

BÉZIER VARIANT OF PHILLIPS-TYPE GENERALISED POSITIVE LINEAR OPERATORS

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Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 41A25; Secondary 41A30; and 41A36.

Keywords and phrases: Positive linear operators, Rate of convergence, Modulus of smoothness, Lipschitz-type spaces.

This article presents a novel Bézier variant within the family of Phillips-type generalized positive linear operators. We derive the moments of these operators, offering a deeper understanding of their properties. The subsequent section delves into the convergence properties in Lipschitz-type spaces, with a particular emphasis on the Ditzian-Totik modulus of smoothness. In the final section, we rigorously analyze the convergence rate for functions whose derivatives exhibit bounded variation, providing valuable results for applications in approximation theory.

1 Introduction

Over the course of the last thirty years, numerous researchers have introduced and examined the approximation properties of the summation-integral of various operators. In 2019, Gupta [12] presented a comprehensive collection of operators, which encompass various types of Durrmeyer operators, hybrid operators, and specific discrete operators that were examined as exceptional instances. The operators are defined for $x \geq 0$

$$\mathcal{H}_{n,r}^{(\mu,\nu)}(\varsigma; x) = \sum_{r=1}^{\infty} p_{n,r}^{(\mu)}(x) \int_0^{\infty} p_{n,r}^{\nu+1,\rho}(v) \varsigma(v) dv + p_{n,0}^{(\mu)} \varsigma(0), \tag{1.1}$$

where,

$$p_{n,r-1}^{(\nu+1,\rho)}(v) = \frac{n}{\nu B(r\rho, \nu\rho + 1)} \frac{\left(\frac{nv}{\nu}\right)^{r\rho-1}}{\left(1 + \frac{nv}{\nu}\right)^{\nu\rho+r\rho+1}},$$

$$p_{n,r}^{(\mu)}(x) = \frac{(\mu)_r}{r!} \frac{\left(\frac{nx}{\mu}\right)^r}{\left(1 + \frac{nx}{\mu}\right)^{\mu+r}}, \quad (\mu)_r = \mu(\mu + 1)\dots(\mu + r - 1) \text{ and } B \text{ is a beta function.}$$

For $x \geq 0$ operators (1.1) can alternatively be written as

$$\mathcal{H}_{n,r}^{(\mu,\nu)}(\varsigma; x) = \sum_{r=1}^{\infty} p_{n,r}^{(\mu)}(x) \mathcal{G}_{n,r}^{\nu,\rho}(\varsigma),$$

where,

$$\mathcal{G}_{n,r}^{\nu,\rho}(\varsigma) = \begin{cases} \int_0^{\infty} p_{n,r-1}^{\nu+1,\rho}(v) \varsigma(v) dv, & 1 \leq k < \infty \\ \varsigma(0), & k = 0. \end{cases}$$

Here, we have some of the following special case of operators (1.1)

- i. If $\mu = \nu \rightarrow \infty, \rho = 1$, the operators (1.1) reduces to Phillips type operators [19].
- ii. If $\mu = \nu = n, \rho = 1$, we obtain Baskakov-Durrmeyer type operators (see [7],[2]).
- iii. If $\mu \neq \nu$ and $\mu = n, \nu \rightarrow \infty, \rho = 1$, the operators (1.1) changes into Baskakov-Szász type operators considered in [1].
- iv. If $\mu \neq \nu$ and $\mu \rightarrow \infty, \nu = n, \rho = 1$, we receive Szász-Beta type operators proposed in [10].
- v. If $\mu \neq \nu$ and $\mu = nx, \nu = n, \rho = 1$, we get Lupaş-Beta type operators introduced in [11].
- vi. If $\mu \neq \nu$ and $\mu = nx, \nu \rightarrow \infty, \rho = 1$, we have Lupaş-Szász type operators defined in [9].
- vii. If $\mu = \nu = -n$ and $\rho = 1$, we realize genuine Bernstein-Durrmeyer operators, proposed by Chen [3] and also by Goldman-Sharma [8].
- viii. If $\mu = \nu = n, \rho > 0$, the operators (1.1) follow the link operators due to Heilmann and Rasa [13] for $c = 1$.

Recently, Neha et al. [17] examined the Bézier variant of Păltănea operators associated with the inverse Pólya-Eggenberger distribution. They also assessed the approximation properties of these operators, focusing on the first- and second-order modulus of smoothness. Additionally, several studies have explored the generalized form of positive linear operators [4, 5, 14, 15, 16, 20, 21].

