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# Generalization of the identity graph of symmetric group $s_\eta$ and some of its topological indices

Muhammad U. Mirza and R. Anjum

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**Corresponding Author: R. Anjum**

**Abstract** *This study presents a comprehensive generalization of the identity graph of the symmetric group $s_\eta$ and introduces novel $\eta$-dependent formulae for calculating key topological indices, including first and second Zagreb indices, multiplicative Zagreb indices, Narumi-Katayama index, reduced second Zagreb index, and forgotten index. These generalizations stem from the analysis of the symmetric group $s_\eta$, where we specifically focus on characterizing elements of order 2, which are self-inverse. By expanding on the conventional understanding of symmetric groups, we provide generalized formulas that allow for the efficient calculation of these indices for any value of $\eta$. Furthermore, this work bridges the gap between theoretical algebra and applied graph theory, offering significant advancements in studying symmetric groups. These advancements hold potential applications in fields such as cryptography, chemical graph theory, and quantum mechanics where the symmetries and structures of algebraic groups play a pivotal role. By defining and generalizing the identity graph and associated topological indices for $s_\eta$, this research offers new directions for theoretical exploration and practical applications across multiple scientific disciplines.*

## 1 Introduction

*Symmetric groups, $s_\eta$ are crucial in algebraic combinatorics due to their role as permutations of a finite set, extensively used in geometry and physics [1, 2]. These groups exhibit essential properties like representing all bijective maps of a set to itself. Moreover, symmetric groups have the intriguing feature that any finite group of order is isomorphic to a subgroup of $s_\eta$ showcasing their versatility and applicability across various mathematical disciplines [2]. Additionally, the study of symmetric groups extends to affine symmetric groups, infinite extensions that describe symmetries of geometric objects and are fundamental in combinatorics and representation theory [3, 25]. These properties make symmetric groups indispensable tools in understanding symmetry, permutations, and their applications in diverse mathematical contexts.*

*Group graphs play a crucial role in graph theory, particularly in the context of group structures and their visual representations. Various types of group graphs have been extensively studied, including conjugacy class graphs, power graphs, enhanced power graphs, deep commuting graphs, and non-generating graphs [4, 5]. These graphs provide insights into the relationships and properties of elements within a group, offering a unique perspective on group theory [6, 26]. Additionally, the research explores the visualization of group structures within graph diagrams, categorizing visualization techniques into visual node attributes, juxtaposed approaches, superimposed techniques, and embedded visualizations [7].*

*Identity graphs, play a significant role in various fields, such as quantum error correction, group theory, and semirings. In the context of quantum error correction, noncommutative operator graphs generated by resolutions of identity are crucial in understanding quantum noise and* *error correction codes [8, 29]. In group theory, identity graphs of finite cyclic groups provide insights into self-inverse and mutual inverse elements, helping determine the number of triangles and edges in these graphs [9]. Additionally, in the study of commutative semirings, the concept of identity-summand graphs explores elements that sum to the identity, offering valuable information about the structure and properties of these semirings [10, 27]. Overall, identity graphs serve as powerful tools for analyzing structures, properties, and relationships within different mathematical and theoretical frameworks.*

*Topological indices play a crucial role in mathematical chemistry, aiding in predicting molecular reactivity and physicochemical properties [11, 12]. Graph theory is extensively utilized to represent molecular structures, with applications in group representation and the study of coprime graphs of dihedral groups [13]. The order divisor graph of cyclic groups is characterized based on group order, with descriptions of various topological indices such as the Wiener index and Harary index [15, 28]. Research focuses on calculating topological descriptors and polynomials for finite graphs associated with quasigroups and cyclic groups, emphasizing the importance of algebraic polynomials in determining accurate topological indices [11]. Additionally, distance-based indices like the Wiener index and hyper-Wiener index are explored for inverse graphs associated with finite cyclic groups, showcasing the diverse applications of topological indices in understanding molecular properties [14].*

*In recent literature, significant advancements have been made in understanding topological indices and graph theory through the study of specialized graph structures and fuzzy systems. Khalid et al. examine degree-based topological indices within two distinct families of graphs, emphasizing configurations with a diameter of three, which are crucial for applications in mathematical physics [30]. Extending this work, Khalid et al. explore the topological indices in more complex graph families, such as bistar and corona products, illustrating the indices' versatility and structural diversity [31]. Meanwhile, Roshini et al. study honeycomb networks, obtaining explicit formulas for non-neighbor topological indices of line graphs of honeycomb subdivisions, aiding the analysis of benzenoid hydrocarbons in chemical graph theory [34]. Additionally, Das and Kumar determine M-Polynomials for two-dimensional Silicon-Carbon structures, enhancing the degree-based topological index computation for molecular networks [35]. Pattabiraman defines a reformulated reciprocal product degree distance invariant for strong products of graphs, contributing to the analysis of connectivity and graph distances [36]. Further extending graph theory within fuzzy contexts, Kalaiarasi et al. investigate the normal product for intuitionistic anti-fuzzy graphs, offering new perspectives in fuzzy logic and intelligent systems [33].*

*Our contribution lies in the formulation of* $\eta$*-dependent expressions for the calculation of topological indices associated with the identity graph of* $s_\eta$*. These indices include but are not limited to, the first and second Zagreb indices, the multiplicative Zagreb indices, the Narumi-Katayama index, the reduced second Zagreb index, and the forgotten index. The generalized formulas allow for the efficient computation of these indices for any value of* $\eta$*, thereby streamlining the process for researchers interested in exploring the structural properties of larger symmetric groups.*

