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Certain Aspects of Lacunary Statistical Convergence in Neutrosophic 2-Normed Spaces

Nesar Hossain

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Corresponding Author: Nesar Hossain

Abstract In this paper, we study the notion of lacunary statistical convergence in neutrosophic 2-normed spaces (shortly N2-NS). Also, we define lacunary statistical Cauchy squence and present lacunary statistical completeness in connection with neutrosophic 2-norm and prove some basic properties of these notions.

1 Introduction

In the study of Freedman et al. [14], functional analytic investigations of the space $|\sigma_1|$, which consists of strongly Cesàro summable sequences, as well as several related spaces of strongly summable sequences, have garnered significant attention. The space is defined as

$$|\sigma_1| = \left\{ x = \{x_k\} \in w : \exists\, l : \frac{1}{n}\sum_{k=1}^{n} |x_k - l| \to 0 \right\}.$$

A key analytical approach, as emphasized in the work of Freedman et al. [14], reveals a notable connection between the structure of $|\sigma_1|$ and the integer sequence $\{2^r\}$, which plays a pivotal role in characterizing its convergence behavior. By a lacunary sequence [16] we mean an increasing integer sequence $\theta = \{n_i\}_{i\in\mathbb{N}}$ such that $n_0 = 0$ and $q_i := n_i - n_{i-1} \to \infty$ as $i \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $J_i = (n_{i-1}, n_i]$ and the ratio $\frac{n_i}{n_{i-1}}$ will be abbreviated by p_i. By generalizing this sequence to a lacunary sequence θ, they have constructed a broad class N_θ of BK-spaces, each inheriting several key characteristics of the space $|\sigma_1|$.

$$N_\theta = \left\{ x = \{x_k\} \in w : \exists\, l : \frac{1}{q_i}\sum_{k\in J_i} |x_k - l| \to 0 \right\}.$$

They examined the fundamental properties of these spaces, placing particular emphasis on their comparison with the space $|\sigma_1|$. Of particular interest is the intersection of all N_θ spaces, comprising sequences that are uniformly strongly Cesàro summable serving as a strong analogue of the space of almost convergent sequences. Furthermore, they explored ordinary summability methods naturally linked to the N_θ spaces, noting that each method is strictly stronger than almost convergence, though none is equivalent to classical Cesàro summability. Notably, their intersection yields almost convergence. And, they presented several additional results, including a characterization of matrices whose convergence fields contain N_θ, and the identification of the duals of these spaces. In 1951, Fast [13] and Steinhaus [44] independently extended the classical notion of convergence for real sequences by introducing the concept of statistical convergence, grounded in the natural density of sets. Let $\mathcal{K} \subset \mathbb{N}$. Then the natural density [15, 46] of $\mathcal{K}$, denoted by $\delta(\mathcal{K})$, is defined as:

$$\delta(\mathcal{K}) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in \mathcal{K}\}|,$$

provided the limit exists, where the vertical bars denote the cardinality of the enclosed set. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ is said to be statistically convergent [15, 46] to $\xi \in \mathbb{R}$ if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |x_k - \xi| \ge \varepsilon\}| = 0.$$

In this case, we write $st - \lim x_k = \xi$. The symbol $\mathcal{S}$ will likewise represent the collection of all statistically convergent sequences. Over the years, this concept has been significantly developed by numerous researchers, including Connor [10], Fridy [15], Šalát [46], Mursaleen [35], Belen and Mohiuddine [5], among many others. Its applications extend widely across various fields, particularly in approximation theory [38, 40, 26], measure theory [23], and beyond. In 1993, Fridy and Orhan [15] introduced the notion of lacunary statistical convergence for real sequences and obtained several noteworthy results concerning $\mathcal{S}$, N_θ and, C_θ-summability. Let $\mathcal{K} \subseteq \mathbb{N}$. Then the number

$$\delta_\theta(\mathcal{K}) = \lim_{i} \frac{1}{q_i} |\{k \in J_i : k \in \mathcal{K}\}|$$

is named to be θ-density of $\mathcal{K}$, provided the limit exists. Let θ be a lacunary sequence. A number sequence $\{x_k\}$ is said to be lacunary statistically convergent [16] to $\xi \in \mathbb{R}$ if for every $\varepsilon > 0$

$$\lim_{i} \frac{1}{q_i} |\{k \in J_i : |x_k - \xi| \ge \varepsilon\}| = 0.$$

In this case, we write $S_\theta - \lim x_k = \xi$ or $x_k \xrightarrow{S_\theta} \xi$ and ξ is called S_θ-limit of $\{x_k\}$. Since its introduction, this notion has continued to gain attention and has been actively developed by numerous researchers, including Dowari and Tripathy [12], Kişi and Dündar [25], Li [33], Mohiuddine and Alamri [39], Nuray [42], Patterson and Savaş [43, 49], Şengül and Et [51], Savaş [52], and Ulusu and Dündar [56]. It has also been extended in various directions, such as in intuitionistic fuzzy normed linear spaces (IFNLS) [8, 36], and fuzzy normed linear spaces [55].

Following the pioneering introduction of fuzzy set theory by Zadeh [57], significant efforts have been made to develop fuzzy analogues of classical theories and to explore their applications across diverse fields of science and engineering. Notable areas include population dynamics [4], chaos control [17], fuzzy logic based programming [19], nonlinear dynamical systems [22], fuzzy physics [34], and many others. Over time, the concept of fuzzy set theory has been effectively developed and generalized into several extended frameworks, including intuitionistic fuzzy sets [1], interval-valued fuzzy sets [54], interval valued intuitionistic fuzzy sets [2], and vague fuzzy sets [3]. As a broader generalization encompassing crisp sets, fuzzy sets, intuitionistic fuzzy sets, and Pythagorean fuzzy sets, Smarandache [47] introduced the notion of neutrosophic sets. Building on this, Bera and Mahapatra later introduced the concepts of neutrosophic soft linear spaces [6] and neutrosophic soft normed linear spaces [7]. Recently, Kirişci and Şimşek [27] introduced the concept of neutrosophic normed spaces, within which various summability methods have been explored. These include statistical convergence [27], statistical convergence of double sequences [20], ideal convergence [28], lacunary statistical convergence [31], and deferred statistical convergence [11]. Numerous researchers have contributed to this growing field, notably Bilgin [8], Gürdal [21], Kişi [29, 30], Kişi et al. [32], and Sharma et al. [53].

Building on recent advancements, Murtaza et al. [41] introduced the neutrosophic 2-normed space (N2-NS) and explored statistical convergence within it. Our work extends this by defining lacunary statistical convergence of sequences in N2-NS. This concept, first introduced by Fridy and Orhan [16] for numbers using θ-density, has since been applied to various spaces, including random 2-normed spaces by Mohiuddine and Aiyub [37], and neutrosophic normed spaces by Khan et al. [31]. In this paper, we delve deeper, investigating the notion of a lacunary statistically Cauchy sequence in N2-NS and establishing key analogous properties related to the neutrosophic 2-norm.

2 Preliminaries

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ indicate the set of natural numbers and the set of reals respectively. $|A|$ denotes the cardinality of the set A. First we recall some basic definitions and notations.