Inspired by the aforementioned notable research, we examine the Bézier variation of the set of operators (1.1) in the following manner

$$\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\varsigma; x) = \sum_{r=1}^{\infty} Q_{n,k}^{(\Theta)}(x) \int_0^{\infty} p_{n,r-1}^{\nu,\rho}(v) \varsigma(v) dv + Q_{n,0}^{(\Theta)}(x) \varsigma(0), \tag{1.2}$$

where,

$$Q_{n,r}^{(\Theta)}(x) = [J_{n,r}(x)]^\Theta - [J_{n,r+1}(x)]^\Theta \text{ and } J_{n,r}(x) = \sum_{j=r}^{\infty} p_{n,j}^{(\mu)}(x). \quad (1.3)$$

Alternatively operators (1.2) could be written as

$$\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\varsigma; x) = \int_0^\infty \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v)\varsigma(v)dv, \quad (1.4)$$

where,

$$\mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v) = \sum_{r=1}^{\infty} Q_{n,r}^{(\Theta)}(x) p_{n,r-1}^{\nu,\rho}(v) + Q_{n,0}^{(\Theta)}(x)\varsigma(0).$$

This article intends to examine the approximation behaviour of the operators (1.2) for the functions defined on $[0, \infty)$. We find a direct result using the modulus of continuity, Lipschitz-type space, and we calculate the rate of convergence using derivatives of bounded variation.

2 Basic Properties

In this section, we discuss some useful lemmas and results.

Remark 2.1. For $x \geq 0$, $m \in \mathbb{N} \cup \{0\}$ and $e_m(x) = x^m$, $m = 0, 1, 2, \dots$, we have

$$\begin{aligned} \mathcal{G}_{n,r}^{\nu,\rho}(e_m) &= \int_0^\infty p_{n,r-1}^{\nu+1,\rho}(v)v^m dv \\ &= \int_0^\infty \frac{n}{\nu.B(r\rho, \nu\rho+1)} \frac{\left(\frac{nv}{\nu}\right)^{r\rho-1}}{\left(1+\frac{nv}{\nu}\right)^{\nu\rho+r\rho+1}} v^m dv \\ &= \frac{\Gamma(\nu\rho-m+1)\Gamma(r\rho+m)}{\Gamma(\nu\rho+1)\Gamma(r\rho)} \left(\frac{\nu}{n}\right)^m. \end{aligned}$$

Remark 2.2. For $\Theta \geq 1$, we have

$$\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(e_0; x) = \sum_{r=0}^{\infty} Q_{n,r}^{(\Theta)}(x) = [J_{n,0}(x)]^\Theta = \left[\sum_{j=0}^{\infty} p_{n,j}^{(\mu)}(x) \right]^\Theta = 1.$$

Lemma 2.3. Using the Remark 2.1, the moments of the operators $\mathcal{H}_{n,r}^{(\mu,\nu)}$ (1.1), may be written as

$$\begin{aligned} \mathcal{H}_{n,r}^{(\mu,\nu)}(e_0; x) &= 1, \\ \mathcal{H}_{n,r}^{(\mu,\nu)}(e_1; x) &= x, \\ \mathcal{H}_{n,r}^{(\mu,\nu)}(e_2; x) &= \frac{\nu}{\mu n(\nu\rho-1)} \left[x^2(1+\mu)n\rho + x(1+\rho)\mu \right], \\ \mathcal{H}_{n,r}^{(\mu,\nu)}(e_3; x) &= \frac{\nu^2}{\mu^2 n^2(\nu\rho-1)(\nu\rho-2)} \left[x^3(2+3\mu+\mu^2)\rho^2 n^2 \right. \\ &\quad \left. + x^2(3+3\mu+3\rho+3\mu\rho)n\rho\mu + x(2+3\rho+\rho^2)\mu^2 \right], \\ \mathcal{H}_{n,r}^{(\mu,\nu)}(e_4; x) &= \frac{\nu^3}{\mu^3 n^3(\nu\rho-1)(\nu\rho-2)(\nu\rho-3)} \left[x^4(6+11\mu+6\mu^2+\mu^3)\rho^3 n^3 \right. \\ &\quad \left. + x^3(12+18\mu+6\mu^2+12\rho+18\mu\rho+6\mu^2\rho)\rho^2 n^2 \mu \right. \\ &\quad \left. + x^2(11+11\mu+18\rho+18\mu\rho+7\rho^2+7\mu\rho^2)\rho n \mu^2 \right. \\ &\quad \left. + x(6+11\rho+6\rho^2+\rho^3)\mu^3 \right]. \end{aligned}$$