*Furthermore, we have generalized the characterization of the self-inverse elements within the symmetric group* $s_\eta$*. This generalization plays a crucial role in the construction of the identity graph for any given* $\eta$*, as it enables the reader to easily determine the vertices and edges of the graph corresponding to elements of order 2 (self-inverse elements). Consequently, our work provides the theoretical framework necessary for generating and studying the identity graph of symmetric groups for all values of* $\eta$*. By extending the analysis beyond specific cases such as* $s_4$*, we offer a robust and scalable method for investigating the identity graphs of symmetric groups, thereby contributing to the broader understanding of algebraic structures and their applications in various scientific fields.*

## 2 Comparison with existing studies on identity graphs

*In the work presented in [18], the identity graph associated with the ring* $Z_\rho$ *is introduced, and various topological indices of this graph are calculated in detail. This analysis includes well-established indices like the Zagreb indices, Wiener index, and others that provide a mathematical characterization of the graph's structure. Building upon this foundation, the study in [19] expands this work by introducing fuzzy topological indices of the identity graph of the ring* $Z_\rho$*,*

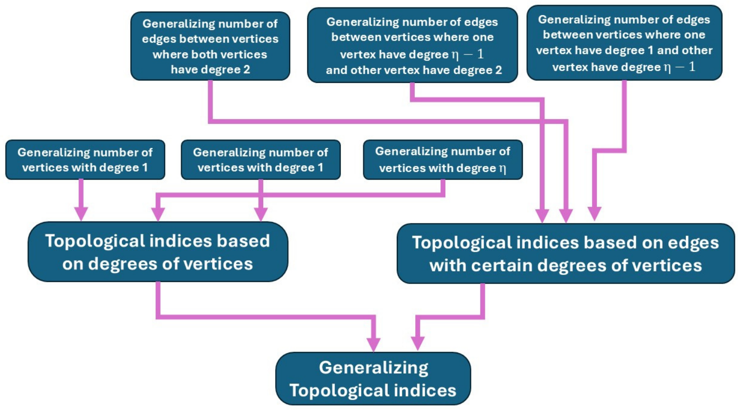


**Figure 1.** Graphical framework for generalizing topological indices

*providing a perspective through fuzzy set theory. This addition not only enhances the understanding of the identity graph but also illustrates how these fuzzy indices can yield insights into the graph's behavior under uncertainty.*

*Furthermore, in prior studies, as presented in [20], the identity graph of the symmetric group $S_4$ has been investigated comprehensively. This work covers the structure, vertex connections, and degree distributions within $S_4$, specifically examining how elements of order 2 form edges, thereby contributing to the graph's identity structure. However, despite these significant advancements, to the best of our knowledge, there has been no effort to generalize the concept of the identity graph for the symmetric group $S_\eta$ for arbitrary values of $\eta$, where $\eta$ represents any positive integer.*

*In this article, we address this gap by proposing a comprehensive generalization of the identity graph for the symmetric group $S_\eta$. Our work presents a novel analysis where we define and extend the identity graph properties of $S_4$ to $S_\eta$, allowing for an exploration of identity graphs across any symmetric group order. This generalization opens new avenues for analyzing the structural properties of identity graphs within the entire family of symmetric groups, enabling comparisons of topological indices as $\eta$ varies. This comparative analysis highlights the structural diversity and mathematical depth of identity graphs as they scale with increasing values of $\eta$, offering insights into their applicability in group theory and topological graph analysis.*

*Our work, therefore, represents the first attempt to generalize the identity graph to symmetric groups beyond $S_4$. This advancement contributes to the theoretical understanding of symmetric groups, establishing a foundation for future studies in exploring topological properties across various symmetric groups $S_\eta$, and provides a valuable resource for researchers interested in graph theory and its applications in abstract algebra.*

## 3 Motivation of the study

*The motivation for this study stems from the fundamental role that symmetric groups play within the broader context of group theory. Symmetric groups are often regarded as a "universal space" for group structures, as any group can be considered a subgroup of a sufficiently large symmetric group. This universality makes symmetric groups a powerful tool for studying and understanding the properties of a wide variety of group structures. By mastering the arithmetic and structural properties of symmetric groups, we can extend these insights to other groups, thereby enriching our understanding of group theory. For example, the dihedral group $D_6$ is isomorphic to the symmetric group $s_3$. As a result, the identity graphs of $s_3$ and $D_6$ are identical, and their corresponding topological indices, such as the Zagreb indices and other structural characteristics, will also be the same. This isomorphism highlights the broader applicability of symmetric group structures to other group types. By generalizing the identity graph of the symmetric group $s_\eta$, this study aims to establish a framework that can be extended to other groups, thus providing a unified approach to the study of group structures through symmetric groups. Furthermore, previ-*

*ous studies, such as [24], have explored the conjugate graph of dihedral groups and uncovered a range of potential applications, including cryptography and theoretical physics. These applications underscore the importance of developing a deeper understanding of group structures and their associated graphs. By focusing on the generalization of the identity graph for symmetric groups, this research builds on these foundational studies and opens up new possibilities for exploring the applications of group theory across multiple scientific disciplines.*