Definition 2.1. [18] Let $\mathcal{Z}$ be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on $\mathcal{Z}$ is a function $\|.,.\| : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ which satisfies the following conditions:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent in $\mathcal{Z}$;

(ii) $\|x, y\| = \|y, x\|$ for all x, y in $\mathcal{Z}$;

(iii) $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ for all α in $\mathbb{R}$ and for all x, y in $\mathcal{Z}$;

(iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all x, y, z in $\mathcal{Z}$.

Example 2.2. [50] Let $\mathcal{Z} = \mathbb{R}^2$. Define $\|\cdot,\cdot\|$ on $\mathbb{R}^2$ by $\|x, y\| = |x_1y_2 - x_2y_1|$, where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Then $(\mathcal{Z}, \|\cdot,\cdot\|)$ is a 2-normed space.

Definition 2.3. [45] A binary operation $\boxdot : [0, 1] \times [0, 1] \to [0, 1]$ is named to be a continuous t-norm if the following conditions hold.

(i) $\boxdot$ is associative and commutative;

(ii) $\boxdot$ is continuous;

(iii) $x \boxdot 1 = x$ for all $x \in [0, 1]$;

(iv) $x \boxdot y \leq z \boxdot w$ whenever $x \leq z$ and $y \leq w$ for each $x, y, z, w \in [0, 1]$.

Definition 2.4. [45] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is named to be a continuous t-conorm if the following conditions are satisfied.

(i) $*$ is associative and commutative;

(ii) $*$ is continuous;

(iii) $x * 0 = x$ for all $x \in [0, 1]$;

(iv) $x * y \leq z * w$ whenever $x \leq z$ and $y \leq w$ for each $x, y, z, w \in [0, 1]$.

Example 2.5. [24] The following are the examples of t-norms:

(i) $x \boxdot y = min\{x, y\}$;

(ii) $x \boxdot y = x.y$;

(iii) $x \boxdot y = max\{x + y - 1, 0\}$. This t-norm is known as Lukasiewicz t-norm.

Example 2.6. [24] The following are the examples of t-conorms:

(i) $x * y = max\{x, y\}$;

(ii) $x * y = x + y - x.y$;

(iii) $x * y = min\{x + y, 1\}$. This is known as Lukasiewicz t-conorm.

Lemma 2.7. *[48] If $\boxdot$ is a continuous t-norm, $*$ is a continuous t-conorm, $r_i \in (0, 1)$ and $1 \leq i \leq 7$, then the following statements hold:*

(i) If $r_1 > r_2$, there are $r_3, r_4 \in (0, 1)$ such that $r_1 \boxdot r_3 \geq r_2$ and $r_1 \geq r_2 * r_4$

(ii) If $r_5 \in (0, 1)$, there are $r_6, r_7 \in (0, 1)$ such that $r_6 \boxdot r_6 \geq r_5$ and $r_5 \geq r_7 * r_7$.

Now we recall the notions of neutrosophic 2-normed space.

Definition 2.8. [41] Let $\mathcal{Y}$ be a vector space and $\mathcal{N}_2 = \{< (e, f), \Theta(e, f), \vartheta(e, f), \psi(e, f) >: (e, f) \in \mathcal{Y} \times \mathcal{Y}\}$ be a 2-normed space such that $\mathcal{N}_2 : \mathcal{Y} \times \mathcal{Y} \times \mathbb{R}^+ \to [0, 1]$. Suppose $\boxdot$ and $*$ be continuous t-norm and t-conorm respectively. Then the four tuple $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ is named to be neutrosophic 2-normed space (N2-NS) if the following conditions hold for all $e, f, g \in \mathcal{Z}, \eta, \zeta > 0$ and $\beta \neq 0$.

(i) $0 \le \Theta(e, f; \eta) \le 1, 0 \le \vartheta(e, f; \eta) \le 1$ and $0 \le \psi(e, f; \eta) \le 1$ for every $\eta > 0$;

(ii) $\Theta(e, f; \eta) + \vartheta(e, f; \eta) + \psi(e, f; \eta) \le 3$;

(iii) $\Theta(e, f; \eta) = 1$ iff e, f are linearly dependent;

(iv) $\Theta(\beta e, f; \eta) = \Theta(e, f; \frac{\eta}{|\beta|})$ for each $\beta \neq 0$;

(v) $\Theta(e, f; \eta) \boxdot \Theta(e, g; \zeta) \le \Theta(e, f + g; \eta + \zeta)$;

(vi) $\Theta(e, f; \cdot) : (0, \infty) \to [0, 1]$ is a non-decreasing function that runs continuously;

(vii) $\lim_{\eta\to\infty} \Theta(e, f; \eta) = 1$;

(viii) $\Theta(e, f; \eta) = \Theta(f, e; \eta)$;

(ix) $\vartheta(e, f; \eta) = 0$ iff e, f are linearly dependent;

(x) $\vartheta(\beta e, f; \eta) = \vartheta(e, f; \frac{\eta}{|\beta|})$ for each $\beta \neq 0$;

(xi) $\vartheta(e, f; \eta) * \vartheta(e, g; \zeta) \ge \vartheta(e, f + g; \eta + \zeta)$;

(xii) $\vartheta(e, f; \cdot) : (0, \infty) \to [0, 1]$ is a non-increasing function that runs continuously;

(xiii) $\lim_{\eta\to\infty} \vartheta(e, f; \eta) = 0$;

(xiv) $\vartheta(e, f; \eta) = \vartheta(f, e; \eta)$;

(xv) $\psi(e, f; \eta) = 0$ iff e, f are linearly dependent;

(xvi) $\psi(\beta e, f; \eta) = \psi(e, f; \frac{\eta}{|\beta|})$ for each $\beta \neq 0$;

(xvii) $\psi(e, f; \eta) * \psi(e, g; \zeta) \ge \psi(e, f + g; \eta + \zeta)$;

(xviii) $\psi(e, f; \cdot) : (0, \infty) \to [0, 1]$ is a non-increasing function that runs continuously;

(xix) $\lim_{\eta\to\infty} \psi(e, f; \eta) = 0$;

(xx) $\psi(e, f; \eta) = \psi(f, e; \eta)$;

(xxi) If $\eta \le 0$, $\Theta(e, f; \eta) = 0, \vartheta(e, f; \eta) = 1, \psi(e, f; \eta) = 1$.

In this case, $\mathcal{N}_2 = (\Theta, \vartheta, \psi)$ is called neutrosophic 2-norm on $\mathcal{Y}$.

Example 2.9. [41] Let $(\mathcal{Y}, \|, ., \|)$ be a 2-normed space. Consider continuous t-norm and continuous t-conorm as $a \boxdot b = ab$ and $a * b = a + b - ab$ for all $a, b \in [0, 1]$ respectively. Now, for $x, y \in \mathcal{Y}$ and $u > 0$ with $u > \|x, y\|$ consider

$$\Theta(x, y; u) = \frac{u}{u + \|x, y\|},\ \vartheta(x, y; u) = \frac{\|x, y\|}{u + \|x, y\|},\ \psi(x, y; u) = \frac{\|x, y\|}{u}.$$

If we take $u \le \|x, y\|$ then

$$\Theta(x, y; u) = 0,\ \vartheta(x, y; u) = 1 \text{ and } \psi(x, y; u) = 1.$$

Then $(\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ is a neutrosophic 2-normed space where $\mathcal{N}_2 : \mathcal{Y} \times \mathcal{Y} \times \mathbb{R}^+ \to [0, 1]$.

