

## ON A DERIVATION OF KU-ALGEBRAS

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**Abstract** The concept of regular  $(\alpha, \beta)$ -derivations in KU-algebras  $\chi$  is introduced, with a discussion of illustrative examples and an examination of associated characteristics. We also present a criteria for  $(\alpha, \beta)$ -derivation to qualify as a regular. Additionally, we present the notions of  $h_{(\alpha, \beta)}$ -invariant and  $\alpha$ -ideal as well as  $\beta$ -ideal along with their corresponding results.

### 1 Introduction

Mathematicians frequently use logic in order to demonstrate theorems and other relevant mathematical concepts. Mathematically, everything depends upon these logical proofs. The purpose of B-algebras i-e BCK and BCI-algebras was to build up the interchange of ideas among the researchers, and to create new concepts in this area. The introduction of Boolean algebras was a great achievement to show that how we can bring logic and mathematics together. So, George Boole created an effective and large area for more developments in this field.

After that, in 1966, Japanese mathematicians Imai, Iseki and Tanaka introduced [1] BCI/BCK-algebras that are important logical algebra's classes, they provide a comprehensive review of BCI/BCK-algebras, highlighting their key algebraic properties. The BCK algebras are recognized as a proper subclass of the BCI algebras. Yisheng covers key theorems and fundamental ideas in his in-depth examination of BCI-algebras in [2]. In algebraic theory, the study provides an extensive resource on the composition and characteristics of BCI-algebras. Both B-algebras i-e BCK or BCI-algebras have their origins in distinct sources that is set propositional calculi and set theory. In 1983, Hu and Li [3] initiated an idea of a class of algebra, known as BCH-algebras, that is generalization of the concept of both BCI and BCK-algebras, also BCI-algebras can be seen as a branch of Boolean Closed Homomorphism-algebras. In [4], the authors studied the branchwise strongly gBCH-algebras and some equivalence conditions to be branch wise commutative.

In [5], mathematicians presented the idea of d-algebras, which is further branch of Bounded Commutative Klein-algebras. In rings, the concept of derivation is classic and it has vital role in many fields of mathematics. Research of derivation in rings was presented earlier, yet received an impulse only after Posner [6] is initiated conclusion in 1957 on derivations in rings. Actually, the idea of derivation in ring is fairly ancient or is important in algebraic geometry, analysis, and other mathematical disciplines. In year 2004, Jun and Xin [7] employed an idea of a derivation to BCI-algebras, therefore, they developed a brand-new idea known as a regular derivation. Jacobson presented the idea of  $(s_1, s_2)$ -derivation in his well-known book "Structures of Rings" [8] which was commonly recognized as either a  $(\theta, \phi)$  or a  $(\sigma, \tau)$ -derivation. The authors formulated the concept of a  $(\alpha, \beta)$ -derivation ( $(\alpha, \beta)$ -der) within Bounded Commutative Integral-algebra [9] and investigated corresponding axioms as inspiration from the idea of  $(\theta, \phi)$  or  $(\sigma, \tau)$ - derivation defined on the rings structures.

In 2009 [10], Prabayak, and Leerawat, introduced a category of algebras referred to as KU-algebras. Within this study, the authors have established definitions for ideals, conducted an examination of congruences within KU-algebras, and explored several interconnected properties. In paper [11], authors introduce the idea of  $(\ell, \gamma)$  and  $(\gamma, \ell)$ -derivations as well as  $t$ -derivations within the context of KU-algebras. Furthermore, the authors explored the  $D$ -invariant and the regular derivations concerning ideals within KU-algebras. Additionally, they provide a characterization of D-Ker through derivations.

In 2019, the authors [12] considered the  $d$ -algebras and then by using the endomorphisms  $\alpha$  and  $\beta$ , some algebraic properties have been investigated. Also some good examples were constructed in the study. In [13], Koam et. al. extended the concept of pseudo-metric for KU-algebras and established a relation between the ideals and pseudo-valuation. In year 2020, [14], the KU-algebras have been studied in the context of  $n$ -ary block code. An interesting result was obtained that KU-algebras which are linked with an  $n$ -ary block code through an established technique, was not unique. The uniqueness was checked up to isomorphism. In [15], a study of extended KU-algebras was introduced and a few relation between the characteristics of extended KU-algebras along with simple KU-algebras were investigated. In [16, 17], a series of results are presented to define and characterize a class of ordered semigroups known as AG-groupoids, which are both non-associative and non-commutative. In recent years, logical algebras were studied with various concepts and invariants, a collection of such study in the context of Fuzzy logic can be seen in [18]-[22]. For further study, we refer the readers to [23, 24, 25, 26, 27, 28].

In the current study, we establish the concept of  $(\alpha, \beta)$ -der within the KU-algebra denoted as  $\chi$  and look into associated characteristics. The inspiration for this research stemmed from the idea of a  $(\theta, \phi)$  or  $(\sigma, \tau)$ -derivation within the field of ring theory. We present a criterion for  $(\alpha, \beta)$ -der to possess regularity. The ideas of  $(\alpha, \beta)$ -der,  $\alpha$ -ideal,  $h_{(\alpha, \beta)}$ -invariant as well as  $\beta$ -ideal and furthermore, we will explore their relationship. Additionally, we'll gain some findings about regular  $(\alpha, \beta)$ -ders. In conclusion, we'll generate numerous examples and uncovered several intriguing discoveries during the course of our study.

