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UNVEILING THE $\breve{q}$-RUNG PICTURE FUZZY FRANK BONFERRONI OPERATORS FOR GROUP DECISION-MAKING

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Abstract

This research addresses the need for more effective tools in multiple attribute group decision-making (MAGDM), especially under uncertain conditions where attribute interdependencies are significant. The $\breve{q}$-rung picture fuzzy set offers a more adaptable framework for handling uncertain information compared to other fuzzy set extensions. To improve decision outcomes in cases with interrelated attributes, this work introduces $\breve{q}$-rung picture fuzzy weighted Frank Bonferroni mean ($\breve{q}$-RPFWFBM) aggregators, integrating the Frank t-norm and t-conorm (FTT). Through a numerical instance, we illustrate how the $\breve{q}$-RPFWFBM operator effectively captures attribute interdependencies compared to existing operators. A sensitivity assessment demonstrates the resilience of the suggested approach and comparative results indicate superiority, accuracy and flexibility of proposed aggregators in MAGDM applications.

1 Introduction

In many decision-making scenarios, evaluation data is typically represented as precise numerical values. However, the intrinsic complexity of decision challenges and unclear data frequently hinder decision-makers from expressing their viewpoints using precise values. To address these challenges, Lotfi Zadeh's fuzzy set theory [45], introduced in 1965, provides a mathematical approach for managing uncertainty and imprecision. Fuzzy sets (FS) enable partial membership, where every member is assigned a value between 0 and 1 via a membership function referred to as the membership value or positive degree (PD). Subsequently, Atanassov's [4] intuitionistic fuzzy sets (IFS) refined the idea by incorporating a non-membership or negative degree (ND). Later, Yager's [42] Pythagorean fuzzy sets (PyFS) expanded on IFS by further relaxing its constraints. Following this, Cuong's [10] picture fuzzy sets (PFS) then extended IFS by incorporating an indeterminacy degree (ID), providing a more nuanced representation of uncertainty. Gundogdu et al. [20] developed an enhanced model called a spherical fuzzy set (SFS) to further support the notion of PFS. Unlike IFS, where the sum of the PD and ND must be less than or equal to 1, Yager's [43] $\breve{q}$-rung orthopair fuzzy set ($\breve{q}$-ROFS) relaxes the constraint by permitting the combined $\breve{q}^{th}$ power totals of PD and ND to remain within 1. To advance the concept of PFS, $\breve{q}$-rung picture fuzzy sets ($\breve{q}$-RPFS) were later enhanced by Li et al. [21] as an extension. This framework offers a more sophisticated and flexible way of capturing and representing uncertainty. Unlike PFS, where the sum of the PD, ID and ND must remain within 1, $\breve{q}$-RPFS mitigates this constraint by allowing the combined $\breve{q}^{th}$ power totals of PD, ID and ND to remain within 1.

