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Introduction to Multi Intuitionistic Fuzzy Lie Algebras

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Abstract In this paper, we introduce for the first time the concept of multi intuitionistic fuzzy (briefly, MIF) Lie sub algebras and multi intuitionistic fuzzy (briefly, MIF) Lie ideals of Lie algebra. Some of their fundamental properties and operations like intersection and Cartesian product are investigated. Moreover, the relationship between MIF Lie sub algebras and MIF Lie ideals are established. Finally, the images and the inverse images of both MIF Lie sub algebras and MIF Lie ideals under Lie homomorphisms are also studied.

1 Introduction

In artificial intelligence era, science and technology deal with intricate phenomena and processes for which there is inadequate information. In such kinds of situations, mathematical models are modified for dealing with different types of systems that have uncertainty components. Most of the models, like fuzzy sets [13], soft sets [6],fuzzy soft sets [16], neutrosophic sets [9], multi neutrosophic sets [8], intuitionistic fuzzy Sets [12], Fermatean fuzzy sets [25] were built for extensions of standard set theocratic model. In this paper we cover the core features of a multi intuitionistic fuzzy Lie subalgebra of Lie algebra. Most probably numerous domains along with signal processing, artificial intelligence, multiagent systems, computer networks, robotics, genetic algorithms, expert systems, neural networks, decision making, medical diagnosis and automata theory shall be benefited with the acquired outcomes. The tangent space at identity element of a Lie group gives us its Lie algebra introduced by Sophus Lie (1842–1899) in an effort to categorise certain smooth subgroups of general linear groups [11]. Sometimes it is convenient and convertible to consider a problem on Lie groups and reduce it into a problem on Lie algebra. Lie algebra has wider applications than others, in different branches of physics and mathematics, such as spectroscopy of molecules, atoms, hyperbolic and stochastic differential equations and so on. Various types of Lie algebras, Lie ideals and their properties were explored and investigated in [2],[14],[15],[20],[22] and [23]. L.A. Zadeh [13] proposed the concept of fuzzy set (FS) in situations that are vague, imprecise and uncertain. As a generalization of fuzzy set, K. Atanassov [12] created intuitionistic fuzzy set (IFS) in 1986. His theory thereafter became widely acknowledged as an essential resource in the fields of engineering [17] and medicine [24]. In 1995, neutrosophic set (in short,NS) was introduced by F. Smarandache [9] as a generalization of intuitionistic fuzzy set. He generalizes IFS into neutrosophic set theory where he added a new component called indeterminacy. He addresses imprecision, indeterminacy, and inconsistent data in his neutrosophic theory.Neutrosophic logic theory and applications were developed by E. Abo EI-Hamed et al. [7]. Moreover, F. Smarandache [8] initiated the multi intuitionistic fuzzy set which is the stronger form of multi neutrosophic sets [8].Diagnosing psychiatric disorder using neutrosophic soft set and its application presented deliberately by V. Chinnadurai et al. [1]. Some aspects on Neutrosophic β-Ideal of β-Algebra were discussed by P. Muralikrishna and S. Manokaran [18] . After the invention of the notion of fuzzy Lie algebra introduced by

B.Davvaz [2] and properties of fuzzy Lie algebra introduced by Q. Keyun, Q. Quanxi and C. Chaoping [19] some important notions had been initiated and investigated such as intuitionistic fuzzy Lie algebra initiated by M. Akram and K.P. Shum [15] and neutrosophic Lie algebra coined by S. Abdullayev and L. Nesibova [20]. In 2023, S. Abdullayev, K. Veliyeva and S. Varamov [21] presented the concept of neutrosophic soft Lie algebra in the light of neutrosophic soft sets and studied their related Lie ideals. Recently, B. Debnath [4], [3] studied and investigated Lie algebric Structure of multi spherical fuzzy sets and multi Fermatean fuzzy sets respectably. In the present paper we introduce for the first time the concept of multi intuitionistic fuzzy Lie subalgebra and Lie ideals.
Our work is organized into the following five sections: we presented the introduction and literature review in the first section. Second section focuses into common definitions and preliminaries. Third section describes the concept of multi IF Lie sub algebras and multi IF Lie ideals of Lie algebra. Some of their fundamental properties and operations like intersection and generalized cartesian product are investigated. Moreover, the relationship between multi IF Lie sub algebras and multi IF Lie ideals are established. In section four, we investigate the images and the inverse images of multi IF Lie sub algebras and multi neutrosophic Lie ideals under Lie homomorphisms. In section five, we give the conclusion of the newly defined concept of multi IF Lie sub algebras and multi IF Lie ideals.

2 Preliminerie s

This section consists of some common notations and definitions which have been involved in the course of the paper.
A Lie algebra is a vector space $\mathcal{L}$ over the field $\mathcal{F}$ ($\mathcal{L}$= R or C) on which $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ defined by (γ, μ)=$[\gamma, \mu]$ for all $\gamma, \mu \in \mathcal{L}$ where $[\gamma, \mu]$ is called Lie Bracket satisfying the following conditions:
(1) $[\gamma, \mu]$ is bilinear
(2) $[\gamma, \gamma]$=0, for all $\gamma \in \mathcal{L}$
(3)$[[\gamma, \mu], \lambda]+[[\mu, \lambda], \gamma]+[[\lambda, \gamma], \mu]=0$, for all $\gamma, \mu, \lambda \in \mathcal{L}$.
(This is called Jacobi identity).
Throughout the paper $\mathcal{L}$ will denote Lie algebra and also we note that the operation Lie bracket [.,.] is neither associative nor commutative i.e,$[[\gamma, \mu], \lambda] \neq [[\mu, \lambda],\gamma]$ and $[\gamma, \mu] \neq [\mu, \gamma]$. But the operation Lie bracket [.,.] is anti-commutative i.e. $[\gamma, \mu] = -[\mu, \gamma]$.
A subspace H of $\mathcal{L}$ is called a Lie sub algebra if it is closed under [. , .].
A subspace I of $\mathcal{L}$ is called a Lie ideal if $[I,L] \subset I$. It is always true that every Lie ideal is Lie sub algebra.

Definition 2.1. [13] Let U be a universe of discourse. Then the fuzzy set on U is described as F= $\{(x, \mu(x))$: x$\in$X, where, $\mu(x) \in [0,1]$, denotes the degree of membership of x$\in$X$\}$.

Definition 2.2. [6] Let U be an initial universe and E be a set of parameters. Let P(U) denotes the power set of U and A$\subseteq$ E. Then the pair (F, A) is called a soft set over U, where F is a mapping given by F: A$\to$ P(U).

Definition 2.3. [16] Let U be an initial universe and E be a set of parameters. Let F(U) denotes the set of all fuzzy sets of U. Then the fuzzy soft set F_A can be represented by the set of ordered pairs $F_A=\{(x,\gamma_A(x)):x\in E, \gamma_A(x)\in F(U)\}$.