Definition 2.10. [41] Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$. Choose $\sigma \in (0, 1)$ and $\varepsilon > 0$. Then $\{l_k\}_{k\in\mathbb{N}}$ is named to be convergent if there exists a $n_0 \in \mathbb{N}$ and $\xi \in \mathcal{Y}$ such that $\Theta(l_k - \xi, z; \varepsilon) > 1 - \sigma$, $\vartheta(l_k - \xi, z; \varepsilon) < \sigma$ and $\psi(l_k - \xi, z; \varepsilon) < \sigma$ for all $n \ge n_0$ and $z \in \mathcal{Z}$ which can be said that $\lim_{k\to\infty} \Theta(l_k - \xi, z; \varepsilon) = 1$, $\lim_{k\to\infty} \vartheta(l_k - \xi, z; \varepsilon) = 0$ and $\lim_{k\to\infty} \psi(l_k - \xi, z; \varepsilon) = 0$. In this case we write $\mathcal{N}_2 - \lim l_k = \xi$ or $l_k \xrightarrow{\mathcal{N}_2} \xi$ and ξ is called $\mathcal{N}_2$-limit of $\{l_k\}_{k\in\mathbb{N}}$.

Definition 2.11. [41] Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$. Choose $\varepsilon \in (0, 1)$ and $\eta > 0$. Then $\{l_k\}_{k\in\mathbb{N}}$ is said to be statistically convergent to ξ if the natural density of the set $\mathcal{A}(\varepsilon, \eta) = \{k \le n : \Theta(l_k - \xi, z; \eta) \le 1 - \varepsilon$ or $\vartheta(l_k - \xi, z; \eta) \ge \varepsilon$ and $\psi(l_k - \xi, z; \eta) \ge \varepsilon\}$ is zero for every $z \in \mathcal{Z}$ i.e. $\delta(\mathcal{A}(\varepsilon, \eta)) = 0$.

Definition 2.12. [41] Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$, $\varepsilon > 0$ and $\eta > 0$. Then $\{l_k\}_{k\in\mathbb{N}}$ is named to be statistical Cauchy if there exists $n_0 \in \mathbb{N}$ such that $\lim_n \frac{1}{n}|\{k \leq n : \Theta(l_k - l_{n_0}, z; \eta) \leq 1-\varepsilon$ or $\vartheta(l_k - l_{n_0}, z; \eta) \geq \varepsilon$ and $\psi(l_k - l_{n_0}, z; \eta) \geq \varepsilon\}| = 0$ for every $z \in \mathcal{Z}$ or equivalently the natural density of the set $\mathcal{A}(\varepsilon, \eta) = \{k \leq n : \Theta(l_k - l_{n_0}, z; \eta) \leq 1-\varepsilon$ or $\vartheta(l_k - l_{n_0}, z; \eta) \geq \varepsilon$ and $\psi(l_k - l_{n_0}, z; \eta) \geq \varepsilon\}$ is zero, i.e., $\delta(\mathcal{A}(\varepsilon, \eta)) = 0$.

3 Main Results

Throughout this section $\mathcal{Z}$, θ and $\delta_\theta(\mathcal{A})$ stand for neutrosophic 2-normed space, lacunary sequence and θ-density of the set $\mathcal{A}$ respectively unless otherwise stated. Also, q_i, J_i and p_i has its own meaning defined earlier. First, We define the following:

Definition 3.1. Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ and θ be a lacunary sequence. Then $\{l_k\}_{k\in\mathbb{N}}$ is named to be lacunary statistically (shortly $S_\theta(\mathcal{N}_2)$) convergent to $\xi \in \mathcal{Y}$ if for every $\varepsilon > 0$, $\sigma \in (0,1)$ and nonzero $z \in \mathcal{Z}$, $\delta_\theta(\{k \in \mathbb{N} : \Theta(l_k - \xi, z; \varepsilon) \leq 1-\sigma$ or $\vartheta(l_k - \xi, z; \varepsilon) \geq \sigma$ and $\psi(l_k - \xi, z; \varepsilon) \geq \sigma\}) = 0$ or equivalently $\lim_i \frac{1}{q_i}|\{k \in J_i : \Theta(l_k - \xi, z; \varepsilon) \leq 1-\sigma$ or $\vartheta(l_k - \xi, z; \varepsilon) \geq \sigma$ and $\psi(l_k - \xi, z; \varepsilon) \geq \sigma\}| = 0$. In this case we write $S_\theta(\mathcal{N}_2) - \lim l_k = \xi$ or $l_k \xrightarrow{S_\theta(\mathcal{N}_2)} \xi$ and ξ is called $S_\theta(\mathcal{N}_2)$-limit of $\{l_k\}_{k\in\mathbb{N}}$.

Lemma 3.2. *Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ and θ be a lacunary sequence. Then for every $\varepsilon > 0$, $\sigma \in (0,1)$ and nonzero $z \in \mathcal{Z}$, the following statements are equivalent:*

(i) $S_\theta(\mathcal{N}_2) - \lim l_k = \xi$;

(ii) $\delta_\theta(\{k \in \mathbb{N} : \Theta(l_k - \xi, z; \varepsilon) \leq 1-\sigma\}) = \delta_\theta(\{k \in \mathbb{N} : \vartheta(l_k - \xi, z; \varepsilon) \geq \sigma\}) = \delta_\theta(\{k \in \mathbb{N} : \psi(l_k - \xi, z; \varepsilon) \geq \sigma\}) = 0$;

(iii) $\delta_\theta(\{k \in \mathbb{N} : \Theta(l_k - \xi, z; \varepsilon) > 1-\sigma$ *and* $\vartheta(l_k - \xi, z; \varepsilon) < \sigma,\ \psi(l_k - \xi, z; \varepsilon) < \sigma\}) = 1$;

(iv) $\delta_\theta(\{k \in \mathbb{N} : \Theta(l_k - \xi, z; \varepsilon) > 1-\sigma\}) = \delta_\theta(\{k \in \mathbb{N} : \vartheta(l_k - \xi, z; \varepsilon) < \sigma\}) = \delta_\theta(\{k \in \mathbb{N} : \psi(l_k - \xi, z; \varepsilon) < \sigma\}) = 1$;

(v) $S_\theta(\mathcal{N}_2) - \lim \Theta(l_k - \xi, z; \varepsilon) = 1$, $S_\theta(\mathcal{N}_2) - \lim \vartheta(l_k - \xi, z; \varepsilon) = 0$ *and* $S_\theta(\mathcal{N}_2) - \lim \psi(l_k - \xi, z; \varepsilon) = 0$.

Theorem 3.3. *Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ and θ be a lacunary sequence. If $\mathcal{N}_2 - \lim l_k = \xi$ then $S_\theta(\mathcal{N}_2) - \lim l_k = \xi$.*

Proof. Let $\mathcal{N}_2 - \lim l_k = \xi$. Then for every $\varepsilon > 0$, $\sigma \in (0,1)$ and nonzero $z \in \mathcal{Z}$ there exists a $n_0 \in \mathbb{N}$ such that $\Theta(l_k - \xi, z; \varepsilon) > 1-\sigma$, $\vartheta(l_k - \xi, z; \varepsilon) < \sigma$ and $\psi(l_k - \xi, z; \varepsilon) < \sigma$ for all $n \geq n_0$. Hence the set $\{k \in \mathbb{N} : \Theta(l_k - \xi, z; \varepsilon) \leq 1-\sigma$ or $\vartheta(l_k - \xi, z; \varepsilon) \geq \sigma$ and $\psi(l_k - \xi, z; \varepsilon) \geq \sigma\}$ is contained in a finite set. Therefore,

$$\delta_\theta(\{k \in \mathbb{N} : \Theta(l_k - \xi, z; \varepsilon) \leq 1-\sigma \text{ or } \vartheta(l_k - \xi, z; \varepsilon) \geq \sigma \text{ and } \psi(l_k - \xi, z; \varepsilon) \geq \sigma\}) = 0,$$

i.e., $S_\theta(\mathcal{N}_2) - \lim l_k = \xi$. This completes the proof. □

But the converse of Theorem 3.3 may not be true which can be shown by the following example.