Lemma 2.4. Using Lemma 2.3, we get

$$\begin{aligned} \mathcal{H}_{n,\mu}^{(\nu,\rho)}((v-x)^2; x) &= \frac{1}{\mu n(\nu\rho-1)} \left[n(\mu+\nu\rho)x^2 + \mu\nu(1+\rho)x \right] \\ &= \frac{\nu(1+\rho)}{n(\nu\rho-1)} \left[x + \frac{n(\mu+\nu\rho)}{\mu\nu(1+\rho)} x^2 \right] \leq \frac{x+\delta x^2}{n} \\ &\leq \frac{\Phi^2(x)}{n}, \end{aligned}$$

where $\Phi^2(x) = x(1+\delta x)$, and $\delta = \frac{n(\mu+\nu\rho)}{\mu\nu(1+\rho)}$.

Lemma 2.5. For real valued continuous and bounded function $\psi \in [0, \infty)$ and the operators $\mathcal{H}_{n,\mu}^{(\nu,\rho)}$, we have

$$|\mathcal{H}_{n,\mu}^{(\nu,\rho)}(\psi; x)| \leq \|\psi\|,$$

where norm of the function on the positive half real line is given by $\|\psi\| = \sup_{x \in [0, \infty)} |\psi(x)|$.

Lemma 2.6. Let $\psi \in C[0, \infty)$, then for any $x \in [0, \infty)$

$$|\tilde{\mathcal{H}}_{n,k}^{(\mu,\nu,\Theta)}(\psi)| \leq \theta \|\psi\|.$$

Proof. Applying the well known property $|a^\lambda - b^\lambda| \leq \lambda|a - b|$, with $0 \leq a, b \leq 1$, $\lambda \geq 1$, and by using (1.3), we get

$$0 < [J_{n,r}(x)]^\Theta - [J_{n,r+1}(x)]^\Theta \leq \theta (J_{n,r}(x) - J_{n,r+1}(x)) \leq \Theta [p_{n,r}^{(\mu)}(x)].$$

Now, by the definition of Bézier variant operators (1.2) and Lemma 2.3, we obtain

$$|\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\psi)| \leq \Theta \|\mathcal{H}_{n,r}^{(\mu,\nu)}(\psi)\| \leq \Theta \|\psi\|.$$

□

3 Convergence Results

For $x \in (0, \infty)$ and $v \in [0, \infty)$, as we have seen in Özarsian and Duman [18], the Lipschitz-type space is described as

$$Lip_M^*(m) = \left\{ \varsigma \in C[0, \infty) : |\varsigma(v) - \varsigma(x)| \leq M \frac{|v - x|^m}{(v + x)^{\frac{m}{2}}} \right\}, \text{ where } 0 < m \leq 1.$$

The following theorem establishes the rate of convergence for the operators $\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}$ for functions in the space $Lip_M^*(m)$.

Theorem 3.1. Let $\varsigma \in Lip_M^*(m)$ and $m \in (0, 1]$. Then for all $x \in (0, \infty)$, we have

$$|\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\varsigma(v); x) - \varsigma(x)| \leq \Theta M \left(\frac{\mathcal{H}_{n,\mu}^{(\nu,\rho)}((v - x)^2; x)}{x} \right)^{\frac{m}{2}}.$$

Proof. From the remark 2.2, we have

$$\begin{aligned} |\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\varsigma(v); x) - \varsigma(x)| &\leq \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(|\varsigma(v) - \varsigma(x)|; x) \\ &\leq \Theta \mathcal{H}_{n,r}^{(\mu,\nu)}(|\varsigma(v) - \varsigma(x)|; x) \\ &\leq \Theta M \mathcal{H}_{n,r}^{(\mu,\nu)} \left(\frac{|v - x|^m}{(v + x)^{\frac{m}{2}}}; x \right) \\ &\leq \frac{\Theta M}{x^{\frac{m}{2}}} \mathcal{H}_{n,r}^{(\mu,\nu)}(|v - x|^m; x). \end{aligned} \tag{3.1}$$

Taking $p = \frac{2}{m}$, $q = \frac{2}{2-m}$ and applying Hölder's inequality, we obtain

$$\begin{aligned} \mathcal{H}_{n,r}^{(\mu,\nu)}(\varsigma)(|v - x|^m; x) &\leq \left\{ \mathcal{H}_{n,r}^{(\mu,\nu)}(|v - x|^2; x) \right\}^{\frac{m}{2}} \cdot \left\{ \mathcal{H}_{n,r}^{(\mu,\nu)}(1^{\frac{2}{2-m}}; x) \right\}^{\frac{2-m}{2}} \\ &\leq \left\{ \mathcal{H}_{n,r}^{(\mu,\nu)}(|v - x|^2; x) \right\}^{\frac{m}{2}}. \end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2), we get the required result

$$|\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\varsigma(v); x) - \varsigma(x)| \leq \Theta M \left(\frac{\mathcal{H}_{n,r}^{(\mu,\nu)}((v - x)^2; x)}{x} \right)^{\frac{m}{2}}.$$