Definition 2.4. [12] Let U be a universe of discourse. Then the intuitonistic fuzzy set on U is described as F= $\{(x, \mu(x), \nu(x))$: x$\in$X, where, $\mu(x), \nu(x) \in [0,1]$, indicating the degrees of membership and non-membership respectively such that $0 \leq \mu_A(x)+\nu_A(x) \leq 1\}$.

Definition 2.5. [15] An IF set W= (τ, η) on $\mathcal{L}$ is said to be IF sub algebra if, $\forall$ $\gamma,\mu \in \mathcal{L}$; $\forall c\in \mathcal{F}$, the following assumptions hold good:
(1) $\tau_W(\gamma+\mu) \geq \text{Min}\{\tau_W(\gamma),\tau_W(\mu)\}$, $\eta_W(\gamma+\mu) \leq \text{Max}\{\eta_W(\gamma), \eta_W(\mu)\}$
(2) $\tau_W(c\gamma) \geq \tau_W(\gamma)$, $\eta_W(c\gamma) \leq \eta_W(\gamma)$
(3) $\tau_W[\gamma,\mu] \geq \text{Min}\{\tau_W(\gamma),\tau_W(\mu)\}$, $\eta_W[\gamma,\mu] \leq \text{Max}\{\eta_W(\gamma), \eta_W(\mu)\}$

Definition 2.6. [15] A IFS W= (τ, λ, η) on $\mathcal{L}$ is said to be IF Lie ideal if, $\forall$ γ,μ $\in\mathcal{L}$; $\forall$c$\in$ $\mathcal{F}$, the conditions (1) and (2) of definition (2.1) and the condition: τ_W ([γ,μ]) $\geq$ τ_W (γ), η_W ([γ,μ]) $\leq$ $\eta_W(\gamma)$, holds good.

Definition 2.7. [25] Let U be a Universe of discourse. Then the Fermatean fuzzy set on U is described as F= {(x, μ(x), ν(x)): x$\in$X, where, μ(x), ν(x) $\in$ [0,1], indicating the degrees of membership and non-membership respectively satisfying 0 $\leq$ μ_F^3(x)+ν_F^3(x)$\leq$1}.

Definition 2.8. [9] Let X be a universe of discourse. Then the neutrosophic set is defined by N= {(x, τ(x), λ(x), η(x)), x$\in$X, where, τ, λ, η $\in$ [0,1], indicating the degrees of truth, indeterminacy and falsehood respectively that satisfy 0 $\leq$ inf(τ) + inf(λ)+ inf(η) $\leq$ sup(τ) + sup(λ)+sup(η) $\leq$3}

Definition 2.9. [8] Let X be a universe of discourse. Then, a multi intuitionistic fuzzy set (shortly, multi IFS) on X is defined by M= {(x, x(τ_1,τ_2,...,τ_p; η_1,η_2,...,η_s)): x$\in$X, where p and s are integers $\geq$ 0,p+s=n $\geq$2, and at least one of p and s is $\geq$2, in order to ensure the existence of multiplicity of at least one intuitionistic component: truth or falsehood; all subsets τ_1,τ_2,...,τ_p; η_1,η_2,...,η_s $\subseteq$ [0,1]; 0 $\leq$ $\sum_{i=1}^{p}$ inf(τ_i) +$\sum_{k=1}^{s}$ inf(η_k) $\leq$ $\sum_{i=1}^{p}$ sup(τ_i) +$\sum_{k=1}^{s}$ sup(η_k) $\leq$ n}

Definition 2.10. [8] Let X be a universe of discourse. Then, a multi neutrosophic set on X is defined by M= {(x, x(τ_1,τ_2,...,τ_p;λ_1,λ_2,...,λ_q; η_1,η_2,...,η_s)): x$\in$X, where p, q, s are integers $\geq$ 0,p+q+s=n $\geq$2, and at least one of p,q,s is $\geq$2, in order to ensure the existence of multiplicity of at least one neutrosophic component: truth, indeterminacy, or falsehood; all subsets τ_1,τ_2,...,τ_p; λ_1,λ_2,...,λ_q; η_1,η_2,...,η_s $\subseteq$ [0,1]; 0 $\leq$ $\sum_{i=1}^{p}$ inf(τ_i) +$\sum_{j=1}^{q}$ inf(λ_j)+$\sum_{k=1}^{s}$ inf(η_k) $\leq$ $\sum_{i=1}^{p}$ sup(τ_i) +$\sum_{j=1}^{q}$ sup(λ_j)+$\sum_{k=1}^{s}$ sup(η_k) $\leq$ n}

3 Multi intuitionistic fuzzy Lie algebra and basic properties

In this section, the notion of multi IF Lie sub algebra is initiated as a stronger version of multi neutrosophic Lie sub algebra. Some characterizations, counter examples, and basic properties are also investigated.

Definition 3.1. Let N$_l$={1,2,....,l}, N$_n$={1,2,....,n} and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. A multi intuitionistic fuzzy set W = (τ_W^1, τ_W^2,...,τ_W^l; η_W^1,η_W^2,...,η_W^n) on $\mathcal{L}$ is said to be multi intuitionistic fuzzy Lie sub algebra over $\mathcal{L}$ if, $\forall$ γ,η $\in\mathcal{L}$; $\forall$ i$\in$N$_l$, $\forall$ j$\in$N$_m$,$\forall$ k$\in$N$_n$, $\forall$c$\in$ $\mathcal{F}$, the following assumptions hold good:
(1) τ_W^i (γ+μ) $\geq$ Min{τ_W^i (γ),τ_W^i (μ)}, η_W^k (γ+μ) $\leq$ Max{$\eta_W^k(\gamma)$, $\eta_W^k(\mu)$}
(2) τ_W^i (cγ) $\geq$ τ_W^i (γ), η_W^k (cγ) $\leq$ $\eta_W^k(\gamma)$
(3) τ_W^i [γ,μ] $\geq$ Min{τ_W^i (γ),τ_W^i (μ)}, η_W^k [γ,μ] $\leq$ Max{$\eta_W^k(\gamma)$, $\eta_W^k(\mu)$}

Definition 3.2. Let N$_l$={1,2,....,l}, N$_n$={1,2,....,n} and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. A multi intuitionistic fuzzy set W = (τ_W^1, τ_W^2,...,τ_W^l; η_W^1,η_W^2,...,η_W^n) on $\mathcal{L}$ is said to be multi intuitionistic fuzzy Lie ideal if,$\forall$ γ,η $\in\mathcal{L}$; $\forall$ i$\in$N$_l$, $\forall$ j$\in$N$_m$,$\forall$ k$\in$N$_n$, $\forall$c$\in$ $\mathcal{F}$, the conditions (1)-(2) of definition (3.1) along with the following conditions are satisfied: $\forall$ γ,η $\in\mathcal{L}$,
(4) τ_W^i [γ,μ] $\geq$ τ_W^i (γ), η_W^k [γ,μ] $\leq$ $\eta_W^k(\gamma)$
It follows from condition (2) that
(5) τ_W^i (0) $\geq$ τ_W^i (γ), η_W^k (0) $\leq$ $\eta_W^k(\gamma)$
(6) τ_W^i (-γ) $\geq$ τ_W^i (γ), η_W^k (-γ) $\leq$ $\eta_W^k(\gamma)$