Example 3.4. Let $\mathcal{Y} = \mathbb{R}^2$ with $\|x, y\| = |x_1 y_2 - x_2 y_1|$, where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. We take continuous t-norm $\boxdot$ and continuous t-conorm $*$ as $a \boxdot b = ab$ and $a * b = \min\{a+b, 1\}$ for $a, b \in [0,1]$ respectively. Now choose $\sigma \in (0,1)$. For $x \in \mathcal{Y}$, $\varepsilon > 0$ with $\varepsilon > \|x, y\|$ and nonzero $z \in \mathcal{Y}$, we Consider $\Theta(x, z; \varepsilon) = \frac{\varepsilon}{\varepsilon + \|x,z\|}$, $\vartheta(x, z; \varepsilon) = \frac{\|x,z\|}{\varepsilon + \|x,z\|}$, $\psi(x, z; \varepsilon) = \frac{\|x,z\|}{\varepsilon}$. Then $\mathcal{N}_2 = (\Theta, \vartheta, \psi)$ is a neutrosophic 2-norm on $\mathcal{Y}$ and the four tuple $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ becomes a neutrosophic 2-normed space. Define a sequence $\{l_k\}_{k\in\mathbb{N}} \in \mathcal{Z}$ by

$$l_k = \begin{cases} (k, 0), & \text{if } n_i - \left[\sqrt{q_i}\right] + 1 \leq k \leq n_i, i \in \mathbb{N} \\ (0, 0), & \text{otherwise.} \end{cases}$$

Let $\xi = (0,0)$. Now we have

$$\begin{aligned}\mathcal{A}(\sigma,\varepsilon) &= \{k \in J_i : \Theta(l_k-\xi,z;\varepsilon) \le 1-\sigma \text{ or } \vartheta(l_k-\xi,z;\varepsilon) \ge \sigma \text{ and } \psi(l_k-\xi,z;\varepsilon) \ge \sigma\}\\ &= \left\{k \in J_i : \frac{\varepsilon}{\varepsilon+\|l_k,z\|} \le 1-\sigma \text{ or } \frac{\|l_k,z\|}{\varepsilon+\|l_k,z\|} \ge \sigma \text{ and } \frac{\|l_k,z\|}{\varepsilon} \ge \sigma\right\}\\ &= \left\{k \in J_i : \|l_k,z\| \ge \frac{\varepsilon\sigma}{1-\sigma} > 0 \text{ or } \|l_k,z\| \ge \varepsilon\sigma > 0\right\}\\ &\subseteq \{k \in J_i : l_k = (k,0)\}\\ &\subseteq \{k \in J_i :\ n_i - [\sqrt{q_i}] + 1 \le k \le n_i\}.\end{aligned}$$

Thus $\frac{1}{q_i}|\mathcal{A}(\sigma,\varepsilon)| \le \frac{1}{q_i}|\{k \in J_i :\ n_i - [\sqrt{q_i}] + 1 \le k \le n_i\}| \le \frac{\sqrt{q_i}}{q_i} \to 0$ as $i \to \infty$. Therefore $S_\theta(\mathcal{N}_2) - \lim l_k = \mathbf{0}$ where $\mathbf{0} = (0,0)$. But $\{l_k\}_{k\in\mathbb{N}}$ is not convergent to $\mathbf{0}$ with regards to $\mathcal{N}_2$.

Theorem 3.5. *Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{Z} = (\mathcal{Y},\mathcal{N}_2,\boxdot,*)$ and θ be a lacunary sequence. If $\{l_k\}_{k\in\mathbb{N}}$ is lacunary statistically convergent with respect to $\mathcal{N}_2$ then $S_\theta(\mathcal{N}_2)$-limit is unique.*

Proof. If possible, let $S_\theta(\mathcal{N}_2) - \lim l_k = \xi_1$ and $S_\theta(\mathcal{N}_2) - \lim l_k = \xi_2$ where $\xi_1 \ne \xi_2$. Let $\sigma \in (0,1)$ and $\varepsilon > 0$. Choose $\lambda \in (0,1)$ such that $(1-\lambda)\boxdot(1-\lambda) > 1-\sigma$ and $\lambda * \lambda < \sigma$. Now, for any nonzero $z \in \mathcal{Z}$ we define the sets as

$$\begin{aligned}\mathcal{A}_{\Theta1}(\lambda,\varepsilon) &= \left\{k\in\mathbb{N} : \Theta\left(l_k-\xi_1,z;\frac{\varepsilon}{2}\right) \le 1-\lambda\right\};\\ \mathcal{A}_{\Theta2}(\lambda,\varepsilon) &= \left\{k\in\mathbb{N} : \Theta\left(l_k-\xi_2,z;\frac{\varepsilon}{2}\right) \le 1-\lambda\right\};\\ \mathcal{A}_{\vartheta1}(\lambda,\varepsilon) &= \left\{k\in\mathbb{N} : \vartheta\left(l_k-\xi_1,z;\frac{\varepsilon}{2}\right) \ge \lambda\right\};\\ \mathcal{A}_{\vartheta2}(\lambda,\varepsilon) &= \left\{k\in\mathbb{N} : \vartheta\left(l_k-\xi_2,z;\frac{\varepsilon}{2}\right) \ge \lambda\right\};\\ \mathcal{A}_{\psi1}(\lambda,\varepsilon) &= \left\{k\in\mathbb{N} : \psi\left(l_k-\xi_1,z;\frac{\varepsilon}{2}\right) \ge \lambda\right\};\\ \mathcal{A}_{\psi2}(\lambda,\varepsilon) &= \left\{k\in\mathbb{N} : \psi\left(l_k-\xi_2,z;\frac{\varepsilon}{2}\right) \ge \lambda\right\}.\end{aligned}$$

Since $S_\theta(\mathcal{N}_2) - \lim l_k = \xi_1$ and $S_\theta(\mathcal{N}_2) - \lim l_k = \xi_2$, using Lemma 3.2 we have

$$\delta_\theta(\mathcal{A}_{\Theta1}(\lambda,\varepsilon)) = \delta_\theta(\mathcal{A}_{\vartheta1}(\lambda,\varepsilon)) = \delta_\theta(\mathcal{A}_{\psi1}(\lambda,\varepsilon)) = 0$$

and

$$\delta_\theta(\mathcal{A}_{\Theta2}(\lambda,\varepsilon)) = \delta_\theta(\mathcal{A}_{\vartheta2}(\lambda,\varepsilon)) = \delta_\theta(\mathcal{A}_{\psi2}(\lambda,\varepsilon)) = 0.$$