## 2 Preliminary Results

In this section, we review the key concepts and characteristics that are necessary for the following discussions.

**Definition 2.1.** [29] A KU-algebra can be defined as non-void set  $\chi$  equipped with constant element 1 and single binary operation  $\circ$  which adheres  $\forall \nu, \omega, z \in \chi$  to the following properties:

- (1)  $(\nu \circ z) \circ [(z \circ \omega) \circ (\nu \circ \omega)] = 1$ ,
- (2)  $\omega \circ 1 = 1$ ,
- (3)  $1 \circ \omega = \omega$ ,
- (4)  $\omega \circ z = 1$  and  $z \circ \omega = 1 \Rightarrow$  that  $\omega = z$ .

For any  $\nu, \omega, z \in \chi$  in any KU-algebra  $\chi$ ,

- (1)  $\nu \circ \nu = 1$
- (2)  $z \circ (\nu \circ \omega) = \nu \circ (z \circ \omega)$
- (3)  $z \circ ((z \circ \omega) \circ \omega) = 1$
- (4)  $(\nu \circ \omega) \circ \omega = \nu$ . Following are a few substructures of KU-algebras.

**Definition 2.2.** [29] A subset  $S(\neq \phi) \subseteq \chi$  is termed as a sub-algebra of the KU-algebra  $\chi$  if  $\omega \circ \nu \in S$ , for all  $\omega, \nu \in S$ .

**Definition 2.3.** [29] For a subset  $A(\neq \phi) \subset \chi$  of a KU-algebra  $\chi$ . The set  $A$  is termed as KU-ideal of the KU-algebra  $\chi$  if

- (1)  $1 \in A$ ,
- (2)  $\forall \omega, \nu, z \in \chi, \omega \circ (\nu \circ z) \in A, \nu \in A \rightarrow \omega \circ z \in A$ .

### 3 $(\alpha, \beta)$ –der on KU-algebras

Within the following part, we will present a notion of  $(\alpha, \beta)$ –der for KU-algebras, accompanied by illustrative instances and outcomes.

**Notation** In KU-algebra  $\chi$ , let's represent the operation of combining elements  $\omega, \nu$  s.t  $\omega \wedge \nu = (\omega \circ \nu) \circ \nu$  and  $\wedge$  stands for meet.

**Definition 3.1.** We define a KU-algebra endomorphism on a KU-algebra  $\chi$  as a mapping  $\alpha$  from  $\chi$  to itself, such as  $\alpha(\omega \circ \nu) = \alpha(\omega) \circ \alpha(\nu)$ .

By using Proposition 2 (1) we have  $\alpha(1) = \alpha(1 \circ 1) = \alpha(1) \circ \alpha(1) = 1$ .

**Definition 3.2.** Consider  $\chi$  as a KU-algebra. A mapping  $h : \chi \rightarrow \chi$  is referred to as a *left–right* der, often abbreviated as  $(\ell, \gamma)$ –der of  $\chi$ , if  $\forall z, \nu \in \chi$  it meets following:

$$h(z \circ \nu) = h(z) \circ \nu \wedge z \circ h(\nu).$$

For the computations, we will first apply  $\circ$  and then  $\wedge$ .

If the mapping  $h$  satisfies the specified identity  $h(\omega \circ \nu) = \omega \circ h(\nu) \wedge h(z) \circ \nu \forall \omega$  and  $\nu$  in the set  $\chi$ , in that case  $h$  is referred to as a *right – left* der of  $\chi$ , often abbreviated as an  $(\gamma, \ell)$ –der. Furthermore, if  $h$  is  $(\ell, \gamma)$ –der and a  $(\gamma, \ell)$ –der, then  $h$  qualifies as a der of  $\chi$ .

**Definition 3.3.** Consider  $\chi$  as a KU-algebra. A mapping  $h : \chi \rightarrow \chi$  is termed as a *left – right*  $(\alpha, \beta)$ –der of  $\chi$ , if it satisfies

$$h(z \circ \nu) = h(z) \circ \alpha(\nu) \wedge \beta(z) \circ h(\nu). \forall z, \nu \in \chi$$

If the mapping  $h$  meets the specified identity  $h(z \circ \nu) = \alpha(z) \circ h(\nu) \wedge h(z) \circ \beta(\nu) \forall z$  and  $\nu$  in the set  $\chi$ , in that case  $h$  is referred to as a *right – left*  $(\alpha, \beta)$ –der of  $\chi$ . Furthermore, if  $h$  is both a  $(\ell, \gamma)$  and  $(\gamma, \ell) - (\alpha, \beta)$ –der, then  $h$  qualifies as  $(\alpha, \beta)$ –der of  $\chi$  and denoted as  $H$

**Example 3.4.** Consider the set  $\chi = \{1, 2, 3, 4\}$  that forms a KU-algebra structure with single operation  $\circ$ , that is defined as:

Define  $H : \chi \rightarrow \chi$  by

$\circ$	1	2	3	4
1	1	2	3	4
2	1	1	2	4
3	1	1	1	4
4	1	2	3	1

$$H(\omega) = \begin{cases} 1 & \omega = 1, 2, 3 \\ 4 & \omega = 4 \end{cases} \tag{3.1}$$

Define endomorphisms  $\alpha$  and  $\beta$  on the set  $\chi$  in the following manner

$$\alpha(\omega) = \begin{cases} 1 & \omega \in \{1, 2, 3\} \\ 4 & \omega = 4 \end{cases} \tag{3.2}$$

$$\beta(\omega) = \begin{cases} 1 & \omega \in \{1, 2, 3\} \\ 3 & \omega = 4 \end{cases} \tag{3.3}$$

Now we can verify simply that  $H$  is a  $(\ell, \gamma) - (\alpha, \beta)$ –der. But, the  $H$  does not qualify as an  $(\gamma, \ell) - (\alpha, \beta)$ –der because  $H(1 \circ 4) = H(4) = 4$  but  $\alpha(1) \circ H(4) \wedge H(1) \circ \beta(4) = 1 \circ 4 \wedge 1 \circ 3 = 1$ .