Theorem 3.3. *Let M = ($\mathcal{L}$, τ_M^1, τ_M^2,...,τ_M^l; η_M^1,η_M^2,...,η_M^n) be a MIF Lie ideal over $\mathcal{L}$ and let N$_l$={1,2,....,l}, N$_n$={1,2,....,n}, then τ_M^i(0)= Sup{$\tau_M^i(\gamma)$: $\gamma\in\mathcal{L}$} and η_M^k(0)= Inf{$\eta_M^k(\gamma)$: $\gamma\in\mathcal{L}$},$\forall$ i$\in$N$_l$,$\forall$ k$\in$N$_n$.*

Proof. From condition (5) of definition 3.2, we have,
τ_M^i (0) $\geq$ τ_M^i (γ)...........(1)
η_M^k (0) $\leq$ $\eta_M^k(\gamma)$.............(2)
As γ runs over $\mathcal{L}$, the results follow just taking supremum on both sides of above inequality (1) and infimum on (2). □

Theorem 3.4. *Let M = ($\mathcal{L}$, $\{\tau_M^i\}_{i=1}^{l}$, $\{\eta_M^k\}_{k=1}^{n}$) be a MIF Lie ideal over $\mathcal{L}$. Then for each ψ, σ $\in$[0,1] satisfying τ_M^i (0) $\geq$ ψ, η_M^k (0) $\leq$ σ and 0 $\leq$ ψ+ σ $\leq$ 1, then (ψ, σ)-level subset $\mathcal{L}_M^{(\psi,\sigma)}$ is a multi intuitionistic fuzzy Lie ideal of $\mathcal{L}$.*

Proof. Straight forward from definition. □

Theorem 3.5. *If δ is a fixed element of $\mathcal{L}$ and W = ($\mathcal{L}$, $\{\tau^i_W\}^l_{i=1}$, $\{\eta^k_W\}^n_{k=1}$) is a multi intuitionistic fuzzy Lie ideal of $\mathcal{L}$. Then the set defined by W^δ = $\{\gamma\in \mathcal{L}$: $\tau^i_W(\gamma) \geq \tau^i_W(\delta)$, $\eta^k_W(\gamma) \leq \eta^k_W(\delta)\}$ is a multi intuitionistic fuzzy Lie ideal of $\mathcal{L}$.*

Proof. Suppose that $\gamma,\mu \in W^\delta$, $i\in N_l$, $k\in N_n$.
Then $\forall\gamma,\mu \in W^\delta$, $\forall$ $i\in N_l$, $\forall$ $k\in N_n$
$\tau^i_W(\gamma+\mu) \geq \text{Min}\{\tau^i_W(\gamma),\tau^i_W(\mu)\}\geq\tau^i_W(\delta)$; $\eta^k_W(\gamma+\mu) \leq \text{Max}\{\eta^k_W(\gamma), \eta^k_W(\mu)\} \leq\eta^i_W(\delta)$
This implies that $\gamma+\mu \in W^\delta$
Now, $\forall\gamma \in W^\delta$,$\forall$ $i\in N_l$,$\forall$ $k\in N_n$, $\forall c\in \mathcal{F}$,
$\tau^i_W(c\gamma) \geq \tau^i_W(\gamma)\geq\tau^i_W(\delta)$; $\eta^k_W(c\gamma) \leq \eta^k_W(\gamma) \leq \eta^i_W(\delta)$
$\Rightarrow c\gamma \in W^\delta$
Also for every $\gamma\in W^\delta$ and for every $\mu \in W^\delta$,$\forall$ $i\in N_l$, $\forall$ $j\in N_m$,$\forall$ $k\in N_n$
$\tau^i_W[\gamma,\mu] \geq \text{Min}\{\tau^i_W(\gamma),\tau^i_W(\mu)\}\geq \tau^i_W(\delta)$;
$\eta^k_W[\gamma,\mu] \leq \text{Max}\{\eta^k_W(\gamma), \eta^k_W(\mu)\} \leq \eta^i_W(\delta)$, which shows that $[\gamma,\eta] \in W^\delta$.
Hence, W^δ is a multi intuitionistic fuzzy Lie ideal of $\mathcal{L}$. □

Theorem 3.6. *If W = ($\mathcal{L}$, $\{\tau^i_W\}^l_{i=1}$, $\{\lambda^j_W\}^m_{j=1}$, $\{\eta^k_W\}^n_{k=1}$) is a multi intuitionistic fuzzy Lie ideal of $\mathcal{L}$. Then the set defined by W^0 = $\{\gamma\in \mathcal{L}$: $\tau^i_W(\gamma) \geq \tau^i_W(0)$, $\lambda^j_W(\gamma) \geq \lambda^j_W(0)$, $\eta^k_W(\gamma) \leq \eta^k_W(0)$, $\forall$ $i\in N_l$, $\forall$ $j\in N_m$,$\forall$ $k\in N_n\}$ is a multi intuitionistic fuzzy Lie ideal of $\mathcal{L}$.*

Proof. Straight forward. □

Theorem 3.7. *Let W = ($\{\tau^i_W\}^l_{i=1}$, $\{\eta^k_W\}^n_{k=1}$) be a multi intuitionistic fuzzy Lie sub algebra of a Lie algebra $\mathcal{L}$ and $\mathcal{R} \subseteq \mathcal{L} \times \mathcal{L}$ be a binary relation on $\mathcal{L}$ defined by $\mathcal{R} =\{(\gamma,\mu) \in \mathcal{L} \times \mathcal{L} \mid \tau^i_W(\gamma\text{-}\mu) =\tau^i_W(0)$, $\lambda^j_W(\gamma\text{-}\mu) =\lambda^j_W(0)$, $\eta^k_W(\gamma\text{-}\mu) =\eta^k_W(0)$, $\gamma,\mu \in \mathcal{L}$, $i\in N_l$, $k\in N_n\}$, then $\mathcal{R}$ is a congruence relation on $\mathcal{L}$.*