Now, let

$$\mathcal{A}_{\Theta,\vartheta,\psi}(\lambda,\varepsilon) = [\mathcal{A}_{\Theta1}(\lambda,\varepsilon)\cup\mathcal{A}_{\Theta2}(\lambda,\varepsilon)] \cap [\mathcal{A}_{\vartheta1}(\lambda,\varepsilon)\cup\mathcal{A}_{\vartheta2}(\lambda,\varepsilon)] \cap [\mathcal{A}_{\psi1}(\lambda,\varepsilon)\cup\mathcal{A}_{\psi2}(\lambda,\varepsilon)].$$

Then, clearly $\delta_\theta(\mathcal{A}_{\Theta,\vartheta,\psi}(\lambda,\varepsilon)) = 0$. So $\delta_\theta(\mathbb{N}\setminus\mathcal{A}_{\Theta,\vartheta,\psi}(\lambda,\varepsilon)) = 1$. Let $k \in \mathbb{N}\setminus\mathcal{A}_{\Theta,\vartheta,\psi}(\lambda,\varepsilon)$. Then there arise three posible cases:
$\underline{Case - I}$: If $k \in \mathbb{N}\setminus\mathcal{A}_{\Theta1}(\lambda,\varepsilon)\cup\mathcal{A}_{\Theta2}(\lambda,\varepsilon)$, then we have

$$\begin{aligned}\Theta(\xi_1-\xi_2,z;\varepsilon) &\ge \Theta\left(l_k-\xi_1,z;\frac{\varepsilon}{2}\right)\boxdot\Theta\left(l_k-\xi_2,z;\frac{\varepsilon}{2}\right)\\ &> (1-\lambda)\boxdot(1-\lambda)\\ &> 1-\sigma.\end{aligned}$$

Since $\sigma \in (0,1)$ is arbitrary, $\Theta(\xi_1-\xi_2,z;\varepsilon) = 1$ which yields $\xi_1 = \xi_2$.

$\underline{Case-II}$: If $k \in \mathbb{N} \setminus \mathcal{A}_{\vartheta 1}(\lambda, \varepsilon) \cup \mathcal{A}_{\vartheta 2}(\lambda, \varepsilon)$ then

$$\begin{aligned}\vartheta(\xi_1 - \xi_2, z; \varepsilon) &\leq \vartheta\left(l_k - \xi_1, z; \frac{\varepsilon}{2}\right) * \vartheta\left(l_k - \xi_2, z; \frac{\varepsilon}{2}\right)\\ &< \lambda * \lambda\\ &< \sigma.\end{aligned}$$

Since $\sigma \in (0,1)$ is arbitrary, $\vartheta(\xi_1 - \xi_2, z; \varepsilon) = 0$ which yields $\xi_1 = \xi_2$.
$\underline{Case-III}$: If $k \in \mathbb{N} \setminus \mathcal{A}_{\psi 1}(\lambda, \varepsilon) \cup \mathcal{A}_{\psi 2}(\lambda, \varepsilon)$ then in the similar way as Case-II we will get $\xi_1 = \xi_2$. Hence $S_\theta(\mathcal{N}_2)$-limit of $\{l_k\}_{k\in\mathbb{N}}$ is unique. This completes the proof. □

Theorem 3.6. *Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ and θ be a lacunary sequence. Then $S_\theta(\mathcal{N}_2) - \lim l_k = \xi$ if and only if there exists a subset $\mathcal{K} = \{k_1 < k_2 < \cdots < k_m < \cdots\}$ such that $\delta_\theta(\mathcal{K}) = 1$ and $\mathcal{N}_2 - \lim l_{k_m} = \xi$.*

Proof. First suppose that $S_\theta(\mathcal{N}_2) - \lim l_k = \xi$. Now, for any $\varepsilon > 0$, $t \in \mathbb{N}$ and nonzero $z \in \mathcal{Z}$ we define

$$\mathcal{A}_{\mathcal{N}_2}(t,\varepsilon) = \left\{k \in \mathbb{N} : \Theta(l_k - \xi, z; \varepsilon) > 1 - \frac{1}{t} \text{ and } \vartheta(l_k - \xi, z; \varepsilon) < \frac{1}{t},\ \psi(l_k - \xi, z; \varepsilon) < \frac{1}{t}\right\} \tag{3.1}$$

and

$$\mathcal{B}_{\mathcal{N}_2}(t,\varepsilon) = \left\{k \in \mathbb{N} : \Theta(l_k - \xi, z; \varepsilon) \leq 1 - \frac{1}{t} \text{ or } \vartheta(l_k - \xi, z; \varepsilon) \geq \frac{1}{t} \text{ and } \psi(l_k - \xi, z; \varepsilon) \geq \frac{1}{t}\right\}. \tag{3.2}$$

Then clearly, $\mathcal{A}_{\mathcal{N}_2}(t+1,\varepsilon) \subset \mathcal{A}_{\mathcal{N}_2}(t,\varepsilon)$ and by our assumption we have $\delta_\theta(\mathcal{B}_{\mathcal{N}_2}(t,\varepsilon)) = 0$. Also, from 3.1 we get $\delta_\theta(\mathcal{A}_{\mathcal{N}_2}(t,\varepsilon)) = 1$. Now we show that for $k \in \mathcal{A}_{\mathcal{N}_2}(t,\varepsilon)$, $\mathcal{N}_2 - \lim l_k = \xi$. If possible, let $\{l_k\}_{k\in\mathcal{A}_{\mathcal{N}_2}(t,\varepsilon)}$ is not convergent with respect to $\mathcal{N}_2$. Then for some $\sigma \in (0,1)$ we have $\Theta(l_k - \xi, z; \varepsilon) \leq 1 - \sigma$, $\vartheta(l_k - \xi, z; \varepsilon) \geq \sigma$ and $\psi(l_k - \xi, z; \varepsilon) \geq \sigma$ for except atmost finite number of terms $k \in \mathcal{A}_{\mathcal{N}_2}(t,\varepsilon)$ and nonzero $z \in \mathcal{Z}$. Define

$$\mathcal{C}_{\mathcal{N}_2}(\sigma,\varepsilon) = \{k \in \mathbb{N} : \Theta(l_k - \xi, z; \varepsilon) > 1 - \sigma \text{ and } \vartheta(l_k - \xi, z; \varepsilon) < \sigma,\ \psi(l_k - \xi, z; \varepsilon) < \sigma\} \tag{3.3}$$

where $\sigma > \frac{1}{t}$. Clearly $\delta_\theta(\mathcal{C}_{\mathcal{N}_2}(\sigma,\varepsilon)) = 0$. Since $\sigma > \frac{1}{t}$, $\mathcal{A}_{\mathcal{N}_2}(t,\varepsilon) \subset \mathcal{C}_{\mathcal{N}_2}(\sigma,\varepsilon)$ and hence $\delta_\theta(\mathcal{A}_{\mathcal{N}_2}(t,\varepsilon)) = 0$ which contradicts $\delta_\theta(\mathcal{A}_{\mathcal{N}_2}(t,\varepsilon)) = 1$. Therefore for $k \in \mathcal{A}_{\mathcal{N}_2}(t,\varepsilon)$, $\mathcal{N}_2 - \lim l_k = \xi$.