## 4 Preliminaries

*For a given set with $\eta$ elements, the symmetric group $s_\eta$ consists of all possible bijections (one-to-one and onto functions) from the set to itself. In simpler terms, each element of $s_\eta$ is a specific way to rearrange or permute the $\eta$ elements of the set. Given a set $\mathfrak{X} = \{1, 2, 3, \ldots, \eta\}$, the symmetric group $s_\eta$ is the set of all permutations of the elements in $\mathfrak{X}$, meaning it contains all functions $\sigma : \mathfrak{X} \to \mathfrak{X}$ such that each element of $\mathfrak{X}$ is mapped to a unique element of $\mathfrak{X}$ (i.e., $\sigma$ is a bijection). The group properties are given below.*

*(i)* ***Closure:*** *If $\sigma$ and $\tau$ are two permutations in $s_\eta$, their composition $\sigma \circ \tau$ (applying $\sigma$ after $\tau$) is also a permutation in $s_\eta$.*

*(ii)* ***Associativity:*** *The composition of functions is associative, meaning for any permutations $\sigma$, $\tau$, and $\rho$ in $s_\eta$, we have $(\sigma \circ \tau) \circ \rho = \sigma \circ (\tau \circ \rho)$.*

*(iii)* ***Identity:*** *There exists an identity permutation, often denoted as $e$, in $s_\eta$, which leaves all elements unchanged. For all $x \in \mathfrak{X}$, $e(x) = x$.*

*(iv)* ***Inverses:*** *Every permutation $\sigma \in s_\eta$ has an inverse $\sigma^{-1} \in s_\eta$, such that $\sigma \circ \sigma^{-1} = e$, the identity permutation.*

*For $s_3$, the set $\mathfrak{X} = \{1, 2, 3\}$, the symmetric group consists of all six possible permutations of $\{1, 2, 3\}$:*

$$s_3 = \{e, (12), (13), (23), (123), (132)\}$$

*where $e$ is the identity permutation, $(12)$ is the permutation swapping 1 and 2, $(13)$ swaps 1 and 3, and so on. The identity graph for the group $s_\eta$, as described in [18], is denoted by $id(s_\eta)$. This graph has a vertex set corresponding to the elements of $s_\eta$. Two distinct vertices, $\sigma$ and $\varsigma$, are connected by an edge if their product satisfies $\sigma\varsigma = 1$. Additionally, each vertex in $id(s_\eta)$ is linked to the identity element of $s_\eta$. The identity graph of $s_4$ is illustrated in Figure 2. .*
*The set of vertices of identity graph of symmetric group is represented by $\mathfrak{V}(\mathbf{id}(\mathfrak{s}_\eta))$ and set of*

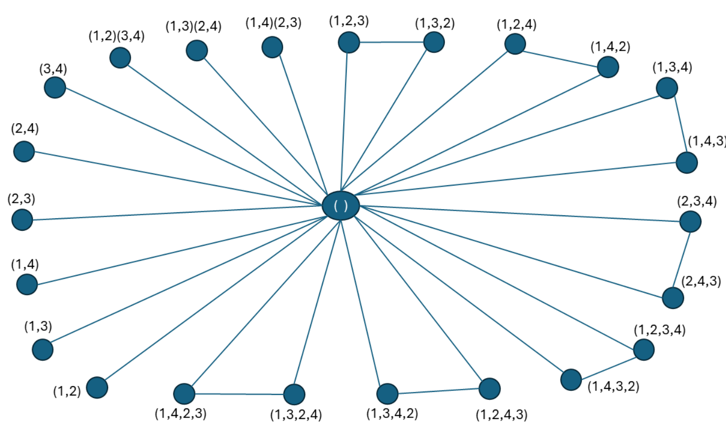


**Figure 2.** The identity graph of $s_4$

*edges of identity graph of symmetric group is represented by $\mathfrak{E}(\mathbf{id}(\mathfrak{s}_\eta))$. Degree of a vertex is defined as the number of edges adjacent to a vertex, in this article the degree of a vertex is denote by $\zeta$.Topological indices are mathematical values computed from the arrangement of atoms* *and bonds within a chemical molecule, represented as a graph. These indices summarize the molecule's structure through simple numeric values. They are extensively utilized in cheminformatics to predict various molecular properties like chemical reactivity, stability, and biological activity without direct experimental measurement. The first Zagreb index is calculated by taking the sum of the degrees of each vertex in a graph. Specifically, for every edge in the graph, the degrees of the two vertices connected by that edge are added together, and this is done for all edges in the graph. The second Zagreb index, on the other hand, is obtained by multiplying the degrees of the two vertices connected by each edge. For each edge, the degrees of the two connected vertices are multiplied, and then these products are summed for all edges in the graph. Mathematically topological indices are given in the table 1.*

