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NEUTROSOPHIC GRADED MEAN INTEGRATION OF FUZZY NON-LINEAR ECONOMIC ORDER QUANTITY WITH POWER AND TIME CONSUMPTION TOTAL COST MODEL

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Abstract This paper presents a novel approach to optimize the Economic Order Quantity (EOQ) model by incorporating Neutrosophic Fuzzy logic and Kuhn-Tucker conditions, aimed at minimizing total inventory costs in complex manufacturing processes. The focus of this research is on addressing the inherent uncertainties and vagueness in inventory management data, which are often challenging to quantify due to the manual nature of data collection and the absence of structured time management systems. Traditional EOQ models often assume crisp, deterministic data, which is rarely the case in real-world scenarios where imprecise and ambiguous information is common. To tackle these challenges, this study proposes a comprehensive methodology that integrates advanced fuzzy logic techniques with Neutrosophic sets to better model the uncertainty and vagueness in inventory variables. The model employs Pentagonal Fuzzy Numbers, known for their flexibility in representing uncertain parameters, alongside the Graded Mean Integration Representation (GMIR) method for defuzzification. This defuzzification process translates fuzzy parameters into crisp values, allowing for more accurate and reliable decision-making. The optimization framework is further enhanced by the application of the Kuhn-Tucker conditions, a powerful tool for solving non-linear programming problems. This mathematical approach is used to identify the optimal conditions for minimizing the total cost associated with inventory management, considering both the holding and ordering costs while accounting for the uncertainties modeled by the fuzzy parameters. By combining Neutrosophic Fuzzy logic with the Kuhn-Tucker conditions, the proposed model provides a robust solution that not only minimizes costs but also improves the efficiency and reliability of inventory management in manufacturing processes. The findings of this research demonstrate that the integrated approach significantly enhances the accuracy of inventory optimization in environments characterized by uncertain and imprecise data. This methodology offers a more realistic and effective framework for inventory management, ultimately leading to reduced costs, improved resource utilization, and enhanced profitability in manufacturing operations.

1 Introduction

1.1 Practical and Methodological Aims of the Study

This research focuses on the application of fuzzy logic in inventory management, particularly in the development of a Dynamic Variable Sensitivity (DVS) framework for improving process inventory data estimation. The study aims to enhance the accuracy and efficiency of resource consumption measurements in manufacturing processes. Specifically, it addresses the uncertainties in data that challenge precise quantification of sustainability performance in production environments, applying fuzzy logic to optimize the Economic Order Quantity (EOQ) in uncertain inventory systems.

1.2 Motivation for Conducting Research

The motivation for this study stems from the increasing environmental concerns associated with industrial manufacturing processes. As the industrial sector seeks more sustainable practices, the ability to accurately assess energy and resource consumption becomes critical. Addressing the challenge of data uncertainty in life cycle assessments, this research contributes by providing a robust methodology for improving inventory management decisions in the face of variable conditions. Incorporating fuzzy logic into traditional EOQ models addresses the limitations of binary logic, which often fails to capture the complexities of real-world inventory scenarios.

1.3 Contributions of the Study

The primary contribution of this study is the development and application of a fuzzy logic-based approach to inventory management that incorporates the DVS framework. This research applies fuzzy arithmetic operations and the Kuhn-Tucker method to optimize the fuzzy EOQ for different inventory models, demonstrating how fuzzy logic can be used to handle uncertainties in the management of production processes. Additionally, the research demonstrates the effectiveness of the Graded Mean Integration Representation (GMIR) method in defuzzifying total costs, further enhancing the practicality of fuzzy logic in optimizing manufacturing systems.

1.4 Organization of the Paper

The paper is organized as follows: Section 2 reviews the foundational concepts of fuzzy logic and its application to decision-making in uncertain environments. Section 3 details the proposed DVS framework and its implementation in a through-feed centerless grinding configuration. Section 4 presents the fuzzy inventory model and the application of fuzzy arithmetic to optimize the EOQ. Section 5 discusses the Kuhn-Tucker method and GMIR for solving the inventory optimization problem. Section 6 provides numerical examples to illustrate the solution procedure. Finally, Section 7 concludes the paper by summarizing the findings and outlining future research directions.

2 Literature Analysis

Fuzzy logic, introduced by Zadeh (1965), marked a paradigm shift from classical logic by allowing for degrees of truth, which proved invaluable in handling uncertainty and imprecision in real-world scenarios. Building on Zadeh's work, Bellman and Zadeh (1970) applied fuzzy logic to decision-making in uncertain environments, furthering its practical applications. Fuzzy sets and systems have since found widespread applications across various fields, from decision-making to inventory management. Harris (1915) introduced the EOQ model, which became a foundational concept in inventory control. This model, however, did not account for uncertainties inherent in inventory systems. Chen, Wang, and Ramer (1996) extended the EOQ model by incorporating fuzzy logic to handle back-orders under uncertain conditions. The use of fuzzy arithmetic allowed for the optimal calculation of inventory replenishment while considering uncertainty. Subsequent research by Kalaiarasi and Gopinath (2020) demonstrated how fuzzy EOQ models could be further optimized by incorporating mathematical models, effectively minimizing costs in uncertain environments.

Recent studies have highlighted new approaches for decision-making in fuzzy environments. The Graded Mean Integration Representation (GMIR) method developed by Chen and Hsieh (1999) plays a critical role in defuzzifying fuzzy total costs, making it essential in solving fuzzy inventory problems. Kalaiarasi et al. (2021) continued this work, applying fuzzy EOQ models to inferior products and maximizing profit per cycle under uncertainty. Furthermore, innovative methodologies for integrating fuzzy logic into manufacturing systems, such as Kim et. al. (2015) decision-guidance framework for sustainability performance analysis, are instrumental in improving resource management and environmental sustainability.