Proof. First of all we need to prove that the relation $\mathcal{R}$ is equivalence relation on $\mathcal{L}$. Now
(i) Reflexivity: Since $\forall$ $\gamma \in \mathcal{L}$, $\tau^i_W(\gamma\text{-}\gamma) =\tau^i_W(0)$, $\eta^k_W(\gamma\text{-}\gamma) =\eta^k_W(0)$, thus, $(\gamma,\gamma) \in \mathcal{R}$, $\forall$ $\gamma \in \mathcal{L}$,$\forall$ $i\in N_l$,$\forall$ $k\in N_n$, and consequently $\mathcal{R}$ is reflexive relation on $\mathcal{L}$.
(ii) Symmetric: Let, $(\gamma,\mu) \in \mathcal{R}$. Then $\tau^i_W(\gamma\text{-}\mu) = \tau^i_W(0)$
$\Rightarrow \tau^i_W(\text{-}(\mu\text{-}\gamma)) \geq \tau^i_W((\mu\text{-}\gamma)) = \tau^i_W(0)$,
$\eta^k_W(\gamma\text{-}\mu) =\eta^k_W(0)$, $\Rightarrow \eta^k_W(\text{-}(\mu\text{-}\gamma)) \leq \eta^j_W((\mu\text{-}\gamma)) = \eta^j_W(0)$,
Thus, $(\mu,\gamma) \in \mathcal{R}$, $\forall$ γ, $\mu \in \mathcal{L}$,$\forall$ $i\in N_l$, $\forall$ $k\in N_n$, so that $\mathcal{R}$ is symmetric relation on $\mathcal{L}$.
(iii) Transitive: Let, (γ,μ), $(\mu,\sigma)\in \mathcal{R}$. Then $\tau^i_W(\gamma\text{-}\mu) = \tau^i_W(0)$, $\tau^i_W(\mu\text{-}\sigma) = \tau^i_W(0)$
$\eta^k_W(\gamma\text{-}\mu) = \eta^k_W(0)$, $\eta^k_W(\mu\text{-}\sigma) = \eta^k_W(0)$
From which we have, $\tau^i_W(\gamma\text{-}\sigma)= \tau^i_W\{(\gamma\text{-}\mu)+ (\mu\text{-}\sigma)\} \geq \text{Min}\ \{\tau^i_W(\gamma\text{-}\mu),\tau^i_W(\mu\text{-}\sigma)\} = \tau^i_W(0)$,
$\eta^k_W(\gamma\text{-}\sigma)= \eta^k_W\{(\gamma\text{-}\mu)+ (\mu\text{-}\sigma)\} \leq \text{Max}\ \{\eta^k_W(\gamma\text{-}\mu),\eta^k_W(\mu\text{-}\sigma)\} = \eta^k_W(0)$. Hence, $(\gamma,\sigma) \in\mathcal{L}$,$\forall$ $i\in N_l$, $\forall$ $k\in N_n$ and consequently, $\mathcal{R}$ is transitive relation on $\mathcal{L}$. Hence, $\mathcal{R}$ is an equivalence relation on $\mathcal{L}$.
We now verify that $\mathcal{R}$ is an congruence relation on $\mathcal{L}$ and for that let us take (γ,μ), $(\mu,\sigma) \in \mathcal{R}$.
Then $\tau^i_W(\gamma\text{-}\mu) = \tau^i_W(0)$, $\tau^i_W(\mu\text{-}\sigma) = \tau^i_W(0)$
$\eta^k_W(\gamma\text{-}\mu) = \eta^k_W(0)$, $\eta^k_W(\mu\text{-}\sigma) = \eta^k_W(0)$
Now if γ_1, γ_2, μ_1, $\mu_2 \in \mathcal{R}$, then we must have,
$\tau^i_W\{(\gamma_1+\gamma_2)\text{-}\ (\mu_1+ \mu_2)\} = \tau^i_W\{(\gamma_1\text{-}\mu_1)+ (\gamma_2\text{-}\mu_2)\} \geq \text{Min}\ \{\tau^i_W(\gamma_1\text{-}\mu_1),\tau^i_W(\gamma_2\text{-}\mu_2)\} = \tau^i_W(0)$,
$\eta^k_W\{(\gamma_1+\gamma_2)\text{-}\ (\mu_1+ \mu_2)\} =\eta^k_W\{(\gamma_1\text{-}\mu_1)+ (\gamma_2\text{-}\mu_2)\} \leq \text{Max}\ \{\eta^k_W(\gamma_1\text{-}\mu_1),\eta^k_W(\gamma_2\text{-}\mu_2)\} = \eta^k_W(0)$,
$\tau^i_W(c\gamma_1\text{-}c\mu_1) =\tau^i_W\{c(\gamma_1\text{-}\mu_1)\}\geq \tau^i_W(\gamma_1\text{-}\mu_1) =\tau^i_W(0)$,
$\eta^k_W(c\gamma_1\text{-}c\mu_1) =\eta^k_W\{c(\gamma_1\text{-}\mu_1)\}\geq \eta^k_W(\gamma_1\text{-}\mu_1) =\eta^k_W(0)$,
$\tau^i_W\{[\gamma_1,\gamma_2]\text{-}\ [\mu_1, \mu_2]\} = \tau^i_W[(\gamma_1\text{-}\mu_1), (\gamma_2\text{-}\mu_2)] \geq \text{Min}\ \{\tau^i_W(\gamma_1\text{-}\mu_1),\tau^i_W(\gamma_2\text{-}\mu_2)\} = \tau^i_W(0)$,
$\eta^k_W\{[\gamma_1+\gamma_2]\text{-}\ [\mu_1+ \mu_2]\} =\eta^k_W[(\gamma_1\text{-}\mu_1), (\gamma_2\text{-}\mu_2)] \leq \text{Max}\ \{\eta^k_W(\gamma_1\text{-}\mu_1),\eta^k_W(\gamma_2\text{-}\mu_2)\} = \eta^k_W(0)$,
Thus, $(\gamma_1+\gamma_2)\ \mathcal{R}\ (\mu_1+ \mu_2)$, $c\gamma_1\ \mathcal{R}\ c\mu_1$ and $[\gamma_1,\gamma_2]\ \mathcal{R}\ [\mu_1, \mu_2]$. Hence, $\mathcal{R}$ is a congruence relation on $\mathcal{L}$. □

Theorem 3.8. *Every multi intuitionistic fuzzy Lie ideal is multi intuitionistic fuzzy Lie sub algebra.*

Proof. Straight forward from definition.
The converse of the above theorem (3.8) is not true which can be seen from the following example: □

Example 3.9. Suppose that F=$\mathbb{R}$, the set of real numbers and $\mathcal{L}$ = {(α,β,γ): α,β,$\gamma$$\in$$\mathbb{R}$} be the Lie aigebra. Let us define the mapping $\mathcal{L}\times\mathcal{L}\to\mathcal{L}$ by [u,v]=u$\times$v+v, where $\times$ denotes the vector product (or cross product). Consider the multi intuitionistic fuzzy set W= (τ^1_W, τ^2_W,...,τ^4_W; η^1_W,η^2_W,...,η^5_W) : $\mathcal{L}\to$ [0,1]$\times$[0,1]$\times$[0,1]$\times$[0,1]$\times$[0,1]$\times$[0,1]$\times$[0,1]$\times$[0,1]$\times$[0,1] described by
τ^i_W(α,β,γ)=
{$\frac{o.8}{i}$, if α=β=γ=0,
$\frac{o.6}{i}$, if $\alpha\neq$0, β=γ=0,
0, otherwise} for i=1,2,3,4.
η^k_W(α,β,γ)=
{0, if α=β=γ=0,
$\frac{o.4}{k}$, if $\alpha\neq$0, β=γ=0,
$\frac{1}{k}$, otherwise} for k=1,2,3,4,5.
Then it is easy to verify that W is a multi intuitionistic fuzzy Lie sub algebra of $\mathcal{L}$ but it is not multi neutrosophic Lie ideal of $\mathcal{L}$ because for all i=1,2,3,4, τ^i_W[(2,0,0),(i,-i,i)]= τ^i_W(i,-3i,-i)=0 $\ngeq$ $\frac{o.6}{i}$ =τ^i_W(2,0,0).