□

In the ensuing outcome, we deduce the rate of convergence when employing the Ditzian-Totik modulus of smoothness $\omega_{\Phi^\lambda}(\varsigma, \delta)$ and Peetre's K -functional $K_{\Phi^\lambda}(\varsigma, \delta)$, $0 \leq \lambda \leq 1$. For $\varsigma \in C_B[0, \infty)$ and $\Phi(x) = \sqrt{x(1 + \delta x)}$, the Ditzian-Totik modulus of smoothness is given by

$$\omega_{\Phi^\lambda}(\varsigma, \delta) = \sup_{0 \leq i \leq \delta} \sup_{x \pm \frac{i\Phi^\lambda(x)}{2} \in [0, \infty)} \left| \varsigma \left(x + \frac{i\Phi^\lambda(x)}{2} \right) - \varsigma \left(x - \frac{i\Phi^\lambda(x)}{2} \right) \right|,$$

and the Peetre's K -functional is defined as

$$K_{\Phi^\lambda}(\varsigma, \delta) = \inf_{h \in W_\lambda} \{ \|\varsigma - h\| + \delta \|\Phi^\lambda h'\| \},$$

where W_λ is a subspace of all real functions defined on $[0, \infty)$ and $h \in W_\lambda$ which is absolutely continuous function with norm $\|\Phi^\lambda h'\| \leq \infty$. In [[6], Theorem 2.1.1], then we have a constant $C \geq 0$ such that

$$C^{-1} \omega_{\Phi^\lambda}(\varsigma, \delta) \leq K_{\Phi^\lambda}(\varsigma, \delta) \leq C \omega_{\Phi^\lambda}(\varsigma, \delta). \tag{3.3}$$

Theorem 3.2. For $\varsigma \in C_B [0, \infty)$ and $\lambda \in [0, 1]$ then, we have

$$\left| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\varsigma; x) - \varsigma(x) \right| \leq \mathcal{C}\omega_{\Phi^\lambda} \left(\varsigma; \Phi^{1-\lambda}(x) \sqrt{\frac{\Theta}{n}} \right).$$

Proof. For $h \in W_\lambda$, we consider

$$h(v) = h(x) + \int_x^v h'(u) du.$$

Applying $\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}$ and Hölder's inequality then, we have

$$\begin{aligned} \left| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(h(v); x) - h(x) \right| &\leq \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)} \left(\int_x^v |h'| du; x \right) \\ &\leq \|\Phi^\lambda h'\| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)} \left(\left| \int_x^v \frac{du}{\Phi^\lambda(u)} \right|; x \right) \\ &\leq \|\Phi^\lambda h'\| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)} \left(|v-x|^{1-\lambda} \left| \int_x^v \frac{du}{\Phi(u)} \right|^\lambda; x \right). \end{aligned} \quad (3.4)$$

Let $I = \left| \int_x^v \frac{du}{\Phi(u)} \right|$, now first we simplify expression I

$$\begin{aligned} I &\leq \left| \int_x^v \frac{du}{\sqrt{u}} \right| \left| \left(\frac{1}{\sqrt{1+\delta x}} + \frac{1}{\sqrt{1+\delta v}} \right) \right| \\ &\leq 2|\sqrt{v} - \sqrt{x}| \left(\frac{1}{\sqrt{1+\delta x}} + \frac{1}{\sqrt{1+\delta v}} \right) \\ &\leq 2 \frac{|v-x|}{\sqrt{v} + \sqrt{x}} \left(\frac{1}{\sqrt{1+\delta x}} + \frac{1}{\sqrt{1+\delta v}} \right) \\ &\leq 2 \frac{|v-x|}{\sqrt{x}} \left(\frac{1}{\sqrt{1+\delta x}} + \frac{1}{\sqrt{1+\delta v}} \right). \end{aligned} \quad (3.5)$$

Now, we use the inequality $|p+q|^\lambda \leq |p|^\lambda + |q|^\lambda$, $0 \leq \lambda \leq 1$ then from (3.5), we get

$$\left| \int_x^v \frac{du}{\Phi(u)} \right|^\lambda \leq 2^\lambda \frac{|v-x|^\lambda}{x^{\frac{\lambda}{2}}} \left(\frac{1}{(1+\delta x)^{\frac{\lambda}{2}}} + \frac{1}{(1+\delta v)^{\frac{\lambda}{2}}} \right). \quad (3.6)$$