**Example 3.5.** Consider  $\chi = \{1, m, n\}$  be a KU-algebra with single binary operation  $\diamond$  is defined: Define  $H : \chi \rightarrow \chi$  by

$\diamond$	1	m	n
1	1	m	n
m	1	1	1
n	1	s	1

$$H(\nu) = \begin{cases} 1 & \nu = 1, n \\ n & \nu = m \end{cases} \tag{3.4}$$

Define endomorphisms  $\alpha$  and  $\beta$  on  $\chi$

$$\alpha(\nu) = \begin{cases} 1 & \nu = 1 \\ m & \nu = m, n \end{cases} \tag{3.5}$$

$$\beta(\nu) = \begin{cases} 1 & \nu = 1 \\ n & \nu = m, n \end{cases} \tag{3.6}$$

Hence, one can simply confirm that  $H$  is a  $(\ell, \gamma)$  as well as  $(\gamma, \ell) - (\alpha, \beta)$ -der on set the  $\chi$ .

**Definition 3.6.** A map  $f$ , that is a self-map (sm) on a KU-algebra  $\chi$  is termed "regular" when it satisfies the condition  $f(1) = 1$ .

**Example 3.7.** Suppose  $(\alpha, \beta)$ -der  $H(\omega)$  on KU-algebra  $\chi$  in example 3.4 and 3.5. The regularity of  $H(\omega)$  can therefore be seen clearly.

In the next two theorems the conditions are presented for  $H(\omega)$  to be regular.

**Theorem 3.8.** Suppose  $(\chi, \diamond, 1)$  is a KU-algebra and  $H : \chi \rightarrow \chi$  is an  $(\ell, \gamma) - (\alpha, \beta)$ -der then  $H$  is regular.

*Proof.*  $H(1) = H(\nu \diamond 1)$   
 $= H(\nu) \diamond \alpha(1) \wedge \beta(\nu) \diamond H(1)$   
 $= H(\nu) \diamond 1 \wedge \beta(\nu) \diamond H(1)$   
 $= 1 \wedge \beta(\nu) \diamond H(1), \text{ since } \nu \diamond 1 = 1$   
 $= (1 \diamond (\beta(\nu) \diamond H(1))) \diamond (\beta(\nu) \diamond H(1)), \text{ since } (a \wedge b) = (a \diamond b) \diamond b$   
 $= (\beta(\nu) \diamond H(1)) \diamond (\beta(\nu) \diamond H(1)), \text{ since } 1 \diamond \nu = \nu$   
 $= 1, \text{ since } \nu \diamond \nu = 1. \quad \square$

**Theorem 3.9.** Suppose  $(\chi, \diamond, 1)$  is KU-algebra and  $H : \chi \rightarrow \chi$  is an  $(\gamma, \ell) - (\alpha, \beta)$ -der then  $H$  is regular.

*Proof.*  $H(1) = H(\nu \diamond 1)$   
 $= \alpha(\nu) \diamond H(1) \wedge H(\nu) \diamond \beta(1)$   
 $= \alpha(\nu) \diamond H(1) \wedge H(\nu) \diamond 1$   
 $= \alpha(\nu) \diamond H(1) \wedge 1, \text{ since } \nu \diamond 1 = 1$   
 $= ((\alpha(\nu) \diamond H(1)) \diamond 1) \diamond 1,$   
 $= ((\alpha(\nu) \diamond 1) \diamond 1) \diamond 1$   
 $= (1 \diamond 1) \diamond 1, \text{ since } \nu \diamond 1 = 1$   
 $= 1 \diamond 1 = 1. \quad \square$

Some more results about the  $H$  for KU-algebras are given in this theorem.

**Theorem 3.10.** For a KU-algebra  $(\chi, \diamond, 1)$ .

- (1) Suppose  $H$  is an  $(\gamma, \ell) - (\alpha, \beta)$ -der of  $\chi$ , then  $H(\nu) = H(\nu) \wedge \beta(\nu), \forall \nu \in \chi$ .
- (2) Suppose  $H$  is an  $(\ell, \gamma) - (\alpha, \beta)$ -der of  $\chi$ , then  $H(\nu) = \alpha(\nu) \wedge H(\nu), \forall \nu \in \chi$ .