**Table 1.** Topological indices.

| Index Name | Formula |
|---|---|
| First Zagreb index [17] | $\xi_1 = \sum_{\sigma\varsigma\in\mathfrak{E}(\mathbf{id}(\mathfrak{s}_\eta))} \zeta(\sigma)+\zeta(\varsigma)$ |
| Second Zagreb index[16] | $\xi_2 = \sum_{\sigma\varsigma\in\mathfrak{E}(\mathbf{id}(\mathfrak{s}_\eta))} \zeta(\sigma)\zeta(\varsigma)$ |
| Forgotten index [21] | $\mathfrak{F} = \sum_{\sigma\in\mathfrak{V}\mathbf{id}(s_\eta)} (\zeta(\sigma))^3$ |
| Reduced second Zagreb index [22] | $\mathfrak{R} = \sum_{\sigma,\varsigma\in\mathfrak{E}\mathbf{id}(s_\eta)} [\zeta(\sigma)-1][\zeta(\varsigma)-1]$ |
| Narumi-Katayama index [23] | $\mathfrak{N} = \prod_{\sigma\in\mathfrak{V}\mathbf{id}(s_\eta)} \zeta(\sigma)$ |
| First multiplicative Zagreb index [22] | $\mathfrak{FM}_1 = \prod_{\sigma\in\mathfrak{V}\mathbf{id}(s_\eta)} (\zeta(\sigma))^2$ |
| Second multiplicative Zagreb index [22] | $\mathfrak{FM}_1 = \prod_{(\sigma,\varsigma)\in\mathfrak{E}\mathbf{id}(s_\eta)} \zeta(\sigma)\zeta(\varsigma)$ |

## 5 Main results

*In this section, we present a series of fundamental results related to the structure and properties of the identity graph of the symmetric group,* $\mathbf{id}(s_\eta)$*. We begin by exploring the calculation of* $\Omega_\eta$*, the number of elements of order 2 in the symmetric group* $s_\eta$*, through a summation formula that reflects the intricacies of the group's structure. This calculation lays the foundation for analyzing various characteristics of the identity graph* $\mathbf{id}(s_\eta)$*, including its order, which is shown to be* $\eta!$*, and the degree distribution of its vertices. Specifically, we determine the number of vertices of degree 1, which is equivalent to* $\Omega_\eta$*, and identify the unique vertex of degree* $\eta!-1$*. Furthermore, the number of vertices with degree 2, denoted by* $\mu$*, is derived, followed by a detailed examination of the edges between vertices of differing degrees. These results are integral to understanding the structural complexity of* $\mathbf{id}(s_\eta)$*, and they lead to the derivation of the first and second Zagreb indices, denoted by* $\xi_1$ *and* $\xi_2$*, respectively. These indices, which are pivotal in characterizing the connectivity properties of* $\mathbf{id}(s_\eta)$*, are computed using closed-form expressions that encapsulate the interplay between the vertex degrees and the graph's combinatorial properties.*

*In this document, we employ a variety of mathematical symbols and abbreviations to streamline the notation and improve readability. For clarity, the following table summarizes each symbol and its corresponding meaning, particularly those frequently referenced in discussions of symmetric groups, identity graphs, and topological indices. This notation will be consistent throughout, ensuring ease of interpretation and a coherent mathematical framework*

**Theorem 5.1.** *Let* $\Omega_\eta$ *be the number of elements of order 2 in a symmetric group* $s_\eta$ *then* $\Omega_\eta$ *can be calculated by the formula*

$$\Omega_\eta = \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \eta!\left(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})}\right)$$

| Symbol | Meaning |
|---|---|
| $s_\eta$ | Symmetric group with parameter $\eta$ |
| $\eta$ | Positive integer parameter in symmetric groups |
| $id(s_\eta)$ | Identity graph of symmetric group $S_\eta$ |
| $\mathfrak{V}(id(s_\eta))$ | Vertex set of the identity graph $S_\eta$ |
| $\mathfrak{E}(id(s_\eta))$ | Edge set of the identity graph $S_\eta$ |
| $\zeta$ | Degree of a vertex in the identity graph |
| $\xi_1$ | First Zagreb index |
| $\xi_2$ | Second Zagreb index |
| $\Omega_\eta$ | Number of elements of order 2 in $S_\eta$ |
| $\Phi_\eta$ | Number of vertices of degree 2 |
| $\mathfrak{p}, \mathfrak{q}$ | Exponents used in generalized Bonferroni mean |
| $\xi$ | Zagreb index parameter |
| $\mathfrak{F}$ | Forgotten index |
| $\mathfrak{R}$ | Reduced second Zagreb index |
| $\mathfrak{N}$ | Narumi-Katayama index |
| $\mathfrak{FM}_1$ | First multiplicative Zagreb index |
| $\mathfrak{FM}_2$ | Second multiplicative Zagreb index |

**Table 2.** List of abbreviations and symbols

*where* $\mathfrak{m} = \frac{\eta-\mathfrak{j}}{2}$ *and* $\eta \equiv j \pmod 2$

*Proof.* Let $s_\eta$ be the group of symmetries of $\{\alpha_1, \alpha_2, \dots \alpha_\eta\}$ then following type of elements are of order 2

$$\begin{aligned}&(\alpha_1, \alpha_2)\\&(\alpha_1, \alpha_2)(\alpha_3, \alpha_4)\\&\vdots\\&(\alpha_1, \alpha_2)(\alpha_3, \alpha_4)\dots(\alpha_{\eta-1}, \alpha_\eta)\end{aligned}$$

The number of elements of the type $(\alpha_1, \alpha_2)$ are the number of elements in the conjugacy class $C_{(\alpha_1,\alpha_2)}$ and can be calculated as

$$|C_{(\alpha_1,\alpha_2)}| = \frac{\eta!}{2(\eta-2)!}$$

The number of elements of the type $(\alpha_1, \alpha_2)(\alpha_3, \alpha_4)$ can be calculated by

$$|C_{(\alpha_1,\alpha_2)(\alpha_3,\alpha_4)}| = \frac{\eta!}{2^2 2!(\eta-4)!}$$

Similarly the number of elements of the type $(\alpha_1, \alpha_2)(\alpha_3, \alpha_4)$ can be calculated by

$$|C_{(\alpha_1,\alpha_2)(\alpha_3,\alpha_4)\dots(\alpha_{\eta-1},\alpha_\eta)}| = \frac{\eta!}{2^{\mathfrak{m}}\mathfrak{m}!(\eta-2\mathfrak{m})!}$$