Theorem 3.10. *Let N$_l$={1,2,....,l}, N$_n$={1,2,....,n} and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. The necessary and sufficient condition for a multi intuitionistic fuzzy set W = (τ^1_W, τ^2_W,...,τ^l_W; η^1_W,η^2_W,...,η^n_W) to be a multi intuitionistic fuzzy Lie sub algebra over $\mathcal{L}$ is that, $\forall$r$\in$[0,1], non-empty upper r-level cut $U_r(\tau^i_W)$ = {$\gamma\in\mathcal{L}$: $\tau^i_W(\gamma)\geq$ r, $\forall$ i$\in$N$_l$} and non-empty lower r-level cut $V_r(\eta^k_W)$ = {$\gamma\in\mathcal{L}$: $\eta^k_W(\gamma)\leq$ r, $\forall$ k$\in$N$_n$} are Lie sub algebra over $\mathcal{L}$.*

Proof. Suppose that W = (τ^1_W, τ^2_W,...,τ^l_W; η^1_W,η^2_W,...,η^n_W) is a multi intuitionistic fuzzy Lie sub algebra over $\mathcal{L}$ and r$\in$[0,1] is such that U$_r(\tau^i_W)\neq\phi$. Let γ,η $\in$ U$_r(\tau^i_W)$. Then $\forall$ γ,η $\in\mathcal{L}$; $\forall$ i$\in$N$_l$,$\forall$ k$\in$N$_n$, $\forall$c$\in$ $\mathcal{F}$,
τ^i_W (γ+μ) $\geq$ Min{τ^i_W (γ),τ^i_W (μ)}$\geq$r, η^k_W (γ+μ) $\leq$ Max{η^k_W(γ), η^k_W(μ)} $\leq$r
τ^i_W (cγ) $\geq$ τ^i_W (γ)$\geq$r , η^k_W (cγ) $\leq$ η^k_W(γ) $\leq$ r,
τ^i_W [γ,μ] $\geq$ Min{τ^i_W (γ),τ^i_W (μ)}$\geq$ r, η^k_W [γ,μ] $\leq$ Max{η^k_W(γ), η^k_W(μ)} $\leq$ r.
Thus, γ+μ,cγ, [γ,μ] $\in$ U$_r(\tau^i_W)$, γ+μ,cγ, [γ,μ] $\in$ V$_r(\eta^k_W)$.
Thus, U$_r(\tau^i_W)$ and V$_r(\eta^k_W)$ constitute Lie sub algebra over $\mathcal{L}$.
Conversely, suppose that $\forall$ i $\in$ i$\in$N$_l$ and $\forall$ r $\in$ [0,1], U$_r(\tau^i_W)\neq\phi$ is a Lie sub algebra over $\mathcal{L}$ and if possible suppose that τ^i_W (γ+μ) $\ngeq$ Min{τ^i_W (γ),τ^i_W (μ)} for some γ,μ $\in$ $\mathcal{L}$. If we choose r$_0$= $\frac{1}{2}$ (τ^i_W (γ+μ) + Min{τ^i_W (γ),τ^i_W (μ)}), the by properties of inequality we must have, τ^i_W (γ+μ) < r$_0$ < Min{τ^i_W (γ),τ^i_W (μ)}. This implies that, γ+μ $\notin$ U$_r(\tau^i_W)$, γ,μ $\in$ U$_r(\tau^i_W)$, which is a contradiction. Hence τ^i_W (γ+μ) $\geq$ Min{τ^i_W (γ),τ^i_W (μ)}, $\forall$ γ,μ $\in$ $\mathcal{L}$.
In a similar manner we can prove that
τ^i_W (cγ) $\geq$ τ^i_W (γ) and τ^i_W [γ,μ] $\geq$ Min{τ^i_W (γ),τ^i_W (μ)}, $\forall$c$\in$ $\mathcal{F}$, $\forall$ i$\in$N$_l$.
The proof is similar for the case V$_r(\eta^k_W)$. This completes the proof. □

Theorem 3.11. *Let N$_l$={1,2,....,l}, N$_n$={1,2,....,n} and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. If V = (τ^1_V, τ^2_V,...,τ^l_V; η^1_V,η^2_V,...,η^n_V) and W = (τ^1_W, τ^2_W,...,τ^l_W; η^1_W,η^2_W,...,η^n_W) are two multi intuitionistic fuzzy Lie sub algebra over $\mathcal{L}$, then their intersection V$\cap$W= H= (τ^1_H, τ^2_H,...,τ^l_H; η^1_H,η^2_H,...,η^n_H) is also multi intuitionistic fuzzy Lie sub algebra over $\mathcal{L}$.*

Proof. Suppose that γ,μ $\in$ $\mathcal{L}$ be arbitrary. Then $\forall$ i$\in$N$_l$, $\forall$ k$\in$N$_n$, $\forall$c$\in$ $\mathcal{F}$, we have,
τ^i_H(γ+μ) = Min{τ^i_V (γ+μ), τ^i_W (γ+μ)}
$\geq$ Min{Min{τ^i_V (γ),τ^i_V (μ)},Min{τ^i_W (γ),τ^i_W (μ)}}
=Min{Min{τ^i_V (γ),τ^i_W (γ)},Min{τ^i_V (μ),τ^i_W (μ)}}
=Min{τ^i_H(γ),τ^i_H(μ)}
η^k_H(γ+μ) = Max{η^k_V (γ+μ), η^k_W(γ+μ)}
$\leq$ Max{Max{η^k_V (γ),η^k_V(μ)},Max{η^k_W (γ),η^k_W(μ)}}
=Max{Max{η^k_V(γ),η^k_W (γ)},Max{η^k_V (μ),η^k_W (μ)}}
=Max{η^k_H(γ),η^k_H (μ)}
τ^i_H (cγ) = Min{τ^i_V (cγ), τ^i_W (cγ)} $\geq$ {Min{τ^i_V(γ), τ^i_W(γ)} =τ^i_H (γ)
η^k_H (cγ)= Max{η^k_V (cγ), η^k_W (cγ)} $\leq$ {Max{η^k_V (γ), η^k_W (γ)} =η^k_H (γ)
τ^i_H [γ,μ] = Min{τ^i_V [γ,μ], τ^i_W [γ,μ]}
$\geq$ Min {Min{τ^i_V (γ),τ^i_V (μ)}, Min{τ^i_W (γ),τ^i_W (μ)}}

= Min {Min{τ^i_V (γ),τ^i_W (γ)},Min {τ^i_V (μ),τ^i_W (μ)}}
= Min {τ^i_H (γ),τ^i_H (μ)}
η^k_H [γ, μ] = Max{ η^k_V [γ, μ], η^k_W [γ, μ]}
$\leq$ Max {Max{η^k_V (γ),η^k_V (μ)}, Max{η^k_W (γ), η^k_W (μ)}}
= Max {Max{η^k_V (γ), η^k_W (γ)},Max {η^k_V (μ), η^k_W (μ)}}
= Max {η^k_H (γ), η^k_H (μ)}
Hence, H= V∩W is multi intuitionistic fuzzy Lie sub algebra over $\mathcal{L}$. □