From (3.4) and (3.6) and using Cauchy inequality, we get

$$\begin{aligned} &\left| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(h(v); x) - h(x) \right| \\ &\leq \frac{2^\lambda \|\Phi^\lambda h'\|}{x^{\frac{\lambda}{2}}} \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)} \left(|v-x| \left(\frac{1}{(1+\delta x)^{\frac{\lambda}{2}}} + \frac{1}{(1+\delta v)^{\frac{\lambda}{2}}} \right); x \right) \\ &= \frac{2^\lambda \|\Phi^\lambda h'\|}{x^{\frac{\lambda}{2}}} \left(\frac{1}{(1+\delta x)^{\frac{\lambda}{2}}} \left(\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)} \left((v-x)^2; x \right) \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left(\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)} \left((v-x)^2; x \right) \right)^{\frac{1}{2}} \cdot \left(\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)} \left((1+\delta v)^{-\lambda}; x \right) \right)^{\frac{1}{2}} \right). \end{aligned} \quad (3.8)$$

From Lemma 2.4, we may write

$$\left(\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)} \left((v-x)^2; x \right) \right)^{\frac{1}{2}} \leq \sqrt{\frac{\Theta}{n}} \Phi(x), \quad (3.9)$$

where $\Phi(x) = \sqrt{x(1+\delta x)}$.

For $x \in [0, \infty)$, $\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)} \left((1+\delta v)^{-\lambda}; x \right)$ approaches to $(1+\delta x)^{-\lambda}$ when n approaches to ∞ . Thus for any $\varepsilon > 0$, we find $n_0 \in \mathbb{N}$ such that

$$\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)} \left((1+\delta v)^{-\lambda}; x \right) \leq (1+\delta x)^{-\lambda} + \varepsilon, \text{ for all } n \geq n_0.$$

By assuming $\varepsilon = (1+\delta x)^{-\lambda}$, above inequality reduces to the following

$$\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)} \left((1+\delta v)^{-\lambda}; x \right) \leq 2(1+\delta x)^{-\lambda}, \text{ for all } n \geq n_0. \quad (3.10)$$

From (3.9) and (3.10), we have

$$\begin{aligned} \left| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(h(v); x) - h(x) \right| &\leq 2^\lambda \|\Phi^\lambda h'\| \sqrt{\frac{\Theta}{n}} \psi(x) \left(\Phi^{-\lambda}(x) \right. \\ &\quad \left. + \sqrt{2} x^{-\frac{\lambda}{2}} (1+\delta x)^{-\frac{\lambda}{2}} \right) \\ &\leq 2^\lambda (1+\sqrt{2}) \|\Phi^\lambda h'\| \sqrt{\frac{\Theta}{n}} \Phi^{1-\lambda}(x). \end{aligned} \quad (3.11)$$

We may write

$$\begin{aligned} \left| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\varsigma(v); x) - \varsigma(x) \right| &\leq \left| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\varsigma(v) - h(v); x) \right| \\ &\quad + \left| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(h(v); x) - h(x) \right| + |h(x) - \varsigma(x)| \\ &\leq 2\|\varsigma - h\| + \left| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(h(v); x) - h(x) \right|. \end{aligned} \quad (3.12)$$

From (3.11) and (3.12) and for very large n , we obtain

$$\begin{aligned} \left| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\varsigma(v); x) - \varsigma(x) \right| &\leq 2\|\varsigma - g\| + 2^\lambda(1 + \sqrt{2})\|\Phi^\lambda h'\| \sqrt{\frac{\Theta}{n}} \Phi^{1-\lambda}(x) \\ &\leq C_1 \{ \|\varsigma - h\| + \Phi^{1-\lambda}(x) \sqrt{\frac{\Theta}{n}} \|\Phi^\lambda h'\| \} \\ &\leq CK_{\Phi^\lambda} \left(\varsigma; \Phi^{1-\lambda}(x) \sqrt{\frac{\Theta}{n}} \right), \end{aligned} \quad (3.13)$$

where $C_1 = \max\{2, 2^\lambda(1 + \sqrt{2})\}$ and $C = 2C_1$. From (3.3) and (3.13), we follow the required result. \square

In the following theorem, we estimate the convergence rate of (1.2) within the function class $DBV[0, \infty)$. This class consists of all absolutely continuous functions ς defined on the interval $[0, \infty)$, which have bounded derivatives on the same interval. For any ς belonging to the set of functions $DBV[0, \infty)$, the following expression holds

$$\varsigma(x) = \int_0^x \psi(v)dv + \varsigma(0),$$

where $\psi(v)$ is a function of bounded variation on each finite partition of $[0, \infty)$.