So the total number of elements of order 2 in symmetric group $s_\eta$ is given bellow

$$\Omega_\eta = \frac{\eta!}{2(\eta-2)!} + \frac{\eta!}{2^2 2!(\eta-4)!} + \cdots + \frac{\eta!}{2^{\mathfrak{m}}\mathfrak{m}!(\eta-2\mathfrak{m})!}$$

$$\Omega_\eta = \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \eta!\left(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\right)$$

□

**Theorem 5.2.** *Let the identity graph of symmetric group $s_\eta$ is denoted by $\mathbf{id}(s_\eta)$ then*
*(1) Order of $\mathbf{id}(s_\eta)$ is equal to $\eta!$.*
*(2) Number of vertices of degree $1$ in $\mathbf{id}(s_\eta)$ is equal to $\Omega_\eta$.*
*(3) There is only one vertex of degree $(\eta!-1)$ in $\mathbf{id}(s_\eta)$ .*
*(4) Number of vertices of degree $2$ in $\mathbf{id}(s_\eta)$ denoted by $\Phi_\eta$ can be calculated by*

$$\Phi_\eta = \eta!\left(1 - \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \left(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\right)\right) - 1$$

*Proof.* (1) Vertices $\mathbf{id}(s_\eta)$ are the elements of symmetric group $s_\eta$.

(2) According to the definition of identity graph, only the self inverse elements of symmetric group are the vertices of degree one in $\mathbf{id}(s_\eta)$. Self inverse elements are the elements of degree 2 in symmetric group hence by theorem 5.1 this part of this theorem is proved.

(3) It is obvious as there is only one identity element in every group.

(4) Total number of vertices in $\mathbf{id}(s_\eta)$ is equal to $\eta!$. There are vertices of three different types in $\mathbf{id}(s_\eta)$. First of all there is only one element of degree $\eta - 1$ which is the identity element. Second type of elements are the element with degree one, which by the second part of this theorem are equal to $\Omega_\eta$. Third and last kind of vertices are the vertices of degree 2 represented by $\Phi_\eta$ and can be calculated as,

$$\begin{aligned}\Phi_\eta &= \eta - \Omega_\eta - 1\\ &= \eta! - \eta!(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\left(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\right) - 1\\ &= \eta!\left(1 - \sum_{\mathfrak{i}=1}^{\mathfrak{m}}\left(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\right)\right) - 1\end{aligned}$$

□

*Based on Theorem 5.1 and 5.2 we can generalize the identity graph of symmetric group $s_\eta$. The generalized identity graph of symmetric group $s_\eta$ is given in figure 3*

**Theorem 5.3.** *(1) Size of $\mathbf{id}(s_\eta)$ denoted by $\psi_\eta$ is given by the formula,*

$$\psi_\eta = \frac{\eta\left(3 - \left(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\left(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\right)\right)\right) - 3}{2}$$

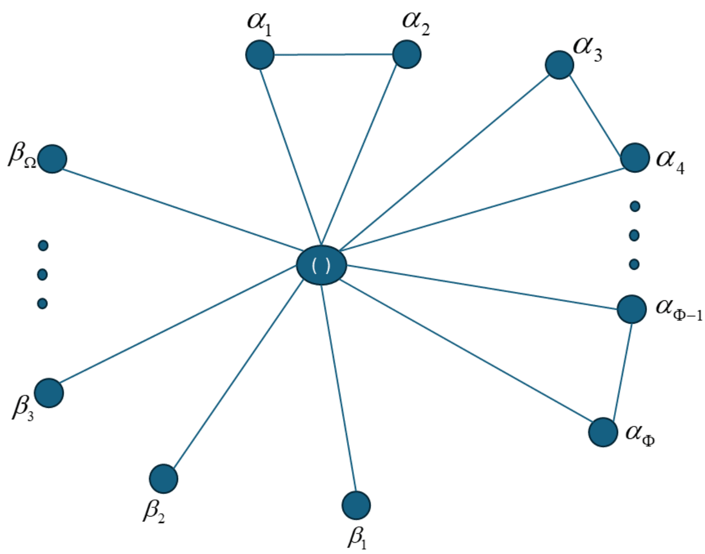


**Figure 3.** The generalized identity graph of $s_\eta$

*Proof.* Theorem 5.2 tells us the degree of each vertex in $\mathbf{id}(s_\eta)$. By the handshaking lemma,

$$\psi_\eta = \frac{\eta!\left(\sum_{i=1}^{m}\left(\frac{1}{2^i i!(n-2i)!}\right) + 2\left(\eta!\left(1-\left(\sum_{i=1}^{m}\left(\frac{1}{2^i i!(n-2i)!}\right)\right)\right)-1\right) + (\eta!-1)\right)}{2}$$

$$= \frac{\eta!\left(\sum_{i=1}^{m}\left(\frac{1}{2^i i!(n-2i)!}\right) + 2\eta! - 2\eta!\sum_{i=1}^{m}\left(\frac{1}{2^i i!(n-2i)!}\right) - 2 + \eta! - 1\right.}{2}$$

$$= \frac{3\eta! - \eta!\sum_{i=1}^{m}\left(\frac{1}{2^i i!(n-2i)!}\right) - 3}{2}$$

$$= \frac{\eta\left(3-\left(\sum_{i=1}^{m}\left(\frac{1}{2^i i!(n-2i)!}\right)\right)\right) - 3}{2}$$