*Proof.* (1) Suppose  $H$  as a  $(\gamma, \ell) - (\alpha, \beta)$ -der of  $\chi$ .

$$\begin{aligned} H(\nu) &= H(1 \diamond \nu) \\ &= \alpha(1) \diamond H(\nu) \wedge H(1) \diamond \beta(\nu) \\ &= 1 \diamond H(\nu) \wedge H(1) \diamond \beta(\nu) \\ &= H(\nu) \wedge H(1) \diamond \beta(\nu), \text{ since, } 1 \diamond \nu = \nu \\ &= H(\nu) \wedge 1 \diamond \beta(\nu) \\ &= H(\nu) \wedge \beta(\nu), \text{ since, } 1 \diamond \nu = \nu \end{aligned}$$

(2) Suppose  $H$  be a  $(\ell, \gamma) - (\alpha, \beta)$ -der of  $\chi$ .

$$\begin{aligned} H(\nu) &= H(1 \diamond \nu) \\ &= H(1) \diamond \alpha(\nu) \wedge \beta(1) \diamond H(\nu) \\ &= H(1) \diamond \alpha(\nu) \wedge 1 \diamond H(\nu) \\ &= H(1) \diamond \alpha(\nu) \wedge H(\nu), \text{ since, } 1 \diamond \nu = \nu \\ &= 1 \diamond \alpha(\nu) \wedge H(\nu) \\ &= \alpha(\nu) \wedge H(\nu), \text{ since, } 1 \diamond \nu = \nu. \end{aligned}$$

□

By considering the condition of self mapping we obtained the following interesting result.

**Theorem 3.11.** Consider  $H$  as a sm of KU-algebra  $\chi$ . If  $H(z) = \alpha(z)$ , then

- (1)  $H$  is an  $(\alpha, \beta)$ -der on  $\chi$ .
- (2)  $H(z \diamond \nu) = H(z) \diamond H(\nu)$ .

*Proof.* (1) Consider  $H$  be a sm of KU-algebra  $\chi$  s.t  $H(z) = \alpha(z)$ , we get

$$\begin{aligned} H(z \diamond \nu) &= \alpha(z \diamond \nu) \\ &= \alpha(z) \diamond \alpha(\nu) \\ &= H(z) \diamond \alpha(\nu) \\ &= ((H(z) \diamond \alpha(\nu)) \diamond (\beta(z) \diamond H(\nu))) \diamond (\beta(z) \diamond H(\nu)), \text{ since } (\nu \diamond z) \diamond z = \nu \\ &= (H(z) \diamond \alpha(\nu)) \wedge (\beta(z) \diamond H(z)), \forall z, \nu \in X \\ &\Rightarrow H(z) \text{ serves as an } (\ell, \gamma) - (\alpha, \beta)\text{-der in } \chi. \end{aligned}$$

Likewise, we can also demonstrate  $H(z)$  serves as an  $(\gamma, \ell) - (\alpha, \beta)$ -der in  $\chi$ .

We can come to the conclusion that  $H(z)$  qualifies as an  $(\alpha, \beta)$ -der on  $\chi$ .

$$\begin{aligned} (2) \forall z, \nu \in \chi, \\ H(z \diamond \nu) &= H(z) \diamond \alpha(\nu) \wedge \beta(z) \diamond H(\nu) \\ &= \alpha(z) \diamond \alpha(\nu) \wedge \beta(z) \diamond \alpha(\nu) \\ &= ((\alpha(z) \diamond \alpha(\nu)) \diamond (\beta(z) \diamond \alpha(\nu))) \diamond (\beta(z) \diamond \alpha(\nu)), \\ &= \alpha(z) \diamond \alpha(\nu), \text{ since, } (\nu \diamond z) \diamond z = \nu \\ &= H(z) \diamond H(\nu). \end{aligned}$$

□

**Definition 3.12.** Consider  $\chi$  as KU-algebra and self maps  $H, \tilde{H}$ , we define  $H \diamond \tilde{H} : \chi \rightarrow \chi$  by  $H \diamond \tilde{H}(\omega) = H(\tilde{H}(\omega)), \forall \omega \in \chi$ .

Next theorem, tells the condition that describes the composition of two  $(\alpha, \beta)$  der are again  $(\alpha, \beta)$  der for KU-algebras.

**Theorem 3.13.** Consider  $\chi$  as a KU-algebra and  $H, \tilde{H}$  are two  $(\ell, \gamma) - (\alpha, \beta)$ -der on  $\chi$  s.t  $\alpha^2 = \alpha$ . Then  $H \circ \tilde{H}$  is a  $(\ell, \gamma) - (\alpha, \beta)$ -der on  $\chi$ .

*Proof.* Let  $\omega, \nu \in \chi$ .

$$\begin{aligned} H \circ \tilde{H}(\omega \circ \nu) &= H(\tilde{H}(\omega \circ \nu)) \\ &= H((\tilde{H}(\omega) \circ \alpha(\nu)) \wedge (\beta(\omega) \circ \tilde{H}(\nu))) \\ &= H(((\tilde{H}(\omega) \circ \alpha(\nu)) \circ (\beta(\omega) \circ \tilde{H}(\nu))) \circ (\beta(\omega) \circ \tilde{H}(\nu))), \text{ since } (\nu \circ \omega) \circ \omega = \nu \\ &= H(\tilde{H}(\omega) \circ \alpha(\nu)) \\ &= (H(\tilde{H}(\omega)) \circ \alpha(\alpha(\nu))) \wedge (\beta(\tilde{H}(\omega)) \circ H(\alpha(\nu))) \\ &= ((H(\tilde{H}(\omega)) \circ \alpha(\alpha(\nu))) \circ (\beta(\tilde{H}(\omega)) \circ H(\alpha(\nu)))) \circ (\beta(\tilde{H}(\omega)) \circ H(\alpha(\nu))) \\ &= H(\tilde{H}(\omega)) \circ \alpha(\alpha(\nu)) \\ &= (H \circ \tilde{H})(\omega) \circ \alpha^2(\nu) \\ &= (H \circ \tilde{H})(\omega) \circ \alpha(\nu) \\ &= (((H \circ \tilde{H})(\omega) \circ \alpha(\nu)) \circ (\beta(\omega) \circ (H \circ \tilde{H})(\nu))) \circ (\beta(\omega) \circ (H \circ \tilde{H})(\nu)), \text{ since } (\nu \circ \omega) \circ \omega = \nu \\ &= ((H \circ \tilde{H})(\omega) \circ \alpha(\nu)) \wedge (\beta(\omega) \circ (H \circ \tilde{H})(\nu)). \end{aligned}$$