Definition 3.12. Let N$_l$={1,2,....,l}, N$_n$={1,2,....,n} and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. If V = (τ^1_V, τ^2_V,...,τ^l_V; η^1_V,η^2_V,...,η^n_V) and W = (τ^1_W, τ^2_W,...,τ^l_W; η^1_W,η^2_W,...,η^n_W) are two multi intuitionistic fuzzy sets on $\mathcal{L}$, then the product H= V×W defined on $\mathcal{L}$ ×$\mathcal{L}$ will be known as generalized cartasian product if (V×W) (γ,μ) =[(τ^1_V, τ^2_V,...,τ^l_V; η^1_V,η^2_V,...,η^n_V) × (τ^1_W, τ^2_W,...,τ^l_W; η^1_W,η^2_W,...,η^n_W)] (γ,μ) = (τ^1_H(γ,μ), τ^2_H(γ,μ),...,τ^l_H(γ,μ); η^1_H(γ,μ), η^2_H(γ,μ),...,η^n_H(γ,μ)), ∀ (γ,μ) ∈ $\mathcal{L}$ ×$\mathcal{L}$, where, τ^i_H(γ,μ)=(τ^i_V×τ^i_W)(γ,μ)= Min { τ^i_V (γ), τ^i_W(μ)}, ∀ i∈N$_l$ and
η^k_H(γ,μ)=(η^k_V×η^k_W)(γ,μ)= Max { η^k_V (γ), η^k_W(μ)}, ∀ k∈N$_n$.
Evidently the generalized cartasian product (V×W) is multi intuitionistic fuzzy set on $\mathcal{L}$ ×$\mathcal{L}$ if $0 \leq \tau^i_H(\gamma,\mu) + \eta^k_H(\gamma,\mu) \leq 2$
i.e., 0 ≤ Min{ τ^i_V (γ), τ^i_W(μ)} + Max { η^k_V (γ), η^k_W(μ)} ≤ 2, where, i∈N$_l$,k∈N$_n$.

Theorem 3.13. *Let $\mathcal{L}$ be the Lie Algebra of vectors over the field $\mathcal{F}$. If V = (τ^1_V, τ^2_V,...,τ^l_V; η^1_V,η^2_V,...,η^n_V) and W = (τ^1_W, τ^2_W,...,τ^l_W; η^1_W,η^2_W,...,η^n_W) are two multi intuitionistic fuzzy Lie sub algebra of $\mathcal{L}$, then the generalized cartasian product (V×W) is multi intuitionistic fuzzy Lie sub algebra of $\mathcal{L}$×$\mathcal{L}$.*

Proof. Let N$_l$={1,2,....,l}, N$_n$={1,2,....,n} and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. Then, ∀ i∈N$_l$, ∀ k∈N$_n$ ∀ γ=(γ_1,γ_2), μ=(μ_1,μ_2) ∈$\mathcal{L}$ ×$\mathcal{L}$ and c∈$\mathcal{F}$, we have,
(τ^i_V×τ^i_W)(γ+μ) =(τ^i_V×τ^i_W) ((γ_1,γ_2)+(μ_1,μ_2))
=(τ^i_V×τ^i_W) ((γ_1 +μ_1),(γ_2 +μ_2))
= Min{τ^i_V (γ_1 +μ_1), τ^i_W (γ_2 +μ_2)}
≥ Min{Min{τ^i_V (γ_1),τ^i_V (μ_1)},Min{τ^i_W (γ_2),τ^i_W (μ_2)}}
=Min{Min{τ^i_V (γ_1),τ^i_W (γ_2)},Min{τ^i_V (μ_1),τ^i_W (μ_2)}}
=Min{(τ^i_V×τ^i_W) (γ_1,γ_2),(τ^i_V×τ^i_W) (μ_1,μ_2) }
=Min{(τ^i_V×τ^i_W) (γ),(τ^i_V×τ^i_W) (μ) }
(η^k_V×η^k_W)(γ+μ) =(η^k_V×η^k_W) ((γ_1,γ_2)+(μ_1,μ_2))
=(η^k_V×η^k_W) ((γ_1 +μ_1),(γ_2 +μ_2))
= Max{η^k_V (γ_1 +μ_1), η^k_W (γ_2 +μ_2)}
≤ Max{Max{η^k_V (γ_1),η^k_V (μ_1)},Min{η^k_W (γ_2),η^k_W (μ_2)}}
=Max{Max{η^k_V (γ_1),η^k_W (γ_2)},Max{η^k_V (μ_1),η^k_W (μ_2)}}
=Max{(η^k_V×η^k_W) (γ_1,γ_2),(η^k_V×η^k_W) (μ_1,μ_2)}}
=Max{(η^k_V×η^k_W) (γ),(η^k_V×η^k_W) (μ)}
(τ^i_V×τ^i_W)(cγ) =(τ^i_V×τ^i_W) {c(γ_1,γ_2)}
=(τ^i_V×τ^i_W) (cγ_1,cγ_2)
= Min {τ^i_V (cγ_1),τ^i_W (cγ_2)}
≥ Min {τ^i_V (γ_1),τ^i_W (γ_2)}
=(τ^i_V×τ^i_W) (γ_1,γ_2)
=(τ^i_V×τ^i_W) (γ)
(η^k_V×η^k_W){(cγ)} =(η^k_V×η^k_W) {c(γ_1,γ_2)}
=(η^k_V×η^k_W) (cγ_1,cγ_2)
= Max {η^k_V (cγ_1),η^k_W (cγ_2)}
≤ Max {η^k_V (γ_1),η^k_W (γ_2)}
=(η^k_V×η^k_W) (γ_1,γ_2)
=(η^k_V×η^k_W) (γ)
(τ^i_V×τ^i_W)[γ,μ] =(τ^i_V×τ^i_W) [(γ_1,γ_2),(μ_1,μ_2)]
≥ Min {Min {τ^i_V (γ_1),τ^i_W (γ_2)},{Min {τ^i_V (μ_1),τ^i_W (μ_2)} }
= Min {(τ^i_V×τ^i_W) (γ_1,γ_2)},{(τ^i_V×τ^i_W) (μ_1,μ_2) }
=Min {(τ^i_V×τ^i_W) (γ), (τ^i_V×τ^i_W) (μ)}
(η^k_V×η^k_W)[γ,μ] =(η^k_V×η^k_W) [(γ_1,γ_2),(μ_1,μ_2)]
≤ Max {Max {η^k_V (γ_1),η^k_W (γ_2)},{Max {η^k_V (μ_1),η^k_W (μ_2)} }

= Max$\{(\eta_V^k \times \eta_W^k)(\gamma_1,\gamma_2)\},\{(\eta_V^k \times \eta_W^k)(\mu_1,\mu_2)\}$
=Max$\{(\eta_V^k \times \eta_W^k)(\gamma), (\eta_V^k \times \eta_W^k)(\mu)\}$
Hence, (V×W) is multi intuitionistic fuzzy Lie sub-algebra of $\mathcal{L}\times\mathcal{L}$. □

4 Multi intuitionistic fuzzy Lie algebra homomorphisms

In this section, the properties of multi intuitionistic fuzzy Lie sub algebra and multi intuitionistic fuzzy Lie ideals under homomorphisms of Lie algebras are investigated. Also the preservation aspects are examined.