□

## 5.1 Remark

*(1) The number of edges between two vertices where one vertex has degree 2 and other vertex has degree $(\eta!-1)$ can be calculated by the formula,*

$$\eta! - \Omega_\eta - 1$$

*(2) the number of edges between two vertices where one vertex has degree 1 and other vertex has degree $(\eta!-1)$ can be calculated by the formula*

$$\Omega_\eta = \eta!\sum_{i=1}^{m}\left(\frac{1}{2^i i!(n-2i)!}\right)$$

*(3) The number of edges between two vertices where both vertices have degree 2 can be calculated by the formula,*

$$\frac{\eta! - \Omega_\eta - 1}{2}$$

**Theorem 5.4.** *First Zagreb index of* $\mathbf{id}(s_\eta)$ *denoted by* $\xi_1$ *can be calculated by the formula,*

$$\xi_1 =(3+\eta!)\Big(\eta!\Big(1-\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)-1\Big)+\eta!\Big(\eta!\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)$$

*Proof.*

$$\begin{aligned}
\xi_1 =& \sum_{(\sigma,\varsigma)\in\mathfrak{E}(\mathbf{id}(\mathfrak{s}_\eta))}(\zeta(\sigma)+\zeta(\varsigma))\\
=&(2+2)\frac{\Big(\eta!\Big(1-\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)-1\Big)}{2}\\
&+(2+\eta!-1)\Big(\eta!\Big(1-\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)-1\Big)\\
&+(1+\eta!-1)\Big(\eta!\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)\\
=&2\Big(\eta!\Big(1-\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)-1\Big)\\
&+(1+\eta!)\Big(\eta!\Big(1-\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)-1\Big)+\eta!\Big(\eta!\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)\\
=&(3+\eta!)\Big(\eta!\Big(1-\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)-1\Big)+\eta!\Big(\eta!\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)
\end{aligned}$$

□

**Theorem 5.5.** *Second Zagreb index of* $\mathbf{id}(s_\eta)$ *denoted by* $\xi_2$ *can be calculated by the formula,*

$$\xi_2 = (3+\eta!)\Big(\eta!\Big(1-\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)-1\Big)+\eta!\Big(\eta!\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)$$

*Proof.*

$$\begin{aligned}
\xi_2 =& \sum_{(\sigma,\varsigma)\in\mathfrak{E}(\mathbf{id}(\mathfrak{s}_\eta))}(\zeta(\sigma)+\zeta(\varsigma))\\
=&(2\times 2)\frac{\Big(\eta!\Big(1-\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)-1\Big)}{2}\\
&+(2(\eta!-1))\Big(\eta!\Big(1-\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)-1\Big)\\
&+(1(\eta!-1))\Big(\eta!\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)\\
=&2\Big(\eta!\Big(1-\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)-1\Big)\\
&+(2\eta!-2)\Big(\eta!\Big(1-\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)-1\Big)+(\eta!-1)\Big(\eta!\Big(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)\Big)\\
=&-\eta!(\eta!+1)\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\Big(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})!}\Big)+(\eta!-1)2\eta!
\end{aligned}$$

□

**Theorem 5.6.** *Forgotten index of* $\mathbf{id}(s_\eta)$ *denoted by* $\mathfrak{F}$ *can be calculated by the formula,*

$$\mathfrak{F} = 8\eta! - 7\Omega_\eta - 8 + (\eta! - 1)^3$$

*Where*

$$\Omega_\eta = \sum_{\mathrm{i}=1}^{\mathfrak{m}} \eta!\left(\frac{1}{2^{\mathrm{i}}\mathrm{i}!(n-2\mathrm{i})}\right) \textit{ and } \mathfrak{m} = \frac{\eta - \mathrm{j}}{2}\ \forall \eta \equiv j \pmod 2$$

*Proof.*

$$\begin{aligned}\mathfrak{F} = &\sum_{\sigma \in \mathfrak{V}\mathbf{id}(s_\eta)} (\zeta(\sigma))^3 \\ =&(1)^3\Omega_\eta + (2)^3(\eta! - \Omega_\eta - 1) + (\eta! - 1)^3(1) \\ =&\Omega_\eta + 8\eta! - 8\Omega_\eta - 8 + (\eta! - 1)^3 \\ =&8\eta! - 7\Omega_\eta - 8 + (\eta! - 1)^3\end{aligned}$$

□

**Theorem 5.7.** *Reduced second Zagreb index of* $\mathbf{id}(s_\eta)$ *denoted by* $\mathfrak{R}$ *can be calculated by the formula,*

$$\mathfrak{R} = (\eta! - 2)(\eta! - \Omega - 1) + \left(\frac{\eta! - \Omega - 1}{2}\right)$$

*Where*

$$\Omega_\eta = \sum_{\mathrm{i}=1}^{\mathfrak{m}} \eta!\left(\frac{1}{2^{\mathrm{i}}\mathrm{i}!(n-2\mathrm{i})}\right) \textit{ and } \mathfrak{m} = \frac{\eta - \mathrm{j}}{2}\ \forall \eta \equiv j \pmod 2$$