Lemma 3.3. For $x \in [0, \infty)$ and very large n , we have

(i) Take $0 \leq y < x$, therefore

$$\xi_n(x, y) = \int_0^y \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v)dv \leq \frac{\Theta x(1 + \delta x)}{n(x - y)^2}.$$

(ii) If $x < z < \infty$, then

$$1 - \xi_n(x, z) = \int_z^\infty \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v)dv \leq \frac{\Theta x(1 + \delta x)}{n(z - x)^2}.$$

Proof. Using Lemma 2.3, 2.4 and the alternative form (1.4) of operators (1.2), for sufficiently large n and $0 \leq y < x$, we have

$$\begin{aligned} \xi_n(x, y) &= \int_0^y \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v)dv \\ &\leq \int_0^y \frac{(x - v)^2}{(x - y)^2} \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v)dv \\ &\leq \frac{1}{(x - y)^2} \mathcal{H}_{n,\mu}^{(\nu,\rho)}((v - x)^2; x) \\ &\leq \frac{\Theta x(1 + \delta x)}{n(x - y)^2}. \end{aligned}$$

In the similar way we can prove the second part of Lemma. \square

Theorem 3.4. Let $\varsigma \in DBV[0, \infty)$ for all $x \in [0, \infty)$ and sufficiently large n , we have

$$\begin{aligned} \left| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\varsigma; x) - \varsigma(x) \right| &\leq \frac{\sqrt{\Theta}}{1 + \Theta} |(\varsigma'(x^+) + \Theta \varsigma'(x^-))| \frac{\Phi(x)}{\sqrt{n}} \\ &\quad + \frac{\Theta^{\frac{3}{2}}}{\Theta + 1} |\varsigma'(x^+) - \varsigma'(x^-)| \frac{\Phi(x)}{\sqrt{n}} \\ &\quad + \frac{\Theta \Phi(x)}{nx} \sum_{r=1}^{[\sqrt{n}]} \bigvee_{x - \frac{x}{r}}^x (\varsigma'_x) + \frac{x}{\sqrt{n}} \bigvee_{x - \frac{x}{\sqrt{n}}}^x (\varsigma'_x) \\ &\quad + \frac{\Theta \Phi(x)}{nx} \sum_{r=1}^{[\sqrt{n}]} \bigvee_x^{x + \frac{x}{r}} (\varsigma'_x) + \frac{x}{\sqrt{n}} \bigvee_x^{x + \frac{x}{\sqrt{n}}} (\varsigma'_x), \end{aligned}$$

where the auxiliary function ς'_x is defined as

$$\varsigma'_x(v) = \begin{cases} \varsigma(v) - \varsigma(x^-), & 0 \leq v < x \\ 0, & v = x \\ \varsigma(v) - \varsigma(x^+), & x < v < \infty. \end{cases}$$

Proof. From Remark 2.2, and using (1.4) second form of the operators (1.2) for every $x \in [0, \infty)$, we have

$$\begin{aligned} \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\varsigma(v); x) - \varsigma(x) &= \int_0^\infty \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v)(\varsigma(v) - \varsigma(x))dv \\ &= \int_0^\infty \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v) \left(\int_x^v \varsigma'(u)du \right) dv. \end{aligned} \quad (3.14)$$

For each function $\varsigma \in DBV[0, \infty)$, we can write

$$\begin{aligned} \varsigma'(u) &= \varsigma'_x(u) + \frac{1}{\Theta+1}(\varsigma'(x^+) + \Theta\varsigma'(x^-)) + \frac{1}{2}(\varsigma'(x^+) - \varsigma'(x^-)) \\ &\quad \times \left(\text{sgn}(u-x) + \frac{\Theta-1}{\Theta+1} \right) + \delta_x(u)(\varsigma'(u) - (\varsigma'(x^+) + \varsigma'(x^-))), \end{aligned} \quad (3.15)$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x \\ 0, & u \neq x. \end{cases}$$

It can be easily seen that

$$\int_0^\infty \left(\int_x^v \left(\varsigma'(u) - \frac{1}{2}(\varsigma'(x^+) + \varsigma'(x^-)) \right) \delta_x(u)du \right) \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v)dv = 0. \quad (3.16)$$