□

**Definition 3.14.** Consider  $\chi$  as an KU-algebra. Binary relation is defined as  $\leq$  on  $\chi$  such that  $\nu \leq \omega$  iff  $\omega \diamond \nu = 1$ . In this instance, the pair  $(\chi, \leq)$  forms a POSET.

Following lemmas give us an interesting characterizations when KU-algebras are partially ordered.

**Lemma 3.15.** Suppose  $\chi$  as a KU-algebra with a defined partial order  $\leq$  and  $H$  is a  $(\ell, \gamma) - (\alpha, \beta)$ -der on  $\chi$ . Then, it follows  $H(\omega) \leq \alpha(\omega)$ , for all,  $\omega, \nu \in \chi$ .

*Proof.* Consider  $H$  as a  $(\ell, \gamma) - (\alpha, \beta)$ -der on  $\chi$ . Thus, according the Theorem 3.10 (2),  $H(\omega) = \alpha(\omega) \wedge H(\omega) = (\alpha(\omega) \diamond H(\omega)) \diamond H(\omega)$ .

This indicates

$$\alpha(\omega) \diamond H(\omega) = \alpha(\omega) \diamond ((\alpha(\omega) \diamond H(\omega)) \diamond H(\omega)),$$

$$\alpha(\omega) \diamond H(\omega) = 1, \text{ since } \nu \diamond ((\nu \diamond \omega) \diamond \omega) = 1.$$

Thus,  $H(\omega) \leq \alpha(\omega)$  as  $\omega \diamond \nu = 1 \Rightarrow \nu \leq \omega$ .  $\square$

**Lemma 3.16.** Suppose  $\chi$  is KU-algebra with  $\leq$  and  $H$  is a  $(\ell, \gamma) - (\alpha, \beta)$ -der on  $\chi$ . In that case, the statements hold  $\forall \omega, \nu \in \chi$ .

$$(1) H(\omega \diamond \nu) \leq H(\omega) \diamond \alpha(\nu),$$

$$(2) H(H(\omega)) \leq \alpha(H(\omega)).$$

*Proof.* (1) Assuming that  $H$  as  $(\ell, \gamma) - (\alpha, \beta)$ -der on  $\chi$ . Here, we obtain

$$H(\omega \diamond \nu) = H(\omega) \diamond \alpha(\nu) \wedge \beta(\omega) \diamond H(\nu) = ((H(\omega) \diamond \alpha(\nu)) \diamond (\beta(\omega) \diamond H(\nu))) \diamond (\beta(\omega) \diamond H(\nu))$$

and

$$(H(\omega) \diamond \alpha(\nu)) \diamond H(\omega \diamond \nu) = (H(\omega) \diamond \alpha(\nu)) \diamond (((H(\omega) \diamond \alpha(\nu)) \diamond (\beta(\omega) \diamond H(\nu))) \diamond (\beta(\omega) \diamond H(\nu)))$$

$$(H(\omega) \diamond \alpha(\nu)) \diamond H(\omega \diamond \nu) = 1.$$

Since,  $((\nu \diamond \omega) \diamond \omega) \diamond \nu = 1$

$$\Rightarrow H(\omega \diamond \nu) \leq H(\omega) \diamond \alpha(\nu), \text{ since, } \omega \diamond \nu = 1 \Rightarrow y \leq \omega.$$

(2) If we assume that  $H$  is a  $(\ell, \gamma)$ -der, according to Theorem 3.10(2)

$$H(H(\omega)) = \alpha(H(\omega)) \wedge H(H(\omega)) = (\alpha(H(\omega)) \diamond H(H(\omega))) \diamond H(H(\omega)),$$

By multiplying from the left by  $\alpha(H(\omega))$ , it yields

$$\alpha(H(\omega)) \diamond H(H(\omega)) = \alpha(H(\omega)) \diamond ((\alpha(H(\omega)) \diamond H(H(\omega))) \diamond H(H(\omega)))$$

$$\alpha(H(\omega)) \diamond H(H(\omega)) = 1, \text{ since, } \nu \diamond ((\nu \diamond \omega) \diamond \omega) = 1 \text{ and } H(H(\omega)) \leq \alpha(H(\omega)). \quad \square$$

The following result shows that the inverse image of special element 1 of KU-algebra form a subalgebra.

**Theorem 3.17.** Suppose  $\chi$  is a KU-algebra with  $\leq$  and  $H$  is a  $(\ell, \gamma) - (\alpha, \beta)$ -der on  $\chi$ . So,  $H^{-1}(1) = \{\omega \in \chi \mid H(\omega) = 1\}$  form a subalgebra of the KU-algebra  $\chi$  if  $\alpha(\omega) = 1$ .