Definition 4.1. Let L_1 and L_2 be two Lie algebras over the field $\mathcal{F}$. A linear transformation φ: $L_1 \to L_2$ is said to be Lie homomorphism if the relationship $\varphi([\gamma,\mu])=[\varphi(\gamma),\varphi(\mu)]$ is true, $\forall\ \gamma,\mu \in L_1$.

Definition 4.2. Let L_1 and L_2 be two Lie algebras over the field $\mathcal{F}$. Then a Lie homomorphism φ: $L_1 \to L_2$ is said have natural extension φ: $I^{L_1} \to I^{L_2}$ if $\forall$ W = $(\tau_W^1, \tau_W^2,...,\tau_W^l; \lambda_W^1,\lambda_W^2....,\lambda_W^m; \eta_W^1,\eta_W^2,...,\eta_W^n) \in I^{L_1}$ and $\mu \in L_2$, the followings hold:
$\varphi(\tau_W^i)(\mu)=$ Sup$\{\tau_W^i(\gamma)$: $\gamma\in \varphi^{-1}(\mu),\gamma\in L_1\}$, for all i=1,2,3,...,l.
$\varphi(\eta_W^k)(\mu)=$ Inf$\{\eta_W^k(\gamma)$: $\gamma\in \varphi^{-1}(\mu),\gamma\in L_1\}$, for all k=1,2,3,...,n.

Theorem 4.3. *Let W = $(\tau_W^1, \tau_W^2,...,\tau_W^l; \eta_W^1,\eta_W^2,...,\eta_W^n) \in I^{L_1}$ be multi intuitionistic fuzzy Lie sub algebras and φ: $L_1 \to L_2$ be Lie homomorphism between L_1 and L_2. Then φ (W) is multi intuitionistic fuzzy Lie sub algebras of L_2.*

Proof. Suppose that μ_1, $\mu_2 \in L_2$. Then $\{\gamma: \gamma\in \varphi^{-1}(\mu_1+\mu_2)\} \supseteq \{\gamma_1+\gamma_2:\gamma_1\in \varphi^{-1}(\mu_1),\gamma_2\in \varphi^{-1}(\mu_2)\}$. Now, for all i=1,2,3,...,l, we have,
$\varphi(\tau_W^i)(\mu_1+\mu_2)=$ Sup$\{\tau_W^i(\gamma)$: $\gamma\in \varphi^{-1}(\mu_1+\mu_2),\gamma\in L_1\}$
$\geq \{\tau_W^i(\gamma_1+\gamma_2) :\gamma_1\in \varphi^{-1}(\mu_1),\gamma_2\in \varphi^{-1}(\mu_2)\}$
$\geq$ Sup$\{$ Min$\{ \tau_W^i(\gamma_1),\tau_W^i(\gamma_2)\} :\gamma_1\in \varphi^{-1}(\mu_1),\gamma_2\in \varphi^{-1}(\mu_2)\}$
= Min$\{$Sup$\{\tau_W^i(\gamma_1) :\gamma_1\in \varphi^{-1}(\mu_1)\}$, Sup$\{\tau_W^i(\gamma_2) :\gamma_2\in \varphi^{-1}(\mu_2)\}\}$
= Min$\{\varphi(\tau_W^i)(\mu_1),\varphi(\tau_W^i)(\mu_2)\}$
and for all k=1,2,3,...,n, we have,
$\varphi(\eta_W^k)(\mu_1+\mu_2)=$ Inf$\{\eta_W^k(\gamma)$: $\gamma\in \varphi^{-1}(\mu_1+\mu_2),\gamma\in L_1\}$
$\leq \{\eta_W^k(\gamma_1+\gamma_2) :\gamma_1\in \varphi^{-1}(\mu_1),\gamma_2\in \varphi^{-1}(\mu_2)\}$
$\leq$ Inf$\{$ Max$\{ \eta_W^k(\gamma_1),\eta_W^k(\gamma_2)\} :\gamma_1\in \varphi^{-1}(\mu_1),\gamma_2\in \varphi^{-1}(\mu_2)\}$
= Max$\{$Inf$\{\eta_W^k(\gamma_1) :\gamma_1\in \varphi^{-1}(\mu_1)\}$, Inf$\{\eta_W^k(\gamma_2) :\gamma_2\in \varphi^{-1}(\mu_2)\}\}$
= Max$\{\varphi(\eta_W^k)(\mu_1),\varphi(\eta_W^k)(\mu_2)\}$
For $\mu \in L_2$ and c$\in \mathcal{F}$, we have,
$\{\gamma$: $\gamma\in \varphi^{-1}(c\mu_1)\} \supseteq \{c\gamma:\gamma\in \varphi^{-1}(\mu)\}$.
Now, for all i=1,2,3,...,l, we have,
$\varphi(\tau_W^i)(c\mu)=$ Sup$\{\tau_W^i(c\gamma)$: $\gamma\in \varphi^{-1}(\mu),\gamma\in L_1\}$
$\geq$ Sup$\{\tau_W^i(c\gamma)$: $\gamma\in \varphi^{-1}(c\mu),\gamma\in L_1\}$
$\geq$ Sup$\{\tau_W^i(\gamma)$: $\gamma\in \varphi^{-1}(\mu),\gamma\in L_1\}$
= $\varphi(\tau_W^i)(\mu)$
Similarly, for all k=1,2,3,...,m, we can we prove that
$\varphi(\eta_W^k)(c\mu)=$ Inf$\{\eta_W^k(c\gamma)$: $\gamma\in \varphi^{-1}(\mu),\gamma\in L_1\}$
$\leq$ Inf$\{\eta_W^k(c\gamma)$: $\gamma\in \varphi^{-1}(c\mu),\gamma\in L_1\}$
$\leq$ Inf$\{\eta_W^k(\gamma)$: $\gamma\in \varphi^{-1}(\mu),\gamma\in L_1\}$
= $\varphi(\eta_W^k)(\mu)$
For, μ_1, $\mu_2 \in L_2$, then $\{\gamma$: $\gamma\in \varphi^{-1}(\mu_1+\mu_2)\} \supseteq \{\gamma_1+\gamma_2:\gamma_1\in \varphi^{-1}(\mu_1),\gamma_2\in \varphi^{-1}(\mu_2)\}$.
Now, for all i=1,2,3,...,l, we have,
$\varphi(\tau_W^i)([\mu_1,\mu_2])=$ Sup$\{\tau_W^i(\gamma)$: $\gamma\in \varphi^{-1}([\mu_1,\mu_2]),\gamma\in L_1\}$
$\geq$ Sup$\{\tau_W^i([\gamma_1,\gamma_2]) :\gamma_1\in \varphi^{-1}(\mu_1),\gamma_2\in \varphi^{-1}(\mu_2)\}$
$\geq$ Sup$\{$ Min$\{ \tau_W^i(\gamma_1),\tau_W^i(\gamma_2)\} :\gamma_1\in \varphi^{-1}(\mu_1),\gamma_2\in \varphi^{-1}(\mu_2)\}$
= Min$\{$Sup$\{\tau_W^i(\gamma_1) :\gamma_1\in \varphi^{-1}(\mu_1)\}$, Sup$\{\tau_W^i(\gamma_2) :\gamma_2\in \varphi^{-1}(\mu_2)\}\}$
= Min$\{\varphi(\tau_W^i)(\mu_1),\varphi(\tau_W^i)(\mu_2)\}$
Similarly, for all k=1,2,3,...,m, we can we prove that
$\varphi(\eta_W^k)([\mu_1,\mu_2])=$ Sup$\{\eta_W^k(\gamma)$: $\gamma\in \varphi^{-1}([\mu_1,\mu_2]),\gamma\in L_1\}$