*Proof.*

$$\begin{aligned}\mathfrak{R} = &\sum_{\sigma,\varsigma \in \mathfrak{E}\mathbf{id}(s_\eta)} [\zeta(\sigma) - 1][\zeta(\varsigma) - 1] \\ =&[1-1][\eta! - 1 - 1](\Omega) + [2-1][\eta! - 1 - 1](\eta! - \Omega - 1) + [2-1][2-1]\left(\frac{\eta! - \Omega - 1}{2}\right) \\ =&(\eta! - 2)(\eta! - \Omega - 1) + \left(\frac{\eta! - \Omega - 1}{2}\right)\end{aligned}$$

□

**Theorem 5.8.** *Narumi-Katayama index of* $\mathbf{id}(s_\eta)$ *denoted by* $\mathfrak{N}$ *can be calculated by the formula,*

$$\mathfrak{N} = (2^{(\eta! - \Omega - 1)})(\eta! - 1)$$

*Where*

$$\Omega_\eta = \sum_{\mathrm{i}=1}^{\mathfrak{m}} \eta!\left(\frac{1}{2^{\mathrm{i}}\mathrm{i}!(n-2\mathrm{i})}\right) \textit{ and } \mathfrak{m} = \frac{\eta - \mathrm{j}}{2}\ \forall \eta \equiv j \pmod 2$$

*Proof.*

$$\begin{aligned}\mathfrak{N} = &\prod_{\sigma \in \mathfrak{V}\mathbf{id}(s_\eta)} \zeta(\sigma) \\ =&(1)^{\eta!} \times (2^{(\eta! - \Omega - 1)}) \times (\eta! - 1) \\ =&(2^{(\eta! - \Omega - 1)})(\eta! - 1)\end{aligned}$$

□

**Theorem 5.9.** *First multiplicative Zagreb index of* $\mathbf{id}(s_\eta)$ *denoted by* $\mathfrak{FM}_1$ *can be calculated by the formula,*

$$\mathfrak{FM}_1 = 4\Omega(\eta! - \Omega - 1)(\eta! - 1)^2$$

*Where*

$$\Omega_\eta = \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \eta!\left(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})}\right) \textit{ and } \mathfrak{m} = \frac{\eta - \mathfrak{j}}{2} \; \forall \eta \equiv j \pmod 2$$

*Proof.*

$$\begin{aligned}\mathfrak{FM}_1 = & \prod_{\sigma \in \mathfrak{V}\mathbf{id}(s_\eta)} (\zeta(\sigma))^2 \\ =& 1^2 \times 2^2(\eta! - \Omega - 1) \times (\eta! - 1)^2(1) \\ =& 4\Omega(\eta! - \Omega - 1)(\eta! - 1)^2\end{aligned}$$

□

**Theorem 5.10.** *Second multiplicative Zagreb index of* $\mathbf{id}(s_\eta)$ *denoted by* $\mathfrak{FM}_2$ *can be calculated by the formula,*

$$\mathfrak{FM}_2 = (\eta! - 1)^{(\eta!-1)} 2^{(\frac{\Omega+\eta!+1}{2})}$$

*Where*

$$\Omega_\eta = \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \eta!\left(\frac{1}{2^{\mathfrak{i}}\mathfrak{i}!(n-2\mathfrak{i})}\right) \textit{ and } \mathfrak{m} = \frac{\eta - \mathfrak{j}}{2} \; \forall \eta \equiv j \pmod 2$$

*Proof.*

$$\begin{aligned}\mathfrak{FM}_1 = & \prod_{(\sigma,\varsigma) \in \mathfrak{E}\mathbf{id}(s_\eta)} \zeta(\sigma)\zeta(\varsigma) \\ =& (1(\eta! - 1)^{\Omega})(2(\eta! - 1))^{(\eta!-\Omega-1)}(2 \times 2)^{(\frac{\eta!-\Omega-1}{2})} \\ =& (\eta - 1)^{(\eta!-\Omega-1+\Omega)} \times 2^{(\Omega+1+\frac{\eta!-\Omega-1}{2})} \\ =& (\eta - 1)^{(\eta!-1)} \times 2^{(\frac{2\Omega+2+\eta!-\Omega-1}{2})} \\ =& (\eta! - 1)^{(\eta!-1)} \times 2^{(\frac{\Omega+\eta!+1}{2})}\end{aligned}$$

□

## 6 Applications of generalized identity graphs and indices in symmetric groups

*The symmetric group* $s_\eta$ *plays a critical role in various mathematical and scientific disciplines, primarily due to its ability to represent permutations and symmetries in systems. By generalizing the identity graph of* $s_\eta$*, along with key graph-theoretical indices-such as the Zagreb indices, multiplicative Zagreb indices, Narumi-Katayama index, reduced second Zagreb index, and forgotten index-this work enhances the toolkit available for analyzing the structural properties of symmetric groups. These generalizations open up new avenues of research and application in the following key areas.*