Using the operators (1.4) and (3.15), we have

$$\begin{aligned} I_1 &= \int_0^\infty \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v) \int_x^v \frac{1}{\Theta+1}(\varsigma'(x^+) + \Theta\varsigma'(x^-))dudv \\ &= \frac{1}{\Theta+1} |(\varsigma'(x^+) + \Theta\varsigma'(x^-))| \int_0^\infty \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v) |v-x| dv \\ &\leq \frac{1}{\Theta+1} (\varsigma'(x^+) + \Theta\varsigma'(x^-)) \left(\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}((v-x)^2; x) \right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{\Theta}}{1+\Theta} |(\varsigma'(x^+) + \Theta\varsigma'(x^-))| \frac{\Phi(x)}{\sqrt{n}}. \end{aligned} \quad (3.17)$$

And

$$\begin{aligned} I_2 &= \int_0^\infty \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v) \int_x^v \frac{1}{2}(\varsigma'(x^+) - \varsigma'(x^-)) \left(\text{sgn}(u-x) + \frac{\Theta-1}{\Theta+1} \right) dudv \\ &\leq \frac{\Theta}{\Theta+1} |\varsigma'(x^+) - \varsigma'(x^-)| \int_0^\infty \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v) |v-x| dv \\ &\leq \frac{\Theta}{\Theta+1} |\varsigma'(x^+) - \varsigma'(x^-)| \left(\tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}((v-x)^2; x) \right)^{\frac{1}{2}} \\ &\leq \frac{\Theta^{\frac{3}{2}}}{\Theta+1} |\varsigma'(x^+) - \varsigma'(x^-)| \frac{\Phi(x)}{\sqrt{n}}. \end{aligned} \quad (3.18)$$

From the relations (3.14 - 3.18), we get the following estimate

$$\begin{aligned} \left| \tilde{\mathcal{H}}_{n,r}^{(\mu,\nu,\Theta)}(\varsigma; x) - \varsigma(x) \right| &\leq \frac{\sqrt{\Theta}}{1+\Theta} |(\varsigma'(x^+) + \Theta\varsigma'(x^-))| \frac{\Phi(x)}{\sqrt{n}} \\ &\quad + \frac{\Theta^{\frac{3}{2}}}{\Theta+1} |\varsigma'(x^+) - \varsigma'(x^-)| \frac{\Phi(x)}{\sqrt{n}} \\ &\quad + A_{n,r}^{(\mu,\nu,\Theta)}(\varsigma'_x; x) + B_{n,r}^{(\mu,\nu,\Theta)}(\varsigma'_x; x), \end{aligned} \quad (3.19)$$

where,

$$A_{n,r}^{(\mu,\nu,\Theta)}(\varsigma'_x; x) = \int_0^x \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v) \left(\int_x^v \varsigma'(u)du \right) dv,$$

and,

$$B_{n,r}^{(\mu,\nu,\Theta)}(\varsigma'_x; x) = \int_x^\infty \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v) \left(\int_x^v \varsigma'(u)du \right) dv.$$

The terms $A_{n,r}^{(\mu,\nu,\Theta)}(\varsigma'_x; x)$ and $B_{n,r}^{(\mu,\nu,\Theta)}(\varsigma'_x; x)$ are to be estimated for a complete proof of the theorem.

Using integration by parts and applying the Lemma 3.3 with $y = x - \frac{x}{\sqrt{n}}$, it follows

$$\begin{aligned}
 |A_{n,r}^{(\mu,\nu,\Theta)}(\varsigma'_x; x)| &= \left| \int_0^x \left(\int_x^v \varsigma'(u) du \right) d_v \xi_{n,\Theta}(x, v) dv \right| \\
 &= \left| \int_0^x \xi_{n,\Theta}(x, v) \varsigma'_x(v) dv \right| \\
 &\leq \int_0^y |\varsigma'_x(v)| |\xi_{n,\Theta}(x, v)| dv + \int_y^x |\varsigma'_x(v)| |\xi_{n,\Theta}(x, v)| dv \\
 &\leq \frac{\Theta\Phi(x)}{n} \int_0^y \bigvee_v^x (\varsigma'_x)(x-v)^{-2} dv + \int_y^x \bigvee_v^x (\varsigma'_x) dv \\
 &\leq \frac{\Theta\Phi(x)}{n} \int_0^y \bigvee_v^x (\varsigma'_x)(x-v)^{-2} dv + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (\varsigma'_x). \tag{3.20}
 \end{aligned}$$