$\geq$ Sup$\{\eta_W^k([\gamma_1,\gamma_2]) : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$
$\geq$ Sup$\{$ Min$\{ \eta_W^k(\gamma_1), \tau_W^k(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$
= Min$\{$Sup$\{\eta_W^k(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}$, Sup$\{\eta_W^i(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\}$
= Min$\{\varphi(\eta_W^k)(\mu_1), \varphi(\eta_W^k)(\mu_2)\}$
Hence, φ (W) is multi intuitionistic fuzzy Lie subalgebras of L_2. □

Theorem 4.4. *Let W = (τ_W^1, τ_W^2,...,τ_W^l; η_W^1,η_W^2,...,η_W^n) $\in I^{L_1}$ be multi intuitionistic fuzzy Lie ideal and φ: $L_1 \rightarrow L_2$ be Lie homomorphism between L_1 and L_2. Then φ (W) is multi intuitionistic fuzzy Lie ideal of L_2.*

Proof. The proof is similar to the proof of Theorem (4.3) □

Theorem 4.5. *Let W = (τ_W^1, τ_W^2,...,τ_W^l; η_W^1,η_W^2,...,η_W^n) $\in I^{L_2}$ be multi intuitionistic fuzzy Lie sub algebra and φ: $L_1 \rightarrow L_2$ be Lie homomorphism between L_1 and L_2. Then φ^{-1}(W) is multi intuitionistic fuzzy Lie sub algebra of L_1.*

Proof. Suppose that μ_1, $\mu_2 \in L_1$. Now, for all i=1,2,3,...,l; k=1,2,3,...,n; we have,
$\varphi^{-1}(\tau_W^i)(\mu_1+\mu_2) = \tau_W^i[\varphi(\mu_1+\mu_2)] = \tau_W^i[\varphi(\mu_1)+\varphi(\mu_2)]$
$\geq$ Min $\{\tau_W^i(\varphi(\mu_1)), \tau_W^i(\varphi(\mu_2))\}$
=Min $\{\varphi^{-1}(\tau_W^i)(\mu_1), \varphi^{-1}(\tau_W^i)(\mu_2)\}$
$\varphi^{-1}(\eta_W^k)(\mu_1+\mu_2) = \eta_W^k[\varphi(\mu_1+\mu_2)] = \eta_W^k[\varphi(\mu_1)+\varphi(\mu_2)]$
$\leq$ Max $\{\eta_W^k(\varphi(\mu_1)), \eta_W^k(\varphi(\mu_2))\}$
=Max $\{\varphi^{-1}(\eta_W^k)(\mu_1), \varphi^{-1}(\eta_W^k)(\mu_2)\}$
For all $\mu \in L_1$ and c$\in \mathcal{F}$, we have,
$\varphi^{-1}(\tau_W^i)(c\mu) = \tau_W^i[\varphi(c\mu)] = \tau_W^i[c\varphi(\mu)] \geq \tau_W^i(\varphi(\mu)) = \varphi^{-1}(\tau_W^i)(\mu)$
$\varphi^{-1}(\eta_W^i)(c\mu) = \eta_W^i[\varphi(c\mu)] = \eta_W^i[c\varphi(\mu)] \leq \eta_W^i(\varphi(\mu)) = \varphi^{-1}(\eta_W^i)(\mu)$
For all μ_1, $\mu_2 \in L_1$.
$\varphi^{-1}(\tau_W^i)[\mu_1,\mu_2] = \tau_W^i(\varphi[\mu_1,\mu_2]) = \tau_W^i[\varphi(\mu_1),\varphi(\mu_2)]$
$\geq$ Min $\{\tau_W^i(\varphi(\mu_1)), \tau_W^i(\varphi(\mu_2))\}$
=Min $\{\varphi^{-1}(\tau_W^i)(\mu_1), \varphi^{-1}(\tau_W^i)(\mu_2)\}$
$\varphi^{-1}(\eta_W^i)[\mu_1,\mu_2] = \eta_W^i(\varphi[\mu_1,\mu_2]) = \eta_W^i[\varphi(\mu_1),\varphi(\mu_2)]$
$\leq$ Max $\{\eta_W^i(\varphi(\mu_1)), \eta_W^i(\varphi(\mu_2))\}$
=Max $\{\varphi^{-1}(\eta_W^i)(\mu_1), \varphi^{-1}(\eta_W^i)(\mu_2)\}$
Hence, φ^{-1}(W) is multi intuitionistic fuzzy Lie sub algebra of L_1. □

Theorem 4.6. *Let W = (τ_W^1, τ_W^2,...,τ_W^l; η_W^1,η_W^2,...,η_W^n) $\in I^{L_2}$ be multi IF Lie ideal and φ: $L_1 \rightarrow L_2$ be Lie homomorphism between L_1 and L_2. Then φ^{-1}(W) is multi intuitionistic fuzzy Lie ideal of L_1.*

Proof. The proof is similar to the proof of theorem (4.5) □

5 Conclusion remarks

The originality of the present paper is that we introduce for the first time the concept of multi intuitionistic fuzzy (briefly, MIF) Lie sub algebras and multi intuitionistic fuzzy Lie ideals of Lie algebra. Some of their fundamental properties and operations like intersection and generalized Cartesian product of multi intuitionistic fuzzy Lie sub algebras are investigated. Moreover, the relationship between MIF Lie sub algebras and MIF Lie ideals are established. Lastly, the image and the inverse image of MIF Lie sub algebras as well as MIF Lie ideals under Lie homomorphisms are also studied. In future study of Lie algebra shall be extended in the light of multi Fermatean fuzzy set and multi picture fuzzy sets respectably. The proposed work is applicable in any multi criterion decision making problem, pattern recognition and classification problems especially problems with more than one decision makers. Therefore this new theory will be a useful tool in decision and ranking problems such as robot selection, green supliers selections, solid waste landfil site section problems etc. In the near future we give some application of the proposed theory to some multi criterion decision making problems such as medical diagnosis and pattern recognition.

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