The most recent advances in fuzzy systems and neutrosophic models, as seen in the work of Murugan Palanikumar and others (2023-2024), have explored complex applications in medical robotics and decision-making processes. The use of neutrosophic logic, a generalization of fuzzy logic, offers even greater flexibility for managing uncertainty. This framework has been successfully applied to various real-world problems, ranging from industrial systems to medical engineering. Studies like Nagarajan et al. (2023) also emphasize the importance of using neutrosophic models in multi-attribute decision-making environments.

Recent research also points to the integration of fuzzy and neutrosophic sets in inventory management, where particle swarm optimization methods were used to solve EOQ models with uncertain variables, as seen in Kalaiarasi et al. (2024). These developments suggest a growing trend toward hybrid models that combine fuzzy logic, neutrosophic sets, and machine learning for more accurate decision-making in uncertain environments. Additionally, Syed Ahmad et al. (2022) discussed the use of fuzzy modeling in dynamic environments to optimize sustainability. This study aligns closely with the current research, particularly in its emphasis on the need for frameworks capable of quantifying resource consumption in uncertain contexts.

This literature analysis highlights the progression from traditional fuzzy logic to more advanced frameworks such as neutrosophic models. These developments have significantly expanded the applicability of fuzzy systems in inventory management, manufacturing processes, and sustainability assessments. Moving forward, integrating fuzzy logic with advanced machine learning techniques and defuzzification methods like GMIR will be essential for optimizing decision-making in uncertain environments.

2.1 Research Gap

Traditional Economic Order Quantity (EOQ) models operate under the assumption that all data such as demand rates, holding costs, and ordering costs are deterministic and can be accurately quantified. However, in real-world manufacturing environments, data are often uncertain, vague, and difficult to quantify due to variability in demand, supply chain disruptions, and manual processes. This uncertainty leads to inefficiencies in inventory management, where optimal inventory levels are difficult to determine. Moreover, traditional EOQ models fail to account for these imprecisions, leading to suboptimal inventory decisions that increase costs.

The research gap lies in addressing this uncertainty by introducing fuzzy logic, particularly Neutrosophic fuzzy sets, to better model imprecise inventory variables. While fuzzy logic has been used in EOQ models, the integration of Neutrosophic sets which allow for degrees of truth, indeterminacy, and falsity provides a more robust framework for handling the inherent vagueness in inventory data. The novelty of this research lies in combining Pentagonal Fuzzy Numbers and the Kuhn-Tucker conditions to optimize inventory costs more effectively.

2.2 Applicability of the Method

This approach is essential for inventory management in industries where uncertainties dominate, such as manufacturing processes with unpredictable demand or supply variability. By using Neutrosophic fuzzy logic, the model can handle not just uncertainty, but also indeterminacy, making it more applicable to complex, real-world scenarios compared to conventional EOQ models. The Pentagonal Fuzzy Numbers allow for flexible representation of fuzzy parameters, making it easier to accommodate varying levels of uncertainty in cost variables. Furthermore, the Graded Mean Integration Representation (GMIR) method ensures that defuzzified values, representing the imprecise parameters, are accurate and reliable. The Kuhn-Tucker conditions provide a rigorous optimization framework that can handle the non-linear relationships often present in inventory cost functions, leading to more efficient decision-making.

2.3 Limitations of the Model

- The integration of Neutrosophic sets and Pentagonal Fuzzy Numbers increases the complexity of the model. Implementing this model in practice requires a deep understanding of fuzzy logic and optimization techniques, which may pose a challenge for practitioners unfamiliar with these concepts.
- Although the model accommodates uncertainty, it still requires accurate estimates of the fuzzy parameters. In cases where data are entirely unavailable or too ambiguous, the model's effectiveness may be limited.

- The use of advanced optimization techniques like the Kuhn-Tucker conditions may increase computational requirements, especially for large-scale inventory systems.
- While the model is robust in uncertain environments, it may not provide significant improvements in more stable or predictable inventory systems where deterministic EOQ models are already effective.

3 Notations

In the context of an inventory model, several key notations are critical for understanding and optimizing the system, particularly when considering factors such as power consumption, investment costs, demand, quantity, operator costs, quality, and time consumption.

- P= Power consumption
- I= Investment cost
- D= Demand
- Q= Quantity
- O= Operator cost
- Qu= Quality
- T= Time consumption

Power consumption (P) represents the total energy required for production, storage, and operational activities within the inventory management process. This includes electricity for machinery, lighting, and other essential equipment. Power consumption is a significant operational expense and directly impacts both costs and environmental sustainability. Effective management of power consumption is vital for reducing operational costs and minimizing carbon emissions, which can be achieved through energy-efficient technologies and renewable energy sources.

Investment cost (I) refers to the initial capital outlay needed to acquire the necessary resources for production and inventory management, such as raw materials, machinery, and facilities. Investment costs play a crucial role in financial planning, as high upfront expenditures can place a financial burden on businesses, particularly those with limited capital or under debt. To manage these costs effectively, businesses may consider strategies such as leasing equipment, adopting just-in-time (JIT) inventory practices, or investing in scalable solutions that allow for incremental growth without significant initial capital expenditure.

Demand (D) is the quantity of goods or services that customers are willing and able to purchase over a specific period. Accurate demand forecasting is essential for maintaining optimal inventory levels, ensuring that businesses can meet customer needs without overstocking or understocking. Inaccurate demand forecasts can lead to excess inventory, increasing holding costs, or shortages that result in lost sales and customer dissatisfaction. Advanced forecasting techniques, including data analytics and machine learning, can help businesses predict demand more accurately, aligning their inventory levels with market requirements.