Conversely, suppose that there exists a subset $\mathcal{K} = \{k_1 < k_2 < \cdots < k_m < \cdots\} \subset \mathbb{N}$ such that $\delta_\theta(\mathcal{K}) = 1$ and $\mathcal{N}_2 - \lim_{m\to\infty} l_{k_m} = \xi$. Then for every $\sigma \in (0,1)$ and $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $\Theta(l_{k_m} - \xi, z; \varepsilon) > 1 - \sigma$, $\vartheta(l_{k_m} - \xi, z; \varepsilon) < \sigma$ and $\psi(l_{k_m} - \xi, z; \varepsilon) < \sigma$ for all $m \geq m_0$ and nonzero $z \in \mathcal{Z}$. Therefore

$$\begin{aligned}\{k \in \mathbb{N} : \Theta(l_k - \xi, z; \varepsilon) \leq 1 - \sigma \text{ or } \vartheta(l_k - \xi, z; \varepsilon) \geq \sigma \text{ and } \psi(l_k - \xi, z; \varepsilon) \geq \sigma\}\\ \subset \mathbb{N} \setminus \{k_{m_0+1}, k_{m_0+2}, \ldots\}.\end{aligned}$$

Hence $\delta_\theta(\{k \in \mathbb{N} : \Theta(l_k - \xi, z; \varepsilon) \leq 1 - \sigma \text{ or } \vartheta(l_k - \xi, z; \varepsilon) \geq \sigma \text{ and } \psi(l_k - \xi, z; \varepsilon) \geq \sigma\}) = 0$, i.e., $S_\theta(\mathcal{N}_2) - \lim l_k = \xi$. This completes the proof. □

Now we proceed with the notion of lacunary statistically Cauchy with respect to $\mathcal{N}_2$ and establish the relation between $S_\theta(\mathcal{N}_2)$-convergence and $S_\theta(\mathcal{N}_2)$-Cauchy sequences.

Definition 3.7. Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ and θ be a lacunary sequence. Then $\{l_k\}_{k\in\mathbb{N}}$ is named to be lacunary statistically Cauchy with respect to $\mathcal{N}_2$ (shortly $S_\theta(\mathcal{N}_2)$-Cauchy) if for every $\sigma \in (0,1)$, $\varepsilon > 0$ and nonzero $z \in \mathcal{Z}$ there exists $k_0 \in \mathbb{N}$ such that $\delta_\theta(\{k \in \mathbb{N} : \Theta(l_k - l_{k_0}, z; \varepsilon) \leq 1 - \sigma \text{ or } \vartheta(l_k - l_{k_0}, z; \varepsilon) \geq \sigma \text{ and } \psi(l_k - l_{k_0}, z; \varepsilon) \geq \sigma\}) = 0$.

Theorem 3.8. *Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ and θ be a lacunary sequence. If $\{l_k\}_{k\in\mathbb{N}}$ is $S_\theta(\mathcal{N}_2)$-convergent then it is $S_\theta(\mathcal{N}_2)$-Cauchy sequence.*

Proof. Let $S_\theta(\mathcal{N}_2)-\lim l_k = \xi$ and $\sigma \in (0,1)$ be given. Choose $\lambda \in (0,1)$ such that $(1-\lambda)\boxdot(1-\lambda) > 1-\sigma$ and $\lambda*\lambda < \sigma$. Then for $\lambda \in (0,1), \varepsilon > 0$ and nonzero $z \in \mathcal{Z}$, $\delta_\theta(\mathcal{A}_{\mathcal{N}_2}(\lambda,\varepsilon)) = 0$ where $\mathcal{A}_{\mathcal{N}_2}(\lambda,\varepsilon) = \left\{k \in \mathbb{N} : \Theta\left(l_k - \xi, z; \frac{\varepsilon}{2}\right) \le 1-\lambda \text{ or } \vartheta\left(l_k - \xi, z; \frac{\varepsilon}{2}\right) \ge \lambda \text{ and } \psi\left(l_k - \xi, z; \frac{\varepsilon}{2}\right) \ge \lambda\right\}$. Then $\delta_\theta(\mathbb{N} \setminus \mathcal{A}_{\mathcal{N}_2}(\lambda,\varepsilon)) = 1$. Let $k_0 \in \mathcal{A}^c_{\mathcal{N}_2}(\lambda,\varepsilon)$. So, we have

$$\Theta\left(l_{k_0} - \xi, z; \frac{\varepsilon}{2}\right) > 1-\lambda,\ \vartheta\left(l_{k_0} - \xi, z; \frac{\varepsilon}{2}\right) < \lambda \text{ and } \psi\left(l_{k_0} - \xi, z; \frac{\varepsilon}{2}\right) < \lambda.$$

Now, we define $\mathcal{B}_{\mathcal{N}_2}(\sigma,\varepsilon) = \{k \in \mathbb{N} : \Theta(l_k - l_{k_0}, z; \varepsilon) \le 1-\sigma \text{ or } \vartheta(l_k - l_{k_0}, z; \varepsilon) \ge \sigma \text{ and } \psi(l_k - l_{k_0}, z; \varepsilon) \ge \sigma\}$ for every nonzero $z \in \mathcal{Z}$. We show that $\mathcal{B}_{\mathcal{N}_2}(\sigma,\varepsilon) \subset \mathcal{A}_{\mathcal{N}_2}(\lambda,\varepsilon)$. Let $m \in \mathcal{B}_{\mathcal{N}_2}(\sigma,\varepsilon)$. Then we get

$$\Theta(l_m - l_{k_0}, z; \varepsilon) \le 1-\sigma,\ \vartheta(l_m - l_{k_0}, z; \varepsilon) \ge \sigma \text{ and } \psi(l_m - l_{k_0}, z; \varepsilon) \ge \sigma.$$

<u>$Case-I$</u> : We consider $\Theta(l_m - l_{k_0}, z; \varepsilon) \le 1-\sigma$. We show $\Theta\left(l_m - \xi, z; \frac{\varepsilon}{2}\right) \le 1-\lambda$. If possible, let $\Theta\left(l_m - \xi, z; \frac{\varepsilon}{2}\right) > 1-\lambda$. Then we have

$$\begin{aligned}
1-\sigma &\ge \Theta(l_m - l_{k_0}, z; \varepsilon) \\
&\ge \Theta\left(l_m - \xi, z; \frac{\varepsilon}{2}\right) \boxdot \Theta\left(l_m - \xi, z; \frac{\varepsilon}{2}\right) \\
&> (1-\lambda) \boxdot (1-\lambda) \\
&> 1-\sigma,
\end{aligned}$$

which is not possible. Therefore $\Theta\left(l_m - \xi, z; \frac{\varepsilon}{2}\right) \le 1-\lambda$.
<u>$Case-II$</u> : We consider $\vartheta(l_m - l_{k_0}, z; \varepsilon) \ge \sigma$. We show $\vartheta\left(l_m - \xi, z; \frac{\varepsilon}{2}\right) \ge \lambda$. If possible, let $\vartheta\left(l_m - \xi, z; \frac{\varepsilon}{2}\right) < \lambda$. Then we have

$$\begin{aligned}
\sigma &\le \vartheta(l_m - l_{k_0}, z; \varepsilon) \\
&\le \vartheta\left(l_m - \xi, z; \frac{\varepsilon}{2}\right) \boxdot \vartheta\left(l_{k_0} - \xi, z; \frac{\varepsilon}{2}\right) \\
&< \lambda * \lambda \\
&< \sigma,
\end{aligned}$$

which is not possible. Therefore we have $\vartheta\left(l_m - \xi, z; \frac{\varepsilon}{2}\right) \ge \lambda$.
<u>$Case-III$</u> : If we consider $\psi(l_m - l_{k_0}, z; \varepsilon) \ge \sigma$ then in the similar way as Case-II we can show that $\psi\left(l_m - \xi, z; \frac{\varepsilon}{2}\right) \ge \lambda$.