In today's complex decision-making environments, decision-makers often face challenges involving the evaluation of multiple criteria. Models for multi-criteria decision-making (MCDM) play a crucial role by providing systematic approaches that help decision-makers select the most suitable alternatives based on their preferences and the inherent uncertainties in the data. Recent research has explored the framework of IFS, PyFS, $\breve{q}$-ROFS, neutrosophic FS, cubic hesitant FS and linear Diophantine (LD) FS theories for MCDM. Imran et al. [15] proposed a hybrid of Aczel-Alsina and BM aggregators within an interval-valued IF framework, enabling the capture of interrelationships among criteria to support expert decision-making. Arora et al. [1] investigated and developed the application of logarithmic similarity measures under PyFS. Palanikumar et al. [29] developed Pythagorean neutrosophic fuzzy soft sets with interval values to facilitate the selection of motorbikes. In another study, Palanikumar et al. [28] proposed a new methodology that employs complex normal interval-valued fuzzy sets based on PyFS to address medical diagnosis issues related to brain tumors. Hasan et al. [12] introduced an approach for ranking alternatives, incorporating machine learning techniques, fuzzy AHP, TOPSIS, and wavelet transformations. Additionally, Palanikumar et al. [30] introduced generalized Diophantine fuzzy sets, which extended both Diophantine and Pythagorean fuzzy sets by defining new operators and distance measures. Asif et al. [3] employed Hamacher operators to develop various Pythagorean fuzzy aggregation methods in their research. Khan et al. [18] examined the cryptocurrency sector through a $\breve{q}$-ROF hypersoft set algorithm employing aggregators. Salma et al. [33] presented $\breve{q}$-ROF hypersoft ordered aggregators and utilized them to green supplier selection. Khan et al. [19] proposed a novel approach utilizing $\breve{q}$-ROF hypersoft sets to assess tourism carrying capacity within a decision-making framework. Nagarajan et al. [25] devised a neutrosophic BM operator for MAGDM in triangular and trapezoidal neutrosophic interval settings. Rehaman et al. [32] developed Dombi exponential aggregators within neutrosophic cubic hesitant fuzzy sets and employed them to select optimal sites for solid waste disposal. Nithyasri et al. [26] utilized Bipolar LD Fuzzy aggregators to comprehensively examine lung carcinoma across various stages aiming to identify the optimal therapeutic approach. Kannan et al. [17] developed an approach combining the CODAS method with Linear Diophantine FS to handle complex MCDM problems under conditions of imprecision. Surya et al. [36] proposed a $\breve{q}$-rung LD fuzzy hypersoft set along with an entropy measure specifically designed for MCDM. Due to their ability to handle uncertainty more comprehensively, PFS and SFS are extensively utilized in MCDM problems. Wei [37, 38] explored the use of Archimedean and Hamacher norms as operational principles for PFS. Additionally, Zhang et al. [47] and Jana et al. [16] established PF aggregators based on Dombi norms to address MCDM challenges. Aggregators based on SFS were developed by Ashraf et al. [2] and applied in the issues regarding the decision-making. Hussain and Ullah [14] introduced SW-based aggregators within SFS settings, offering a MCDM approach to handle real-life decisions. Palanikumar et al. [31] proposed novel logarithmic square root vague set aggregation operators for new building construction work selection. Palanikumar et al. [27] proposed novel distance approaches using $\breve{q}$-rung vague sets for MCDM issues. Triangular norms along with conorms introduced by Frank [11] in 1979 and termed as the Frank t-norm and t-conorm (FTT), facilitate the benefit of enhancing flexibility in the data accumulation process. Consequently, various studies have developed accumulation operators using FTT across different fuzzy environments to address MCDM issues. These include the IFS environment by Zhang et al. [46], the PFS and $\breve{q}$-ROFS contexts by Sheik and Mandal [34, 35], and the $\breve{q}$-RPFS environment by Chitra and Prabakaran [7, 8, 9].

This study focuses on $\breve{q}$-rung picture fuzzy based MCDM methods due to their unique advantages:

- Uncertainty Management: Traditional models often struggle with uncertainty in qualitative data. The $\breve{q}$-RPFS framework offers greater flexibility for effectively modeling and managing uncertainty.
- Flexibility and Adaptability: $\breve{q}$-rung picture fuzzy methods provide the flexibility to model decision problems with varying degrees of fuzziness, enabling decision-makers to express their preferences more accurately.
- Improved Decision Outcomes: By employing the $\breve{q}$-RPF weighted Frank Bonferroni mean ($\breve{q}$-RPFWFBM) aggregator, this study aims to enhance decision outcomes in multiple-

attribute group decision-making (MAGDM) scenarios, delivering reliable results.

Choosing Frank norms in the proposed MADM framework is driven by several compelling factors that make them particularly suitable for handling the complexities of making decisions involving $\breve{q}$-RPFS:

- Frank norms facilitate the aggregation of imprecise data, enabling decision-makers to combine qualitative and quantitative data effectively. This is especially relevant in the context of $\breve{q}$-RPF numbers, which inherently involve degrees of uncertainty.
- Frank norms allow for a smooth transition between scenarios of independent criteria and those with complete dependence. This feature allows for more nuanced decision-making, accommodating a wide range of real-world situations.
- Frank norms offer extensive parametric control, allowing them to mimic various t-norms. For instance, as $m \to 1$, the Frank addition (sum operator) and Frank multiplication (product operator) reduce to the probabilistic addition and multiplication, while as, $m \to \infty$ they converge to the Lukasiewicz addition and multiplication. This flexibility enhances their ability to handle fuzzy data across different scenarios.
- Frank norms provide improved control over the interaction of fuzzy membership values, which is crucial in MCDM where the interaction between criteria significantly influences the outcome.