*Proof.* According to the Theorem 3.9, we obtain  $H$  is regular.

$$\text{That is } H(1) = 1 \Rightarrow 1 \in H^{-1}(1) \Rightarrow H^{-1}(1) \neq \varphi.$$

Let  $\omega, \nu \in H^{-1}(1)$ . Then by Lemma 3.16,

$$H(\omega \diamond \nu) \leq H(\omega) \diamond \alpha(\omega) = 1$$

$$\Rightarrow (H(\omega) \diamond \alpha(\omega)) \diamond H(\omega \diamond \nu) = 1$$

Since,  $H(\omega) = 1$  and  $\alpha(\omega) = 1$ . So that,  $H(\omega \diamond \nu) = 1$

$$\Rightarrow \omega \diamond \nu \in H^{-1}.$$

This indicates that,  $H^{-1}(1)$  form a subalgebra of the KU-algebra  $\chi$ .  $\square$

Some more characteristics of  $H$  are given in the following result.

**Theorem 3.18.** Consider a KU-algebra  $\chi$  s.t.  $\alpha.H(\nu) = H(\nu)$  with  $\leq$  and  $H$  be a  $(\alpha, \beta)$ -der on  $\chi$ . So, the statements exist,  $\forall, \nu \in \chi$

$$(1) H(\nu \diamond H(\nu)) = 1,$$

$$(2) H(H(\nu) \diamond \nu) = 1.$$

*Proof.* (1) Suppose  $H(\nu)$  is  $(\ell, \gamma)$ -der on  $\chi$ .

$$H(\nu \diamond H(\nu)) = (H(\nu) \diamond \alpha(H(\nu))) \wedge (\beta(\nu) \diamond H(H(\nu)))$$

$$= (H(\nu) \diamond \alpha.H(\nu)) \wedge (\beta(\nu) \diamond H(H(\nu)))$$

$$= (H(\nu) \diamond H(\nu)) \wedge (\beta(\nu) \diamond H(H(\nu))), \text{ since, } \alpha.H(\nu) = H(\nu)$$

$$\begin{aligned}
 &= 1 \wedge (\beta(\nu) \diamond H(H(\nu))) \\
 &= (1 \diamond (\beta(\nu) \diamond H(H(\nu)))) \diamond (\beta(\nu) \diamond H(H(\nu))) \\
 &= (\beta(\nu) \diamond H(H(\nu))) \diamond (\beta(\nu) \diamond H(H(\nu))) = 1. \\
 (2) \text{ Suppose } H(\nu) \text{ is } (\gamma, \ell) - (\alpha, \beta)\text{-der on } \chi. \\
 &H(H(\nu) \diamond \nu) = (\alpha(H(\nu)) \diamond H(\nu)) \wedge (H(H(\nu)) \diamond \beta(\nu)) \\
 &= (\alpha.H(\nu) \diamond H(\nu)) \wedge (H(H(\nu)) \diamond \beta(\nu)) \\
 &= (H(\nu) \diamond H(\nu)) \wedge (H(H(\nu)) \diamond \beta(\nu)) \\
 &= 1 \wedge (H(H(\nu)) \diamond \beta(\nu)) \\
 &= (1 \diamond (H(H(\nu)) \diamond \beta(\nu))) \diamond (H(H(\nu)) \diamond \beta(\nu)), \\
 &= (H(H(\nu)) \diamond \beta(\nu)) \diamond (H(H(\nu)) \diamond \beta(\nu)) = 1.
 \end{aligned}$$

□

**Theorem 3.19.** Consider  $\chi$  as KU-algebra and suppose that  $H$  is a  $(\ell, \gamma) - (\alpha, \beta)$ -der on  $\chi$ . So,  $H^\zeta(H^{\zeta-1}(\dots(H^2(H^1(\omega))))\dots) \leq \alpha(\omega)$ .

*Proof.* Let  $n = 1$ . We obtain through Lemma 3.15 that  $H^1(\nu) \leq \alpha(\nu)$ .

Consider  $\zeta \in \mathbb{N}$

$$H^\zeta(H^{\zeta-1}(\dots(H^2(H^1(\nu))))\dots) \leq \alpha(\nu).$$

Suppose  $\Delta_\zeta = H^\zeta(H^{\zeta-1}(\dots(H^2(H^1(\nu))))\dots) \leq \alpha(\nu)$ .

$$\begin{aligned}
 \text{Now, } H^{\zeta+1}(\Delta_\zeta) &= H^{\zeta+1}(1 \diamond \Delta_\zeta) \\
 &= H^{\zeta+1}(1) \diamond \alpha(\Delta_\zeta) \wedge \beta(1) \diamond H^{\zeta+1}(\Delta_\zeta) \\
 &= 1 \diamond \alpha(\Delta_\zeta) \wedge 1 \diamond H^{\zeta+1}(\Delta_\zeta) \\
 &= \alpha(\Delta_\zeta) \wedge H^{\zeta+1}(\Delta_\zeta), \text{ since, } 1 \diamond \nu = \nu \\
 &= (\alpha(\Delta_\zeta) \diamond H^{\zeta+1}(\Delta_\zeta)) \diamond H^{\zeta+1}(\Delta_\zeta).
 \end{aligned}$$