Therefore, $m \in \mathcal{A}_{\mathcal{N}_2}(\lambda,\varepsilon)$. Hence $\mathcal{B}_{\mathcal{N}_2}(\sigma,\varepsilon) \subset \mathcal{A}_{\mathcal{N}_2}(\lambda,\varepsilon)$. Since $\delta_\theta(\mathcal{A}_{\mathcal{N}_2}(\lambda,\varepsilon)) = 0$, $\delta_\theta(\mathcal{B}_{\mathcal{N}_2}(\sigma,\varepsilon)) = 0$. So, $\{l_k\}_{k\in\mathbb{N}}$ is $S_\theta(\mathcal{N}_2)$-Cauchy sequence. □

Theorem 3.9. *Let $\{l_k\}_{k\in\mathbb{N}}$ be a sequence in a N2-NS $\mathcal{Z} = (\mathcal{Y}, \mathcal{N}_2, \boxdot, *)$ and θ be a lacunary sequence. If $\{l_k\}_{k\in\mathbb{N}}$ is $S_\theta(\mathcal{N}_2)$-Cauchy sequence then it is $S_\theta(\mathcal{N}_2)$-convergent sequence.*

Proof. Let $\{l_k\}_{k\in\mathbb{N}}$ is $S_\theta(\mathcal{N}_2)$-Cauchy but not it is $S_\theta(\mathcal{N}_2)$-convergent to $\xi \in \mathcal{Z}$. Then for $\sigma \in (0,1)$, $\varepsilon > 0$ and nonzero $z \in \mathcal{Z}$, there exist $k_0 = k_0(\sigma) \in \mathbb{N}$ such that $\delta_\theta(\mathcal{K}) = 0$, where

$$\mathcal{K} = \{k \in \mathbb{N} : \Theta(l_k - l_{k_0}, z; \varepsilon) \le 1-\sigma \text{ or } \vartheta(l_k - l_{k_0}, z; \varepsilon) \ge \sigma \text{ and } \psi(l_k - l_{k_0}, z; \varepsilon) \ge \sigma\}$$

and $\delta_\theta(\mathcal{M}) = 0$, where

$$\mathcal{M} = \left\{k \in \mathbb{N} : \Theta(l_k - \xi, z; \frac{\varepsilon}{2}) > 1-\sigma \text{ or } \vartheta(l_k - \xi, z; \frac{\varepsilon}{2}) < \sigma \text{ and } \psi(l_k - \xi, z; \frac{\varepsilon}{2}) < \sigma\right\}.$$

Since $\Theta(l_k - l_{k_0}, z; \varepsilon) \ge 2\Theta\left(l_k - \xi, z; \frac{\varepsilon}{2}\right) > 1-\sigma$ and $\vartheta(l_k - l_{k_0}, z; \varepsilon) \le 2\vartheta\left(l_k - \xi, z; \frac{\varepsilon}{2}\right) < \sigma$, $\psi(l_k - l_{k_0}, z; \varepsilon) \le 2\psi\left(l_k - \xi, z; \frac{\varepsilon}{2}\right) < \sigma$, if $\Theta\left(l_k - \xi, z; \frac{\varepsilon}{2}\right) > \frac{1-\sigma}{2}$ and $\vartheta\left(l_k - \xi, z; \frac{\varepsilon}{2}\right) < \frac{\sigma}{2}$, $\psi\left(l_k - l_{k_0}, z; \varepsilon\right) < \frac{\sigma}{2}$. This gives

$$\delta_\theta(\{k \in \mathbb{N} : \Theta(l_k - l_{k_0}, z; \varepsilon) > 1-\sigma \text{ and } \vartheta(l_k - l_{k_0}, z; \varepsilon) < \sigma,\ \psi(l_k - l_{k_0}, z; \varepsilon) < \sigma\}) = 0.$$

This implies $\delta_\theta(\mathbb{N} \setminus \mathcal{K}) = 0$ and so $\delta_\theta(\mathcal{K}) = 1$ which leads to a contradiction. Therefore $\{l_k\}_{k\in\mathbb{N}}$

is $S_\theta(\mathcal{N}_2)$-convergent to $\xi \in \mathcal{Z}$. This completes the proof. □

Definition 3.10. A N2-NS $\mathcal{Z}$ is named to be lacunary statistically complete with respect to $\mathcal{N}_2$ if every lacunary statistically Cauchy sequence is lacunary statistically convergent with respect to $\mathcal{N}_2$.

Remark 3.11. In the light of Theorem 3.8 and 3.9, we see every N2-NS is lacunary statistically complete.

Conclusion and future developments

In this paper, we investigated lacunary convergent sequences within the context of neutrosophic 2-normed spaces (N2-NS), specifically focusing on θ-density. Our findings demonstrate that every N2-NS is lacunary statistically complete. These results lay the groundwork for future research, potentially enabling the generalization of this concept through the lens of an ideal. Furthermore, this methodological approach can be extended to double sequences for advanced developments and holds significant promise for addressing convergence related challenges across various scientific and engineering disciplines.

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References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Syst., **20** (1986), 87-96.

[2] K.T. Atanassov and G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets Syst., **31(3)** (1989), 343–349.

[3] H. Bustince and P. Burillo, *Vague sets are intuitionistic fuzzy sets*, Fuzzy Sets Syst., **79(3)** (1996), 403-405.

[4] L. C. Barros, R. C. Bassanezi and P. A. Tonelli, *Fuzzy modelling in population dynamics*, Ecol. Model., **128(1)** (2000), 27-33.

[5] C. Belen and S. A. Mohiuddine, *Generalized weighted statistical convergence and application*, Appl. Math. Comput., **219(18)** (2013), 9821-9826.

[6] T. Bera and N. K. Mahapatra, *On neutrosophic soft linear spaces*, Fuzzy Inf. Eng., **9(3)** (2017), 299-324.

[7] T. Bera and N. K. Mahapatra, *Neutrosophic soft normed linear space*, Neutrosophic Sets Syst., **23** (2018), 52-71.

[8] N. G. Bilgin and G. Bozma, *Fibonacci lacunary statistical convergence of order γ in IFNLS*, Int. J. Adv. Appl. Math. Mech., **8(4)** (2021), 28-36.

[9] N. G. Bilgin, *Hibrid Δ-statistical convergence for Neutrosophic normed space*, J. Math. Vol. 2022, Art. ID 3890308, pp-10, https://doi.org/10.1155/2022/3890308.

[10] J. S. Connor, *The statistical and strong p-Cesàro convergence of sequences*, Analysis, **8(1-2)** (1988), 47-63.

[11] S. Debnath, S. Debnath and C. Choudhury, *On deferred statistical convergence of sequences in neutrosophic normed spaces*, Sahand Commun. Math. Anal., **19(4)** (2022), 81-96.

[12] P. J. Dowari and B. C. Tripathy, *Lacunary statistical convergence of complex uncertain variables*, Bol. Soc. Parana. Mat., **41** (2023), 1-10.

[13] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2 (3-4)**, (1951), 241-244.