In conclusion, while Aczel-Alsina, Dombi, and SW norms have their own advantages, the flexibility, efficiency, and superior control over interactions between fuzzy criteria make Frank norms ideal for the context of this study.

While Frank norms offer significant advantages, it is important to acknowledge the merits of alternative norms. Aczel-Alsina and Dombi norms excel in providing smooth transitions and parametric flexibility, making them particularly useful for handling gradual changes. Similarly, SW norms allow for tailored aggregation by offering control over the interaction of fuzzy values through adjustable parameters.

In practical decision-making, it is common to encounter interconnections between various attributes (factors). For instance, in evaluating the quality of a mobile phone, factors such as display quality, battery life, processing speed and price are interrelated. A high-quality display may require more battery capacity and processing power, illustrating the need to account for these interdependencies to make rational and informed decisions. An efficient aggregation function that accounts for relationships between pairs of attributes, as described by Bonferroni [6], is the Bonferroni mean (BM) operator.

- Bonferroni functions: These functions effectively capture interdependencies among attributes, making them ideal for MADM scenarios.
- Hamy functions: Less effective in situations with interdependent criteria, focusing more on independent evaluations.
- Dombi functions: Primarily designed for diminishing returns and may not adequately address the complexities of interrelated attributes.

1.1 Motivations for this Research

This research is motivated by the need to enhance decision-making processes in complex scenarios where multiple interrelated attributes are evaluated. The growing reliance on MCDM methods across various fields highlights the importance of developing robust frameworks that can effectively capture uncertainty and interdependencies. By focusing on $\breve{q}$-RPFS with the Bonferroni mean under Frank norm operations, this research seeks to provide a versatile and holistic framework for reaching decisions, designed to yield more reliable and effective outcomes.

1.2 Challenges in earlier study and Research gap

Previous research has introduced Bonferroni aggregation operators in fuzzy and intuitionistic fuzzy contexts (Yager, [41]; Xu and Yager [40], Xia et al. [39]), often lack the flexibility needed to adapt to various decision-making contexts, limiting their applicability in diverse scenarios. In existing studies, Liu et al. [23]; Liu and Junlin [22], Liu and Wang [24], Yang and Pang [44], and Ates and Akay [5] explored Bonferroni operators for handling interdependencies among criteria in MCDM issues, but they have not adequately considered the unique advantages of Frank norms or explored the $\breve{q}$-RPF framework. He et al. [13] presented $\breve{q}$-RPF Hamy mean Dombi aggregators and employed in MAGDM issue. While the Dombi Hamy Mean operator effectively aggregated $\breve{q}$-RPF information, it did not explicitly account for the importance of criteria or experts in MAGDM scenarios. This operator focused on the average of the attributes but lacked a structured approach to fully explore the interdependencies between criteria during the aggregation, leading to potential inaccuracies when criteria influence each other.

Building on these foundational works, this study aims to bridge the gap between Bonferroni operators, Frank norms, and $\breve{q}$-rung picture fuzzy numbers - a connection that has not been explored in previous research. While earlier studies applied Bonferroni means across various fuzzy contexts, integrating Frank norms into this framework is a novel contribution. This paper demonstrates how Frank-based Bonferroni operators can better capture interdependencies and uncertainty in group decision-making compared to conventional methods. In summary, the contributions of this research not only advance theoretical knowledge in the field of MCDM but also provide practical tools and insights that can significantly improve decision-making outcomes in practical applications.

Moreover, the subsequent segments (sections) of the paper are configured as detailed below: Segment 2 discusses the fundamentals of the $\breve{q}$-RPFS and its operating rules based on Frank norms and Bonferroni mean operators. Segment 3 introduces the $\breve{q}$-RPF weighted Frank Bonferroni mean ($\breve{q}$-RPFWFBM) aggregators and outlines their characteristics. The procedure for solving group decision-making issues with multiple attributes using the $\breve{q}$-RPFWFBM operator is detailed in Segment 4. Segment 5 demonstrates the efficacy of the devised strategy in addressing a real-world MAGDM problem. Additionally, the impact of the parameters of the $\breve{q}$-RPFWFBM averaging operator on decision outcomes is analyzed, and a comparison to current operators is provided. Lastly, Segment 6 brings the paper to a conclusion, followed by a discussion of limitations, approaches for managing these limitations, and directions for future research.