Multiply both side by  $\alpha(\Delta_\zeta)$ , we get,

$$\alpha(\Delta_\zeta) \diamond H^{\zeta+1}(\Delta_\zeta) = \alpha(\Delta_\zeta) \diamond ((\alpha(\Delta_\zeta) \diamond H^{\zeta+1}(\Delta_\zeta)) \diamond H^{\zeta+1}(\Delta_\zeta)), y \diamond ((\nu \diamond \nu) \diamond \nu) = 1$$

$$\alpha(\Delta_\zeta) \diamond H^{\zeta+1}(\Delta_\zeta) = 1, \text{ since, } \nu \diamond ((\nu \diamond \nu) \diamond \nu) = 1$$

$$\Rightarrow H^{\zeta+1}(\Delta_\zeta) \leq \alpha(\Delta_\zeta).$$

However, based on the assumption, we obtain,

$$H^{\zeta+1}(\Delta_\zeta) \leq \alpha(\Delta_\zeta) \leq \alpha(\nu).$$

□

**Definition 3.20.** Consider  $\chi$  as KU-algebra and  $H$  as an  $(\alpha, \beta)$ -der of  $\chi$ . An ideal  $K$  of  $\chi$  is defined as  $\alpha$ -ideal (approp.  $\beta$ -ideal) if  $\alpha(K) \subseteq K$  (approp.  $\beta(K) \subseteq K$ ).

**Example 3.21.** Consider a KU-algebra  $\chi = \{1, r, s\}$  with the  $\diamond$  operator as defined in Example 3.5. We describe an endomorphism in terms of  $\alpha$  and  $\beta$ .

$$\alpha(\nu) = \begin{cases} 1 & \nu = 1 \\ s & \nu = r, s \end{cases} \tag{3.7}$$

$$\beta(\nu) = \begin{cases} 1 & \nu = 1 \\ s & \nu = r, s \end{cases} \tag{3.8}$$

It can be readily verified that  $\chi$  satisfies the properties of a KU-algebra, and the set  $A = \{1, s\}$  qualifies as both  $\alpha$ -ideal and  $\beta$ -ideal of  $\chi$ .

**Definition 3.22.** Consider  $\chi$  as KU-algebra, and  $H$  as  $(\alpha, \beta)$ -der of  $\chi$ . An ideal  $A$  of  $\chi$  is referred to as  $H$ -invariant if  $H(A) \subseteq A$ , where  $H(A) = \{H(\omega) \mid \omega \in A\}$ .

**Example 3.23.** Suppose  $H$  is  $(\alpha, \beta)$ -der of  $\chi$  as given in example 3.4 that is  $(\ell, \gamma) - (\alpha, \beta)$ -der and suppose  $A = \{1, 2, 3\}$ . It is an easy exercise to see that  $A$  is not only  $\alpha$ -ideal and  $\beta$ -ideal but also an  $H$ -invariant on  $\chi$ .

**Example 3.24.** Suppose  $H$  as  $(\alpha, \beta)$ -der of  $\chi$  from example 3.5 that is both  $(\gamma, \ell)$  as well as  $(\ell, \gamma) - (\alpha, \beta)$ -der and suppose  $A = \{1, s\}$ . It is apparent that  $A$  is  $\beta$ -ideal,  $A$  is not  $\alpha$ -ideal, hence we are able to observe that  $A$  is an  $H$ -invariant in  $\chi$ .

If and only if condition is given for an  $(\alpha, \beta)$ -der of KU-algebras to be regular in this theorem.

**Theorem 3.25.** Consider  $\chi$  as KU-algebra,  $A$  as KU-ideal of  $\chi$ , and  $H$  as  $(\ell, \gamma) - (\alpha, \beta)$ -der of  $\chi$ . Then, the statement "H is regular iff every  $\alpha$ -ideal of  $\chi$  is  $(\alpha, \beta)$ -invariant" holds true.

*Proof.* Consider  $A$  as  $\alpha$ -ideal of KU-algebra  $\chi$ ,  $H$  be a regular and let's consider  $\nu \in H(A)$ . In that case for some elements  $\omega$  in  $A$ , we have  $\nu = H(\omega)$ , thus according to lemma 3.15, we obtain,  $H(\omega) \leq \alpha(\omega)$  this suggested,  $\alpha(\omega) \diamond H(\omega) = 1 \in A$ . Notice,  $\alpha(\omega) \in \alpha(A) \subseteq A$  and consequently, it can be concluded that  $\nu \in A$ .

Therefore,  $H(\omega) \subseteq A$ , which demonstrates that  $A$  is indeed  $H$ -invariant.

On the other hand, assume that each  $\alpha$ -ideal of  $\chi$  demonstrates  $H$ -invariant. We are aware the zero ideal is  $\alpha$ -ideal as well as  $H$ -invariant, so we obtain,  $H(1) \subseteq 1$  and thus,  $H(1) = 1 \Rightarrow H$  is regular.  $\square$

## 4 Conclusion

The concept of KU-algebras has recently emerged as one of the important topic of research in the field of algebraic structures. In this article we introduced the  $(\alpha, \beta)$ -der of the KU-algebras. Further we proved some algebraic properties of the  $(\alpha, \beta)$ -der of the KU-algebras. By applying the conditions of self mapping, partial ordering and inverse image of special element, we obtained some interesting results. Study of the logical algebras are important due to their applications in the research fields of Artificial Intelligence, theoretical physics, engineering and graph algebras. Our aim in the future is to introduce the  $(\alpha, \beta)$ -der for the other logical algebras.