Quantity (Q) represents the number of units produced, stored, or ordered within a given timeframe. This metric is directly influenced by demand forecasts and plays a crucial role in determining the balance between supply and demand. The goal is to produce or order a quantity that meets customer demand while minimizing inventory holding costs and avoiding stockouts. Each of these factors power consumption, investment cost, demand, and quantity interacts dynamically within the inventory management process. By carefully monitoring and optimizing these variables, businesses can achieve a more efficient, cost-effective, and sustainable operation.

3.1 The Kuhn Tucker Conditions

Taha[5] (1997) discussed how to solve the optimum solution of nonlinear programming problem subject to inequality constraints by using the Kuhn-Tucker conditions. The development of the Kuhn-Tucker conditions is based on the Lagrangian Method.
Suppose that the problem is given by Minimize $y = f(x)$

Subject to $g_i(x) \geq 0, i = 1, 2, \ldots, m$
The non-negativity constraints $x \geq 0$, if any, are included in the m constraints. The inequality constraints may be converted into equations by using nonnegative surplus variables. Let P_i^2 be the surplus quantity added to the i^{th} constraints $g_i(x) \geq 0$. Let

$$\mu = (\mu_1, \mu_2, \ldots, \mu_m), g(x) = (g_1(x), g_2(x), \ldots, g_m(x)) \ldots \& \ P_2 = (P_1^2, P_2^2, \ldots, P_m^2)$$

The Kuhn-Tucker conditions need and λ to be a stationary point of the minimization problem, which can be summarized as follows:

$$\left\{\begin{array}{r} \mu \leq 0 \\ \nabla f(x) - \mu \nabla g(x) = 0, \\ \mu_i g_i(x) = 0, i = 1, 2, \ldots, m \\ g_i(x) \geq 0, i = 1, 2, \ldots, m \end{array}\right\}$$

3.2 Graded Mean Integration Representation Method

Let $\tilde{Z}$ be a Pentagonal Fuzzy number and denoted as

$$\tilde{Z} = (Z_1, Z_2, Z_3, Z_4, Z_5)$$

$$\begin{aligned} P(\tilde{Z}) &= \frac{\int_0^1 \frac{h}{2}\left[(z_1 + z_5) + h\left(z_2 - z_1 - z_4 - z_3 - z_5 - z_4\right)\right] dh}{\int_0^1 h dh} \\ &= \frac{Z_1 + 2Z_2 + 2Z_3 + 2Z_4 + Z_5}{8} \end{aligned}$$

4 Mathematical Model

A mathematical model is a structured representation that describes a real-life situation using equations to simulate the behavior of real-world objects or processes. The primary goal of such a model is to replicate the dynamics of the system it represents, enabling analysis and prediction of outcomes under various scenarios. By manipulating variables within the model, it is possible to explore different conditions, such as changes in measurements, demand levels, or costs, to observe how these alterations impact the system. This approach is widely used across various fields to solve complex real-life problems by providing a structured framework for decision-making and optimization.

In inventory management, one of the key aspects to consider is the total cost associated with production, storage, and ordering. The total cost can be expressed as a function that encompasses various elements such as production costs, holding costs, order costs, and possibly penalty costs for shortages or delays. In scenarios where data is uncertain or imprecise, traditional crisp models may not adequately capture the complexities involved. This is where a Neutrosophic Fuzzy Inventory Model comes into play, allowing for a more realistic representation of uncertainties in the system.

In this model, the total cost function is extended to incorporate Neutrosophic fuzzy variables, which account for the indeterminacy and vagueness inherent in real-world inventory systems. These fuzzy variables are represented using Pentagonal Fuzzy Numbers, which provide a flexible and nuanced way to model uncertainty in the parameters. The defuzzification process, typically performed using the Graded Mean Integration Representation (GMIR) method, translates these fuzzy parameters into crisp values that can be used for precise calculations.

$$TC_s = \frac{2QP}{T} + \frac{DOT}{Qu} + \frac{2ID}{T} + \frac{TQuO}{P}$$

Differentiating with respect to T and $\frac{\partial TC_s}{\partial T}$, we get

$$T^* = \sqrt{\frac{QuP[2QP + 2ID]}{PDO + OQu^2}}$$

To optimize the total cost within the Neutrosophic Fuzzy Inventory Model, we differentiate the fuzzy cost function with respect to certain variables, such as time (T) or other relevant parameters. This process involves taking the derivative of the fuzzy cost function with respect to T and setting it to zero to find the critical points. These critical points help identify the optimal values for the variables that minimize or maximize the total cost, depending on the objective of the model.

Time consumption is a significant factor in this model, as it directly impacts both production efficiency and cost. By analyzing the time-related components within the Neutrosophic Fuzzy Inventory Model, we can determine the optimal time allocation for various stages of the manufacturing process. The derived values from the differentiation process indicate the most efficient use of time, considering the inherent uncertainties in the production process, thus helping to minimize unnecessary delays and reduce overall costs. The integration of Neutrosophic fuzzy logic into the inventory model enhances the decision-making process by providing a more comprehensive understanding of the system's behavior under uncertain conditions, ultimately leading to more effective inventory management strategies.