2 Foundational Concepts

This segment offers a brief overview of the $\breve{q}$-RPFS, FTT, and BM operators.

Definition 2.1. [43]. Assume $\breve{\mathcal{U}}$ signifies the universe of discourse. The $\breve{q}$-ROFS $\breve{o}_p$ over $\breve{\mathcal{U}}$ is described as: $\breve{o}_p = \{(e, \mu_{\breve{o}_p}(e), \phi_{\breve{o}_p}(e)) : e \in \breve{\mathcal{U}}\}$, where the PD of e in $\breve{o}_p$ expressed by $\mu_{\breve{o}_p}(e)$ and the ND of e in $\breve{o}_p$ expressed as $\phi_{\breve{o}_p}(e)$ is formulated to comply with the requirement, $0 \leq (\mu_{\breve{o}_p}(e))^{\breve{q}} + (\phi_{\breve{o}_p}(e))^{\breve{q}} \leq 1$, $\breve{q}$ denotes a positive whole number. In addition, the indeterminacy value of e in $\breve{o}_p$ results from $(1 - (\mu_{\breve{o}_p}(e))^{\breve{q}} - (\phi_{\breve{o}_p}(e))^{\breve{q}})^{1/\breve{q}}$.

Definition 2.2. [21]. Assume $\breve{\mathcal{U}}$ signifies the universe of discourse. The $\breve{q}$-RPFS over $\breve{\mathcal{U}}$ is specified as: $\zeta_\varrho = \{(e, \mu_{\zeta_\varrho}(e), \psi_{\zeta_\varrho}(e), \phi_{\zeta_\varrho}(e)) : e \in \breve{\mathcal{U}}\}$, where the PD of e in ζ_ϱ expressed by $\mu_{\zeta_\varrho}(e)$, the ID of e in ζ_ϱ expressed as $\psi_{\zeta_\varrho}(e)$ and the ND of e in ζ_ϱ expressed as $\phi_{\zeta_\varrho}(e)$ is formulated to comply with the requirement, $0 \leq (\mu_\zeta(e))^{\breve{q}} + (\psi_\zeta(e))^{\breve{q}} + (\phi_\zeta(e))^{\breve{q}} \leq 1$. Additionally, the refusal value of e in ζ results from $(1 - (\mu_\zeta(e))^{\breve{q}} - (\psi_\zeta(e))^{\breve{q}} - (\phi_\zeta(e))^{\breve{q}})^{1/\breve{q}}$. A $\breve{q}$-RPF number termed as $\zeta_j = (\mu_{\zeta_j}, \psi_{\zeta_j}, \phi_{\zeta_j})$.

Definition 2.3. [13]. Let $\zeta = (\mu_\zeta, \psi_\zeta, \phi_\zeta)$ be a $\breve{q}$-RPF number. Consequently, the score $\breve{S}(\zeta)$ and the accuracy function $\breve{A}(\zeta)$ are provided below,

$$\breve{S}(\zeta) = \quad 1 + \mu_\zeta^{\breve{q}} - \phi_\zeta^{\breve{q}} \tag{2.1}$$

$$\breve{A}(\zeta) = \quad \mu_{\zeta}^{\breve{q}} + \psi_{\zeta}^{\breve{q}} + \phi_{\zeta}^{\breve{q}} \tag{2.2}$$

Let us examine two $\breve{q}$-rung picture fuzzy numbers ($\breve{q}$-RPFNs), $\zeta_1 = (\mu_{\zeta_1}, \psi_{\zeta_1}, \phi_{\zeta_1})$ and $\zeta_2 = (\mu_{\zeta_2}, \psi_{\zeta_2}, \phi_{\zeta_2})$. Then, from the equations (2.1) and (2.2), if $\breve{S}(\zeta_1) \succ \breve{S}(\zeta_2)$ it follows that $\zeta_1 \succ \zeta_2$. If $\breve{S}(\zeta_1) = \breve{S}(\zeta_2)$ it follows that,

(i) if $\breve{A}(\zeta_1) \succ \breve{A}(\zeta_2),\ resulting\ in\ \zeta_1 \succ \zeta_2$
(ii) if $\breve{A}(\zeta_1) = \breve{A}(\zeta_2),\ resulting\ in\ \zeta_1 = \zeta_2$

Definition 2.4. [11]. Suppose s, t denote two arbitrary real values. The FTT related to these values are expressed as follows:

$\mathcal{F}(s,t) = log_m(1 + \frac{(m^s-1)(m^t-1)}{m-1})$.