Again, using Lemma 3.3 and taking $u = \frac{x}{x-v}$, we get

$$\begin{aligned}
 \frac{\Theta\Phi(x)}{n} \int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_v^x (\varsigma'_x)(x-v)^{-2} dv &= \frac{\Theta\Phi(x)}{nx} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x (\varsigma'_x) du \\
 &\leq \frac{\Theta\Phi(x)}{nx} \sum_{r=1}^{[\sqrt{n}]} \int_r^{r+1} \bigvee_{x-\frac{x}{u}}^x (\varsigma'_x) du \\
 &\leq \frac{\Theta\Phi(x)}{nx} \sum_{r=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{r}}^x (\varsigma'_x). \tag{3.21}
 \end{aligned}$$

On combining (3.20) and (3.21), we get the following

$$|A_{n,r}^{(\mu,\nu,\Theta)}(\varsigma'_x; x)| \leq \frac{\Theta\Phi(x)}{nx} \sum_{r=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{r}}^x (\varsigma'_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (\varsigma'_x). \tag{3.22}$$

Now, using the integration by parts and applying the Lemma 3.3 with $z = x + \frac{x}{\sqrt{n}}$, we get

$$\begin{aligned}
 |B_{n,r}^{(\mu,\nu,\Theta)}(\varsigma'_x; x)| &= \left| \int_x^\infty \mathcal{M}_{n,r}^{(\nu,\rho,\Theta)}(x, v) \left(\int_x^v \varsigma'_x(u) du \right) dv \right| \\
 &= \left| \int_x^z \left(\int_x^v \varsigma'_x(u) du \right) d_v (1 - \xi_{n,\Theta}(x, v)) dv \right. \\
 &\quad \left. + \int_z^\infty \left(\int_x^v \varsigma'_x(u) du \right) d_v (1 - \xi_{n,\Theta}(x, v)) dv \right| \\
 &= \left| \left[\left(\int_x^v \varsigma'_x(u) du \right) (1 - \xi_{n,\Theta}(x, v)) \right]_x^z - \int_x^z \varsigma'_x(v) (1 - \xi_{n,\Theta}(x, v)) dv \right. \\
 &\quad \left. + \int_z^\infty \left(\int_x^v \varsigma'_x(u) du \right) d_v (1 - \xi_{n,\Theta}(x, v)) dv \right| \\
 &= \left| \left(\int_x^z \varsigma'_x(u) du \right) (1 - \xi_{n,\Theta}(x, z)) - \int_x^z \varsigma'_x(v) (1 - \xi_{n,\Theta}(x, v)) dv \right. \\
 &\quad \left. + \left[\left(\int_x^v \varsigma'_x(u) du \right) (1 - \xi_{n,\Theta}(x, v)) \right]_z^\infty - \int_z^\infty \varsigma'_x(v) (1 - \xi_{n,\Theta}(x, v)) dv \right| \\
 &= \left| \int_x^z \varsigma'_x(v) (1 - \xi_{n,\Theta}(x, v)) dv + \int_z^\infty \varsigma'_x(v) (1 - \xi_{n,\Theta}(x, v)) dv \right| \\
 &< \frac{\Theta\Phi(x)}{n} \int_x^\infty \bigvee_x^v (\varsigma'_x)(v-x)^{-2} dv + \int_x^z \bigvee_x^v (\varsigma'_x) \\
 &\leq \frac{\Theta\Phi(x)}{n} \int_{x+\frac{x}{\sqrt{n}}}^\infty \bigvee_x^v (\varsigma'_x)(v-x)^{-2} dv + \frac{x}{\sqrt{n}} \bigvee_{x+\frac{x}{\sqrt{n}}}^x (\varsigma'_x), \tag{3.23}
 \end{aligned}$$

on substituting $u = \frac{x}{v-x}$, we get

$$\begin{aligned} \frac{\Theta\Phi(x)}{n} \int_{x+\frac{x}{\sqrt{n}}}^{\infty} \bigvee_x(\zeta'_x)(v-x)^{-2} dv &= \frac{\Theta\Phi(x)}{nx} \int_0^{\sqrt{n}} \bigvee_x^{x+\frac{x}{u}}(\zeta'_x) du \\ &\leq \frac{\Theta\Phi(x)}{nx} \sum_{r=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_x^{x+\frac{x}{u}}(\zeta'_x) du \\ &\leq \frac{\Theta\Phi(x)}{nx} \sum_{r=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{r}}(\zeta'_x). \end{aligned} \quad (3.24)$$

Using (3.23) and (3.24), we get the following

$$\left| B_{n,r}^{(\mu,\nu,\Theta)}(\zeta'_x; x) \right| \leq \frac{\Theta\Phi(x)}{nx} \sum_{r=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{r}}(\zeta'_x) + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}}(\zeta'_x). \quad (3.25)$$

The relations (3.19), (3.22) and (3.25), leads to the required result. \square

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Received: 2023-09-20

Accepted: 2025-07-16