4.1 An Integrated Neutrosophic Inventory Model for Crisp Production Quantity

An Integrated Inventory Model for Neutrosophic Fuzzy Production Quantity We introduce an integrated inventory model incorporating Neutrosophic fuzzy parameters for determining the optimal production quantity. In this model, the production quantity is considered crisp, while key parameters influencing the inventory system are treated as Neutrosophic fuzzy variables, which account for the inherent uncertainties, indeterminacies, and vagueness in real-world data. The inventory model is formulated as follows

$$
\begin{aligned}
J\tilde{T}C(T) = {} & \frac{2QP}{T} + \frac{D_1O_1T}{Qu} + \frac{2ID_1}{T} + \frac{TQuO_1}{P}, \\
& \frac{2QP}{T} + \frac{D_2O_2T}{Qu} + \frac{2ID_2}{T} + \frac{TQuO_2}{P}, \\
& \frac{2QP}{T} + \frac{D_3O_3T}{Qu} + \frac{2ID_3}{T} + \frac{TQuO_3}{P}, \\
& \frac{2QP}{T} + \frac{D_4O_4T}{Qu} + \frac{2ID_4}{T} + \frac{TQuO_4}{P}, \\
& \frac{2QP}{T} + \frac{D_5O_5T}{Qu} + \frac{2ID_5}{T} + \frac{TQuO_5}{P}
\end{aligned}
$$

Suppose $\tilde{D} = (D_1, D_2, D_3, D_4, D_5)$ and $\tilde{O} = (O_1, O_2, O_3, O_4, O_5)$
Are non - negative trapezoidal neutrosophic fuzzy numbers. Then we solve the optimal production quantity of formula as the following steps. Next, we defuzzify the fuzzy total production inventory for the maximizing profit by graded mean integration representation.

$$
P(J\tilde{T}C(T)) = \frac{1}{8}\begin{bmatrix}
\left[\frac{2QP}{T} + \frac{D_1O_1T}{Qu} + \frac{2ID_1}{T} + \frac{TQuO_1}{P}\right] + \\
2\left[\frac{2QP}{T} + \frac{D_2O_2T}{Qu} + \frac{2ID_2}{T} + \frac{TQuO_2}{P}\right] + \\
2\left[\frac{2QP}{T} + \frac{D_3O_3T}{Qu} + \frac{2ID_3}{T} + \frac{TQuO_3}{P}\right] + \\
2\left[\frac{2QP}{T} + \frac{D_4O_4T}{Qu} + \frac{2ID_4}{T} + \frac{TQuO_4}{P}\right] + \\
\left[\frac{2QP}{T} + \frac{D_5O_5T}{Qu} + \frac{2ID_5}{T} + \frac{TQuO_5}{P}\right]
\end{bmatrix}
$$

To find the minimization of $P(J\tilde{T}C(T))$
The partial derivative of $P(J\tilde{T}C(T))$ with respect to T is

$$\frac{\partial P(J\tilde{T}C(T))}{\partial T} = \frac{1}{8}\begin{bmatrix} \left[-\frac{2QP}{T^2} + \frac{O_1D_1T}{Qu} - \frac{2ID_1}{T^2} + \frac{TQuO_1}{P}\right] + \\ 2\left[-\frac{2QP}{T^2} + \frac{O_2D_2T}{Qu} - \frac{2ID_2}{T^2} + \frac{TQuO_2}{P}\right] + \\ 2\left[-\frac{2QP}{T^2} + \frac{O_3D_3T}{Qu} - \frac{2ID_3}{T^2} + \frac{TQuO_3}{P}\right] + \\ 2\left[-\frac{2QP}{T^2} + \frac{O_4D_4T}{Qu} - \frac{2ID_4}{T^2} + \frac{TQuO_4}{P}\right] + \\ \left[-\frac{2QP}{T^2} + \frac{O_5D_5T}{Qu} - \frac{2ID_5}{T^2} + \frac{TQuO_5}{P}\right] \end{bmatrix}$$

Let $\frac{\partial P(J\tilde{T}C(T))}{\partial T} = 0$, we get

$$T^* = \sqrt{\frac{QuP\left\{\begin{array}{r}[2QP + 2ID_1] + 2[2QP + 2ID_2] + 2[2QP + 2ID_3] \\ +2[2QP + 2ID_4] + [2QP + 2ID_5]\end{array}\right\}}{\begin{array}{r}[PD_1O_1 + Qu^2O_1] + 2[PD_2O_2 + Qu^2O_2] + 2[PD_3O_3 + Qu^2O_3] \\ +2[PD_4O_4 + Qu^2O_4] + [PD_5O_5 + Qu^2O_5]\end{array}}}$$

We now have the value for the crisp model in Kuhn - Tucker method. The result will yield the crisp value of the production quantity that minimizes the total cost while accommodating the uncertainties represented by fuzzy parameters. This value is derived from solving the fuzzy model using the methods mentioned, including defuzzification and applying optimization techniques.

4.2 An Integrated Inventory Model for Neutrosophic Fuzzy Production Quantity

Suppose fuzzy production quantity $\tilde{T}$ be a neutrosophic pentagonal fuzzy number $\tilde{T} = (T_1, T_2, T_3, T_4, T_5)$ with $0 < T_1 \leq T_2 \leq T_3 \leq T_4 \leq T_5$. We get the fuzzy total production inventory cost.

$$P(J\tilde{T}C(T)) = \frac{1}{8}\begin{bmatrix} \left[\frac{2QP}{T} + \frac{O_1D_1T}{Qu} + \frac{2ID_1}{T} + \frac{TQuO_1}{P}\right] + \\ 2\left[\frac{2QP}{T} + \frac{O_2D_2T}{Qu} + \frac{2ID_2}{T} + \frac{TQuO_2}{P}\right] + \\ 2\left[\frac{2QP}{T} + \frac{O_3D_3T}{Qu} + \frac{2ID_3}{T} + \frac{TQuO_3}{P}\right] + \\ 2\left[\frac{2QP}{T} + \frac{O_4D_4T}{Qu} + \frac{2ID_4}{T} + \frac{TQuO_4}{P}\right] + \\ \left[\frac{2QP}{T} + \frac{O_5D_5T}{Qu} + \frac{2ID_5}{T} + \frac{TQuO_5}{P}\right] \end{bmatrix}$$