## References

- [1] M. Kaviyarasu and K. Indhira, *Review on BCI/BCK-algebras and development*, International Journal of Pure and Applied Mathematics, **117(11)**, 377–383, (2017).
- [2] H. Yisheng, *BCI-algebra*, Beijing: Science Press, **16**, (2006).
- [3] Q. P. Hu and X. Li, *On BCH-algebra*, Math. Sem. Notes Kobe University, **11(2)**, 313–320, (1983).
- [4] M. A. Chaudhry, A. Fahad, Y. Rao, M. I. Qureshi and S. Gulzar, *Branchwise solid generalized BCH-algebras*, AIMS Mathematics, **5(3)**, 2424–2432, (2020).
- [5] J. Neggers and H. S. Kim, *On d-algebras*, Mathematica Slovaca, **49(1)**, 19–26, (1999).
- [6] E. C. Posner, *Ders in prime rings*, Proceedings of the American Mathematical Society, **8(6)**, 1093–1100, (1957).
- [7] Y. B. Jun and X. L. Xin, *On ders of BCI-algebras*, Information Sciences, **159(3-4)**, 167–176, (2004).
- [8] N. Jacobson, *Structure of rings*, American Mathematical Society, **37**, 299, (1956).
- [9] G. Muhiuddin and A. M. Al-Roqi,  $\alpha, \beta$ -ders in BCI-algebras, Discrete Dynamics in Nature and Society, **2012**, 403209, (2012).
- [10] C. Prabpayak and U. Leerawat, *On ideals and congruences in KU-algebras*, Scientia Magna., **5(1)**, 54–57, (2009).
- [11] S. M. Mostafa, R. A. K. Omar and A. Abd-eldayem, *Properties of ders on KU-algebras*, Journal of Advances in Mathematics, **10(1)**, 3085–3097, (2010).
- [12] R. M. Al Omary, *On  $(\alpha, \beta)$ -ders in d-algebras*, Bollettino dell'unione Matematica Italiana, **12**, 549–556, (2019).
- [13] A. N. Koam, A. Haider and M. A. Ansari, *Pseudo-metric on KU-algebras*, Korean Journal of Mathematics, **27(1)**, 131–140, (2019).
- [14] A. N. Koam, M. A. Ansari and A. Haider, *n-ary block codes related to KU-algebras*, Journal of Taibah University for Science, **14(1)**, 172–176, (2020).
- [15] A. N. Koam, A. Haider and M. A. Ansari, *On an extension of KU-algebras*, AIMS Mathematics, **6**, 1249–1257, (2021).
- [16] N. Kausar, M. Munir and K. Anita, *Left Regular Ordered AG-Groupoid*, Advanced Mathematical Models & Applications, **5**, 307–312, (2020).
- [17] N. Kausar1, M. Munir, M. Gulzar and G. M. Addis, *Characterizing the ordered AG- groupoids through the properties of their different classes of ideals*, Bulletin of the karaganda university. Mathematics series, **100**, 101–112, (2020).
- [18] A. Riaz, S. Kousar, N. Kausar, D. Pamucar and G. M. Addis, *An Analysis of Algebraic Codes over Lattice Valued Intuitionistic Fuzzy Type-3 R-Submodules*, Computational Intelligence and Neuroscience, 1–13, (2022).
- [19] G. M. Addis, N. Kausar, M. Munir and Y. M. Chu, *The Commutator of Fuzzy Congruences in Universal Algebras*, International Journal of Computational Intelligence Systems, **14(1)**, 1322–1336, (2021).

- [20] K. Kalaiarasi, V. Manimozhi, N. Kausar, D. Pamucar, S. Kousar and Y. U. Gaba, *Fuzzy Product KM-Subalgebras and Some Related Properties*, Complexity, 1–8, (2021).
- [21] S. Kousar, S. Arshad, N. Kausar and T. P. Hong. *Construction of Nilpotent and Solvable Lie Algebra in Picture Fuzzy Environment*, International Journal of Computational Intelligence Systems, **16:37**, 1–10, (2023).
- [22] S. Kousar, S. Shaheen, N. Kausar, D. Pamucar, V. Simic, M. A. Salman, *On the Construction of Congruences over Generalized Fuzzy G-Acts*, International Journal of Computational Intelligence Systems, **17:242**, 1–16, (2024).
- [23] H. K. Jawad and O. I. Al-Shaher, *New types of ideals in KU-algebra*, In AIP Conference Proceedings, **2398(1)**, 060026, (2022).
- [24] A. Almuhaimeed, *Applications on Fuzzy Translation and Fuzzy Multiplication of KU-algebras*, Computer Science, **17(2)**, 667–677, (2022).
- [25] J. Golzarpoor and S. Mehrshad, *State KU-Algebra*, Current Applied Sciences, 91–100, (2022).
- [26] M. M. Jansirani and S. Sujatha, *Lukasiewicz Fuzzy Ideals Applied to BCK/BCI-Algebras*, Palestine Journal of Mathematics, **13**, 62–72, (2024).
- [27] V. V. Kumar, M. S. Rao and S. K. Vali, *Radical of Filters of Transitive BE-Algebras*, Palestine Journal of Mathematics, **11(2)**, 243–255, (2022).
- [28] M. D. Siddiqi, S. A. Siddiqui and M. Ahmad, *r-almost Newton-Yamabe solitons on submanifolds of Kenmotsu space forms*, Palestine Journal of Mathematics, **11(1)**, 427–438, (2022).
- [29] S. M. Mostafa, M. A. Abd-Elnaby and M. M. M. Yousef, *Fuzzy ideals of KU-Algebras*, Int. Math. Forum., **6(63)**, 3139–3149, (2011).

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