[14] A. R. Freedman, J. J. Sember and M. Raphael, *Some Cesàro-type summability spaces*, Proc. London Math. Soc., **37** (1978), 508-520.

[15] J. A. Fridy, *On statistical convergence*, Analysis, **5(4)** (1985), 301-313.

[16] J. A. Fridy and C. Orhan, *Lacunary statistical convergence*, Pacific J. Math., **160(1)** (1993), 43-51.

[17] A. L. Fradkov and R. J. Evans, *Control of chaos: Methods and applications in engineering*, Annu. Rev. Control, **29(1)** (2005), 33-56.

[18] S. Gähler, 2-*normed spaces*, Math. Nachr., **28** (1964), 1-43.

[19] R. Giles, *A computer program for fuzzy reasoning*, Fuzzy Sets Syst., **4(3)** (1980), 221-234.

[20] C. Granados and A. Dhital, *Statistical convergence of double sequences in neutrosophic normed spaces*, Neutrosophic Sets Syst., **42** (2021), 333-344.

[21] V. Gürdal, *Triple lacunary* Δ-*statistical convergence in neutrosophic normed spaces*, Konuralp J. Math., **10(1)** (2022), 127-133.

[22] L. Hong and J. Q. Sun, Bifurcations of fuzzy nonlinear dynamical systems, Commun. Nonlinear Sci. Numer. Simul., **11(1)** (2006), 1-12.

[23] B. Hazarika, A. Alotaibi and S. A. Mohiuddine, *Statistical convergence in measure for double sequences of fuzzy-valued functions*, Soft Comput., **24** (2020), 6613-6622.

[24] E. P. Klement, R. Mesiar and E. Pap, *Triangular norms. Position paper I: basic analytical and algebraic properties*, Fuzzy Sets Syst., **143** (2004), 5-26.

[25] Ö. Kişi and E. Dündar, *Rough* $\mathcal{I}_2$-*lacunary statistical convergence of double sequences*, J. Inequal. Appl., 2018, 1-16, https://doi.org/10.1186/s13660-018-1831-7.

[26] U. Kadak and S. A. Mohiuddine, *Generalized statistically almost convergence based on the difference operator which includes the* (p,q)-*gamma function and related approximation theorems*, Results Math., **73** (2018), Article 9.

[27] M. Kirişci and N. Şimşek, *Neutrosophic normed spaces and statistical convergence*, J. Anal., **28** (2020), 1059-1073.

[28] Ö. Kişi, *Ideal convergence of sequences in neutrosophic normed spaces*, J. Intell. Fuzzy Syst., **41(2)** (2021), 2581-2590.

[29] Ö. Kişi, *Convergence methods for double sequences and applications in neutrosophic normed spaces*, Soft Computing Techniques in Engineering, Health, Mathematical and Social Sciences, pp. 137-154. CRC Press, 2021.

[30] Ö. Kişi, *On* $\mathcal{I}_\theta$-*convergence in neutrosophic normed spaces*, *Fundam. J. Math. Appl.*, **4(2)** (2021), 67-76.

[31] V. A. Khan, M. D. Khan and M. Ahmad, *Some new type of lacunary statistically convergent sequences in neutrosophic normed space*, Neutrosophic Sets Syst., **42** (2021), 241-252.

[32] Ö. Kişi, M. Gürdal and B. Çakal, *Certain aspects of Nörlund* $\mathcal{I}$-*statistical convergence of sequences in neutrosophic normed spaces*, Demonstr. Math., **56(1)** (2023), 20220194.

[33] J. Li, *Lacunary statistical convergence and inclusion properties between lacunary methods*, Int. J. Math. Math. Sci., **23** (2000), 175-180.

[34] J. Madore, *Fuzzy physics*, Ann. Physics, **219(1)** (1992), 187-198.

[35] M. Mursaleen, λ-*statistical convergence*, Math. Slovaca, **50(1)**, (2000), 111-115.

[36] M. Mursaleen and S. A. Mohiuddine, *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space*, J. Comput. Appl. Math., **233(2)** (2009), 142-149.

[37] S. A. Mohiuddine and M. Aiyub, *Lacunary statistical convergence in random* 2-*normed spaces*, Appl. Math. Inf. Sci., **6(3)** (2012), 581-585.

[38] S. A. Mohiuddine, *Statistical weighted* A-*summability with application to Korovkin's type approximation theorem*, J. Inequal. Appl., (2016), Article 101.

[39] S. A. Mohiuddine and B. A. S. Alamri, *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. RACSAM, **113(3)** (2019), 1955-1973.

[40] S. A. Mohiuddine, A. Asiri and B. Hazarika, *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*, Int. J. Gen. Syst., **48(5)** (2019) 492-506.

[41] S. Murtaza, A. Sharma and V. Kumar, *Neutrosophic* 2-*normed spaces and generalized summability*, Neutrosophic Sets Syst., **55(1)** (2023), Article 25.

[42] F. Nuray, *Lacunary statistical convergence of sequences of fuzzy numbers*, Fuzzy Sets Syst., **99(3)** (1998), 353-355.

[43] R. F. Patterson and E. Savaş, *Lacunary statistical convergence of double sequences*, Math. Commun., **10(1)** (2005), 55-61.

[44] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2(1)** (1951), 73-74.

[45] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math., **10(1)** (1960), 313-334.

[46] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30(2)** (1980), 139-150.

[47] F. Smarandache, *Neutrosophic set, a generalisation of the intuitionistic fuzzy sets*, Int. J. Pure. Appl. Math., **24** (2005), 287–297.

[48] R. Saadati and J.H. Park, *On the intuitionistic fuzzy topological spaces*, Chaos Solitons Fractals, **27(2)** (2006), 331-344.

[49] E. Savaş and R. F. Patterson, *Lacunary statistical convergence of multiple sequences*, Appl. Math. Lett., **19(6)** (2006), 527-534.

[50] A. Şahiner, M. Gürdal, S. Saltan and H. Gunawan, *Ideal convergence in* 2*-normed spaces*, Taiwanese J. Math., **11(5)** (2007), 1477-1484.

[51] H. Şengül and M. Et, *On lacunary statistical convergence of order* α, Acta Math. Sci., **34(2)** (2014), 473-482.

[52] E. Savaş, *On* $\mathcal{I}$*-lacunary Statistical Convergence of Order* α *for Sequences of Sets*, Filomat, **29(6)** (2015), 1223-1229.

[53] A. Sharma, V. Kumar and I. R. Ganaie, *Some remarks on* $\mathcal{I}(S\theta)$*-summability via neutrosophic norm*, Filomat, **37(20)** (2023), 6699-6707.

[54] I. B. Turksen, *Interval valued fuzzy sets based on normal forms*, Fuzzy Sets Syst., **20(2)** (1986), 191-210.

[55] M. R. Türkmen and M. Çınar, *Lacunary statistical convergence in fuzzy normed linear spaces*, Appl. Comput. Math., **6(5)** (2017), 233-237.

[56] U. Ulusu and E. Dündar, $\mathcal{I}$*-lacunary statistical convergence of sequences of sets*, Filomat, **28(8)** (2014), 1567-1574.

[57] L. A. Zadeh, *Fuzzy sets*, Inform. Control, **8** (1965), 338-353.

Author information

Nesar Hossain, Department of Mathematics, The University of Burdwan, Burdwan - 713104, West Bengal, India.
E-mail: nesarhossain24@gmail.com

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