To optimize the total production inventory cost within this framework, we first express the fuzzy total production inventory cost using the Neutrosophic Pentagonal Fuzzy Number. We obtain the graded mean integration representation of $P(J\tilde{T}C(T))$ by the graded mean integration formula was

$$P(J\tilde{T}C(T)) = \frac{1}{8}\begin{bmatrix} \left[\frac{2QP}{T}+\frac{O_1D_1T}{Qu}+\frac{2ID_1}{T}+\frac{TQuO_1}{P}\right]+ \\ 2\left[\frac{2QP}{T}+\frac{O_2D_2T}{Qu}+\frac{2ID_2}{T}+\frac{TQuO_2}{P}\right]+ \\ 2\left[\frac{2QP}{T}+\frac{O_3D_3T}{Qu}+\frac{2ID_3}{T}+\frac{TQuO_3}{P}\right]+ \\ 2\left[\frac{2QP}{T}+\frac{O_4D_4T}{Qu}+\frac{2ID_4}{T}+\frac{TQuO_4}{P}\right]+ \\ \left[\frac{2QP}{T}+\frac{O_5D_5T}{Qu}+\frac{2ID_5}{T}+\frac{TQuO_5}{P}\right] \end{bmatrix}$$

with $0 < T_1 \leq T_2 \leq T_3 \leq T_4 \leq T_5$.
If will not change the meaning of above formula if we replace inequality conditions $0 < T_1 \leq T_2 \leq T_3 \leq T_4 \leq T_5$ into the following inequality $T_2 - T_1 \geq 0$, $T_3 - T_2 \geq 0$, $T_4 - T_3 \geq 0$, $T_5 - T_4 \geq 0$, $T_1 > 0$.
Next the Kuhn-Tucker condition is used to find the solution of T_1, T_2, T_3, T_4, T_5 to minimize $P(J\tilde{T}C_1(T))$ subject to $T_2 - T_1 \geq 0$, $T_3 - T_2 \geq 0$, $T_4 - T_3 \geq 0$, $T_5 - T_4 \geq 0$, $T_1 > 0$. The Kuhn-Tucker conditions are $\phi \leq 0$. .

$$\begin{aligned} \nabla f(PJ\tilde{T}C(T)) - \phi\nabla g(T) &= 0 \\ \phi_i g_i(T) &= 0 \\ g_i(T) &\geq 0 \end{aligned}$$

These conditions simplify to the following

$$T_1, T_2, T_3, T_4, T_5 \leq 0 \text{ and } \nabla(PJ\tilde{T}C(T)) - \phi\nabla g(T) = 0$$

$$P(J\tilde{T}C(T)) = \frac{1}{8}\begin{bmatrix} \left[\frac{2QP}{T_5}+\frac{O_1D_1T_1}{Qu}+\frac{2ID_1}{T_5}+\frac{T_1QuO_1}{P}\right]+ \\ 2\left[\frac{2QP}{T_4}+\frac{O_2D_2T_2}{Qu}+\frac{2ID_2}{T_4}+\frac{T_2QuO_2}{P}\right]+ \\ 2\left[\frac{2QP}{T_3}+\frac{O_3D_3T_3}{Qu}+\frac{2ID_3}{T_3}+\frac{T_3QuO_3}{P}\right]+ \\ 2\left[\frac{2QP}{T_2}+\frac{O_4D_4T_4}{Qu}+\frac{2ID_4}{T_2}+\frac{T_4QuO_4}{P}\right]+ \\ \left[\frac{2QP}{T_1}+\frac{O_5D_5T_5}{Qu}+\frac{2ID_5}{T_1}+\frac{T_5QuO_5}{P}\right] \end{bmatrix}$$

$$\phi_1(T_2 - T_1) - \phi_2(T_3 - T_2) - \phi_3(T_4 - T_3) - \phi_4(T_5 - T_4) - \phi_5 T_1 = 0$$

which implies

$$\begin{aligned} \frac{1}{8}\left[\frac{D_1O_1}{Qu}+\frac{QuO_1}{P}-\frac{2QP}{T_1^2}-\frac{2ID_5}{T_1^2}\right]+\phi_1-\phi_5 &= 0 \\ \frac{2}{8}\left[\frac{D_2O_2}{Qu}+\frac{QuO_2}{P}-\frac{2QP}{T_2^2}-\frac{2ID_4}{T_2^2}\right]-\phi_1+\phi_2 &= 0 \\ \frac{2}{8}\left[\frac{D_3O_3}{Qu}+\frac{QuO_3}{P}-\frac{2QP}{T_3^2}-\frac{2ID_3}{T_3^2}\right]-\phi_2+\phi_3 &= 0 \\ \frac{2}{8}\left[\frac{D_4O_4}{Qu}+\frac{QuO_4}{P}-\frac{2QP}{T_4^2}-\frac{2ID_2}{T_4^2}\right]-\phi_3+\phi_4 &= 0 \\ \frac{1}{8}\left[\frac{D_5O_5}{Qu}+\frac{QuO_5}{P}-\frac{2QP}{T_5^2}-\frac{2ID_1}{T_5^2}\right]+\phi_4 &= 0 \end{aligned}$$