### 6.1 Cryptography

*Symmetric groups are widely used in cryptography, particularly in the design of group-based encryption systems. The elements of* $s_\eta$*, specifically those of order 2, are integral to many cryptographic protocols that rely on the algebraic properties of these groups. By generalizing the* *identity graph and introducing $\eta$-dependent indices, this work provides refined tools for analyzing the structural complexity of these groups. The generalized Zagreb and multiplicative indices can be used to measure the robustness and security of cryptographic systems by assessing the interconnectivity and resilience of the underlying algebraic structures. This can lead to the development of more secure encryption protocols based on symmetric group properties.*

### 6.2 Chemical graph theory

*In chemical graph theory, symmetric groups help model the symmetries in molecular structures, which are essential for predicting molecular properties. The identity graph of $s_\eta$ and its generalized indices, such as the Zagreb indices, offer a novel approach to quantifying molecular stability and reactivity. The $\eta$-dependent formulas for these indices provide a more detailed understanding of the molecular graph's structural intricacies. For instance, the generalized forgotten index can be applied to predict the stability of molecules with complex symmetry, offering chemists an enhanced method for evaluating potential chemical reactions and designing new molecules with desired properties.*

### 6.3 Quantum mechanics

*Symmetric groups appear prominently in quantum mechanics, particularly in systems involving identical particles, where the symmetries of the wavefunction are described by elements of $s_\eta$. The generalization of the identity graph of these groups allows for a deeper examination of the symmetries present in quantum states. By utilizing the generalized graph-theoretical indices, such as the Zagreb and multiplicative Zagreb indices, researchers can better understand the structural properties of quantum systems, leading to new insights into quantum entanglement and particle interactions. These tools are particularly beneficial for optimizing quantum algorithms and studying the behavior of quantum systems with high degrees of symmetry.*

## 7 Limitations and future directions

*In previous studies, such as [24] and [19], topological indices and graph energies were explored using machine learning algorithms, particularly polynomial regression. These studies aimed to model and predict various properties of algebraic graph structures by applying polynomial regression techniques to analyze topological indices and energies. While polynomial regression is a powerful tool, it is most effective when applied to datasets that are closely and equally spaced. This assumption holds for many systems, particularly in physical and chemical contexts where changes occur gradually or in relatively predictable patterns.*

*However, when considering the symmetric group $s_\eta$, the number of elements grows extremely rapidly with respect to $\eta$, following a factorial progression. This rapid, non-linear increase in the number of elements creates significant challenges for applying polynomial regression effectively. In symmetric groups, the data points, representing different values of $\eta$ and the associated topological indices are neither closely spaced nor follow a predictable, smooth trajectory. Instead, the gaps between successive values increase exponentially. As a result, polynomial regression does not provide accurate or reliable models for predicting topological indices or graph energies within these groups. The steep growth and irregularity of the data cause polynomial regression to overfit or underfit, leading to significant errors in both interpolation and extrapolation. In our research, we initially attempted to apply polynomial regression to generalize the energies of symmetric group's identity graphs. However, the results were not satisfactory. The inherent nature of the symmetric group data specifically, its factorial growth prevented polynomial regression from capturing the true complexity of the system. This finding underscores the limitations of polynomial regression in cases where the underlying data does not conform to the algorithm's assumptions of smoothness and equal spacing.*

*Furthermore, while our current research focuses on the crisp topological indices of the symmetric group's identity graphs, studies like [24] and [19] have also ventured into the realm of fuzzy topological indices. Fuzzy indices are particularly useful in systems that exhibit uncertainty or imprecision, as they offer a more flexible framework for modeling real-world phenomena. In graph theory, fuzzy indices allow for the representation of uncertainties within graph structures,* *which can be especially beneficial in fields like chemical graph theory, where molecular structures and interactions often involve degrees of uncertainty.*

*Given the successful application of fuzzy indices in previous works, our research offers a clear path for future extension. While we have concentrated on crisp indices, the techniques and generalizations we have developed could be expanded to incorporate fuzzy logic. By applying fuzzy set theory to our generalized identity graphs and topological indices, we could account for uncertainties or variations within the graph structures of symmetric groups. This would make the indices even more versatile and applicable in a wider range of scientific and mathematical contexts.*

## 8 Conclusion

- *This study successfully generalizes the identity graph of the symmetric group $s_\eta$, offering $\eta$-dependent formulas for efficiently calculating key topological indices, including the first and second Zagreb indices, multiplicative Zagreb indices, Narumi-Katayama index, reduced second Zagreb index, and forgotten index.*

- *The generalization of the number of self-inverse elements in $s_\eta$ allows for a deeper understanding of the structural properties of symmetric groups, enabling the construction of identity graphs for any value of $\eta$.*

- *The symmetric group $s_\eta$, being a universal space for group structures, provides a foundation for extending these generalized results to other groups, such as dihedral groups, which can be studied as subgroups or isomorphic structures.*

- *This work enhances the theoretical understanding of symmetric groups and expands the application of topological indices, particularly in fields like cryptography, chemical graph theory, and quantum mechanics, where group structures are critical.*

- *The research lays the groundwork for further studies into fuzzy topological indices, extending beyond crisp indices and offering potential for future exploration in fuzzy systems.*

- *The findings contribute to bridging algebraic theory with practical applications, providing a unified framework for the study of identity graphs across various group structures.*

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### Author information


*Muhammad U. Mirza, Department of Mathematics and Statistics, University of Lahore, Pakistan.*
*E-mail:* umarmirza096@gmail.com

*R. Anjum, Department of Mathematics and Statistics, University of Lahore, Pakistan.*
*E-mail:* rukhshanda.anjum@math.uol.edu.pk



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