consider $x \neq y$, then there exists $m \in \mathbb{N}$ for which $x \in K_m(\{y\})$. So there exist $n \in \mathbb{N}$, $k_i \in \mathbb{Z}_0^+$ for $i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2k_i + 1$ such that $x \in \sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}$ and $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap K_{m-1}(\{y\}) \neq \phi$. Let $a_1 \in \sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap K_{m-1}(\{y\})$. Thus $x \delta a_1$ and $a_1 \in K_{m-1}(\{y\})$. In similar way, there exists $a_2 \in S$ such that $a_1 \delta a_2$, continuing this process m-times we get $x = a_0, a_1, a_2, ..., a_m = y$ such that $a_i \delta a_{i+1}$ for $i \in \{1, 2, ..., m-1\}$. Hence $x\delta^* y$.

Conversely, let $x \delta y$. Then there exist $n \in \mathbb{N}, k_i \in \mathbb{Z}_0^+, i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2k_i + 1$ such that

$$x = y \text{ or } \{x, y\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}.$$

Thus $x \in K_2(\{y\})$ and so $x \in K(\{y\})$. Hence $\delta \subseteq \mu$. Since μ is an equivalence relation, we have $\delta^* \subseteq \mu$ and the proof is complete. □

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