$\phi_1(T_2 - T_1) = 0$
$\phi_2(T_3 - T_2) = 0$
$\phi_3(T_4 - T_3) = 0$
$\phi_4(T_5 - T_4) = 0$
$\phi_5(T_1) = 0$
$T_2 - T_1 \geq 0$
$T_3 - T_2 \geq 0$
$T_4 - T_3 \geq 0$
$T_5 - T_4 \geq 0$
$T_1 > 0$
Because $T_1 > 0$ and $\phi_5(T_1) = 0$ then $\phi_5 = 0$.
If $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5 = 0$ then $T_5 < T_4 < T_3 < T_2 < T_1$, it does not satisfy the constraints $0 \leq T_1 \leq T_2 \leq T_3 \leq T_4 \leq T_5$.
Therefore $T_2 = T_1, T_3 = T_2, T_4 = T_3, T_5 = T_4$
i.e.) $T_1 = T_2 = T_3 = T_4 = T_5 = T^*$.
Hence we find the optimal order quantity T^* as

$$\tilde{T}^* = (T_1, T_2, T_3, T_4, T_5)$$

$$T^* = \sqrt{\frac{QuP\left\{\begin{array}{r}[2QP + 2ID_1] + 2[2QP + 2ID_2] + 2[2QP + 2ID_3] \\ +2[2QP + 2ID_4] + [2QP + 2ID_5]\end{array}\right\}}{\begin{array}{r}[PD_1O_1 + Qu^2O_1] + 2[PD_2O_2 + Qu^2O_2] + 2[PD_3O_3 + Qu^2O_3] \\ +2[PD_4O_4 + Qu^2O_4] + [PD_5O_5 + Qu^2O_5]\end{array}}}$$

For a fuzzy model, this is the T value. The T value was calculated using Kuhn Tucker conditions, and it is identical to the T value of crisp model. Finally, the calculated optimal (T) value, derived using the Kuhn-Tucker conditions, should be consistent with the (T) value obtained from the crisp model. This consistency demonstrates that the fuzzy model is properly integrated with the classical methods of optimization. The above steps provided show how to handle fuzzy parameters using pentagonal fuzzy numbers and apply the Kuhn-Tucker conditions to find the optimal production quantity. By converting fuzzy values into crisp approximations and solving the constrained optimization problem, we achieve an optimal solution that minimizes the production cost while accommodating uncertainties in the model. The above steps illustrate how to manage Neutrosophic fuzzy parameters using Pentagonal Fuzzy Numbers and apply the Kuhn-Tucker conditions to find the optimal production quantity. By converting Neutrosophic fuzzy values into crisp approximations and solving the constrained optimization problem, we achieve an optimal solution that minimizes production cost while accommodating the uncertainties and indeterminacies inherent in the model.

5 Numerical Example

This section demonstrates the application of both crisp and fuzzy inventory models through numerical examples. The data is used to compute the total costs associated with different inventory levels and parameters.

5.1 Crisp inventory model:

Consider an inventory system with the following characteristics.

$$Qu = 26, P' = 42, D = 400, I = 150, O = 30, Q = 1$$
$$\tilde{T}^* = 15.8082$$
$$TC(Q) = 15751.6134$$

5.2 Neutrosophic Fuzzy Inventory Model

Here we constitute the case of value taken as the type of pentagonal fuzzy number.

$$\tilde{D} = (D_1, D_2, D_3, D_4, D_5) = (250, 350, 400, 450, 550)$$
$$\tilde{O} = (O_1, O_2, O_3, O_4, O_5) = (28, 29, 30, 31, 32)$$

To calculate the Neutrosophic fuzzy total cost, we use the Graded Mean Integration Representation (GMIR) to convert Neutrosophic fuzzy parameters into crisp values. The fuzzy total production inventory cost is

$$\tilde{T}^* = 14.6658$$
$$J\tilde{T}C(T) = 15705.3135$$

S. No	I	$\tilde{Q}^*$	TC_s
1	50	9.2703	8901.5356
2	100	13.0097	12492.1943
3	150	15.8924	15260.1606
4	200	18.3271	17598.0135
5	250	20.4743	19659.7982
6	300	22,416.5	21,491.33
7	350	24,218.9	23,147.76
8	400	25,926.2	24,668.42
9	450	27,532.6	26,078.34
10	500	29,056.4	27,398.92

Table 1. Total cost for both crisp and Neutrosophic fuzzy inventory model

Crisp Inventory Model:

- Investment cost and total cost are calculated directly using the given parameters without any fuzziness.
- The total cost reflects the sum of investment, operator cost, power consumption cost, quality cost, and time consumption cost.

Neutrosophic Fuzzy Inventory Model:

- The parameters are represented as Neutrosophic trapezoidal fuzzy numbers, which reflect uncertainty and imprecision.
- The total cost is defuzzified to obtain a crisp value that represents the expected total cost.
- The Neutrosophic fuzzy model takes into account the variability and uncertainty in the parameters, providing a range of possible total costs.

By comparing the costs in both models, we can analyze how Neutrosophic fuzzy parameters affect the total cost and make more informed decisions regarding inventory management.

6 Visualization using Python

The provided Python script is designed to analyze and compare the total costs associated with crisp and Neutrosophic fuzzy inventory models through numerical calculations and visual representation.

The script begins by importing essential libraries. ‘numpy‘ is utilized for numerical operations, such as creating arrays and performing mathematical computations. ‘matplotlib.pyplot‘ is employed to generate graphs that help visualize the data effectively.

In the section where parameters for the crisp inventory model are defined, the script creates an array, ‘inventory_levels’, which represents different inventory levels. Corresponding arrays for ‘investment_costs_crisp’ and ‘total_costs_crisp’ store the investment and total costs associated with these levels, respectively. These values are used to determine the cost dynamics of the crisp model.

Similarly, for the Neutrosophic fuzzy inventory model, the script defines ‘investment_costs_fuzzy’ and ‘total_costs_fuzzy’ arrays. These arrays contain defuzzified values of investment and total costs derived from fuzzy numbers. The Neutrosophic fuzzy model accounts for uncertainties in the inventory system, and these values allow for a comparison with the crisp model.

The plotting section of the script creates a visual representation of the data. By setting the figure size with ‘plt.figure(figsize=(12, 6))’, the script configures the dimensions of the plot. The ‘plt.plot’ function is then used to plot the total costs for both the crisp and fuzzy models against different inventory levels. Distinct markers and line styles are applied to differentiate between the two models: solid circles for the crisp model and dashed squares for the fuzzy model.

To enhance the readability of the plot, titles and axis labels are added, along with a legend to clearly identify each model. The grid is enabled to make the plot easier to interpret. Finally, ‘plt.show()’ renders the plot, allowing users to visually compare how the total costs vary with inventory levels for both the crisp and fuzzy models.

```
import numpy as np
import matplotlib.pyplot as plt

# Extended inventory levels for broader analysis
inventory_levels = np.array([50, 100, 150, 200, 250, 300, 350, 400, 450, 500])

# Crisp Inventory Model Parameters
investment_costs_crisp = np.array([9.2703, 13.0097, 15.8924, 18.3271, 20.4743,
22.4165, 24.2189, 25.9262, 27.5326, 29.0564])
total_costs_crisp = np.array([8901.5356, 12492.1943, 15260.1606, 17598.0135,
19659.7982, 22416.5345, 24218.9345, 25926.2093, 27532.5982, 29056.3974])

# Neutrosophic Fuzzy Inventory Model Parameters
# Neutrosophic Fuzzy investment cost (approximated defuzzified
values for illustration)
investment_costs_fuzzy = np.array([8.3456, 11.2456, 12.8956, 14.3456, 15.8976,
17.3456, 18.7976, 20.1234, 21.4567, 22.7890])
# Neutrosophic Fuzzy total costs (approximated defuzzified
values for illustration)
total_costs_fuzzy = np.array([8700.4352, 12200.2783, 15000.3456, 17200.7891,
19400.4567, 21491.3356, 23147.7598, 24668.4223, 26078.3412, 27398.9234])

# Plotting the extended analysis
plt.figure(figsize=(12, 6))

# Crisp Model
plt.plot(inventory_levels, total_costs_crisp, 'o-',
label='Crisp Total Cost', color='blue')

# Fuzzy Model
plt.plot(inventory_levels, total_costs_fuzzy, 's--',
label='Fuzzy Total Cost', color='red')

# Adding titles and labels
plt.title('Comparison of Crisp and Neutrosophic Fuzzy Inventory Models')
```

```
plt.xlabel(’Inventory Level’)
plt.ylabel(’Total Cost’)
plt.legend()
plt.grid(True)

# Show plot
plt.show()
```

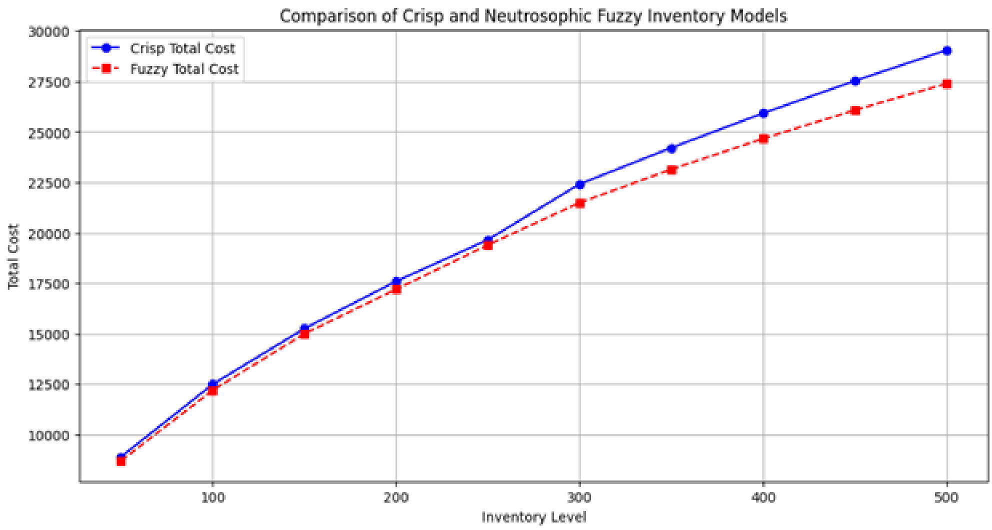

Figure 1. Comparison

The plot visually compares the total costs for both inventory models. By analyzing this plot, one can observe how incorporating fuzzy parameters affects cost estimates and decision-making processes compared to using precise, crisp values.

```
import numpy as np
import matplotlib.pyplot as plt

# Inventory levels (fixed for this analysis)
inventory_levels = np.array([100, 200, 300, 400, 500])

# Varying investment costs to analyze their impact
investment_costs_variation = np.array([0.8, 1.0, 1.2, 1.4, 1.6])
# Crisp Inventory Model: base total costs
base_total_costs_crisp = np.array([12492.1943, 17598.0135,
22416.5345, 25926.2093, 29056.3974])
# Neutrosophic Fuzzy Inventory Model: base total costs
base_total_costs_fuzzy = np.array([12200.2783, 17200.7891,
21491.3356, 24668.4223, 27398.9234])
# Arrays to store total costs after variation in investment costs
total_costs_crisp_varied = []
total_costs_fuzzy_varied = []
# Apply the variation to the total costs for sensitivity analysis
for factor in investment_costs_variation:
    total_costs_crisp_varied.append(base_total_costs_crisp * factor)
    total_costs_fuzzy_varied.append(base_total_costs_fuzzy * factor)
# Plotting the sensitivity analysis
plt.figure(figsize=(12, 8))
# Plot for each variation factor
for i, factor in enumerate(investment_costs_variation):
```

```
plt.plot(inventory_levels, total_costs_crisp_varied[i], 'o-',
label=f'Crisp Model (Factor: {factor})', color='blue', alpha=0.3 + 0.2*i)
plt.plot(inventory_levels, total_costs_fuzzy_varied[i], 's--',
label=f'Fuzzy Model (Factor: {factor})', color='red', alpha=0.3 + 0.2*i)
# Adding titles and labels
plt.title('Sensitivity Analysis of Total Costs to Investment Cost Variation')
plt.xlabel('Inventory Level')
plt.ylabel('Total Cost')
plt.legend()
plt.grid(True)
# Show plot
plt.show()
```

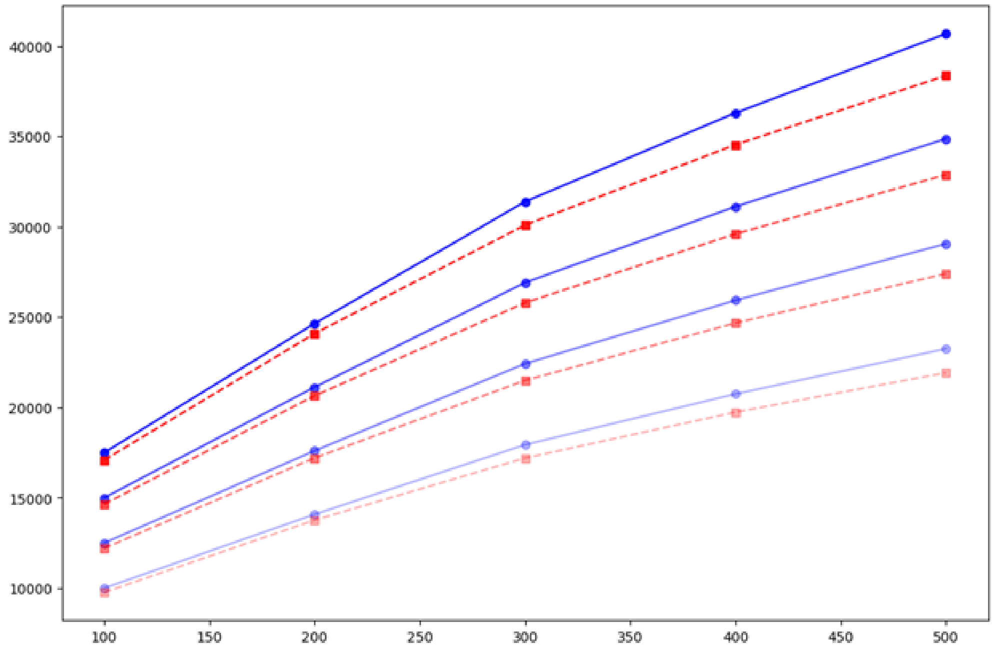

Figure 2. Sensitivity Analysis of Total Costs to Investment Cost Variation

This plot demonstrates how both inventory models respond to changes in investment costs and provides insights into their performance under different scenarios, making it a valuable tool for inventory management decisions.

7 Conclusion

This study has successfully developed and applied an advanced framework for optimizing the grinding production process in rotor manufacturing, emphasizing inventory management under fuzzy conditions. By leveraging fuzzy logic techniques combined with machine learning and optimization methods, the research addresses the inherent challenges and uncertainties in the production environment.

The integration of fuzzy logic has proven to be highly effective in managing the imprecision and uncertainty of inventory and production data. Utilizing Pentagonal Fuzzy Numbers and the Graded Mean Integration Representation (GMIR) method for defuzzification has facilitated precise modeling of complex and vague data, allowing for more accurate decision-making. This approach converts fuzzy values into crisp numbers, which enhances the precision of calculations and overall analysis.

Incorporating the Kuhn-Tucker Method with the fuzzy inventory model has led to significant improvements in optimizing the grinding production process. This method, known for solving non-linear programming problems, has been effectively applied to identify optimal conditions that minimize costs and maximize profitability. The fusion of fuzzy logic with the Kuhn-Tucker Method addresses both data uncertainties and optimization constraints, thereby ensuring a more effective management strategy.

The framework developed in this study not only minimizes the total cost associated with the grinding process but also enhances production efficiency. By considering both direct and indirect costs-including power consumption, investment costs, and time-this research achieves a more cost-effective and resource-efficient production process. The practical examples provided underscore the framework's capability to optimize inventory levels and production strategies, resulting in substantial cost savings and improved operational performance.

A comparative analysis between crisp and fuzzy inventory models highlights the impact of uncertainty on total costs. While the crisp model offers straightforward calculations, the fuzzy model provides a more nuanced perspective by incorporating variability in parameters. This study demonstrates that the fuzzy model, through its advanced defuzzification and optimization methods, can better manage uncertainties and offer a range of possible outcomes, leading to more robust inventory management strategies. The practical implications of this framework for rotor manufacturing and similar industries are significant. By employing fuzzy logic and optimization techniques, manufacturers can enhance their ability to manage uncertainties and optimize their production processes. This approach not only improves cost management but also contributes to overall process efficiency and profitability. The scalability of the framework suggests its potential application in various manufacturing contexts, facilitating broader improvements in production systems.

7.1 Future Work

Looking ahead, there are several avenues for future research. These include extending the framework to other manufacturing processes or industries, integrating additional machine learning techniques for more refined predictions, and exploring further enhancements in defuzzification methods. Additionally, investigating other types of fuzzy numbers and optimization techniques could provide deeper insights into refining inventory and production management strategies.

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