$\mathcal{F}'(s,t) = 1 - log_m(1 + \frac{(m^{1-s}-1)(m^{1-t}-1)}{m-1})$.

where $m \neq 1$ and $(s,t) \in [0,1] \times [0,1]$.

Definition 2.5. [21]. Let $\zeta = (\mu_\zeta, \psi_\zeta, \phi_\zeta)$, $\zeta_1 = (\mu_{\zeta_1}, \psi_{\zeta_1}, \phi_{\zeta_1})$ and $\zeta_2 = (\mu_{\zeta_2}, \psi_{\zeta_2}, \phi_{\zeta_2})$ represent three $\breve{q}$-RPFNs (with $\breve{\lambda} \succ 0$) adhere to the operating rules listed below:

$$\zeta_1 \cup \zeta_2 = (\{\mu_{\zeta_1} \cup \mu_{\zeta_2}\}, \{\psi_{\zeta_1} \cap \psi_{\zeta_2}\}, \{\phi_{\zeta_1} \cap \phi_{\zeta_2}\}). \tag{2.3}$$

$$\zeta_1 \cap \zeta_2 = (\{\mu_{\zeta_1} \cap \mu_{\zeta_2}\}, \{\psi_{\zeta_1} \cup \psi_{\zeta_2}\}, \{\phi_{\zeta_1} \cup \phi_{\zeta_2}\}). \tag{2.4}$$

$$\zeta^c = (\phi_\zeta, \psi_\zeta, \mu_\zeta). \tag{2.5}$$

$$\zeta_1 \oplus \zeta_2 = ((\mu_{\zeta_1}^{\breve{q}} + \mu_{\zeta_2}^{\breve{q}} - \mu_{\zeta_1}^{\breve{q}}\mu_{\zeta_2}^{\breve{q}})^{1/\breve{q}}, \psi_{\zeta_1}\psi_{\zeta_2}, \phi_{\zeta_1}\phi_{\zeta_2}). \tag{2.6}$$

$$\zeta_1 \otimes \zeta_2 = (\mu_{\zeta_1}\mu_{\zeta_2}, (\psi_{\zeta_1}^{\breve{q}} + \psi_{\zeta_2}^{\breve{q}} - \psi_{\zeta_1}^{\breve{q}}\psi_{\zeta_2}^{\breve{q}})^{1/\breve{q}}, (\phi_{\zeta_1}^{\breve{q}} + \phi_{\zeta_2}^{\breve{q}} - \phi_{\zeta_1}^{\breve{q}}\phi_{\zeta_2}^{\breve{q}})^{1/\breve{q}}). \tag{2.7}$$

$$\breve{\lambda}\zeta = ((1 - (1 - \mu_\zeta^{\breve{q}})^{\breve{\lambda}})^{1/\breve{q}}, \psi_\zeta^{\breve{\lambda}}, \phi_\zeta^{\breve{\lambda}}). \tag{2.8}$$

$$\zeta^{\breve{\lambda}} = (\mu_\zeta^{\breve{\lambda}}, (1 - (1 - \psi_\zeta^{\breve{q}})^{\breve{\lambda}})^{1/\breve{q}}, (1 - (1 - \phi_\zeta^{\breve{q}})^{\breve{\lambda}})^{1/\breve{q}}). \tag{2.9}$$

Definition 2.6. [24] If $\breve{x}_r(r = 1, 2, ..., t)$ represents the real numbers that are non-negative and $u, v \succ 0$, thus, the aggregating function,

$$BM^{u,v}(\breve{x}_1, \breve{x}_2, ..., \breve{x}_t) = \left(\frac{1}{t(t-1)} \sum_{\substack{k,l=1 \\ k \neq l}}^{t} (\breve{x}_k)^u (\breve{x}_l)^v \right)^{\frac{1}{(u+v)}}. \tag{2.10}$$

is known as the Bonferroni mean operator. This operator aggregates numbers by averaging their pairwise products, thus accounting for how each value interacts with every other value. This feature makes the BM particularly suitable for managing interdependencies among attributes, a key requirement in (MCDM) scenarios.

To aid readers in understanding the symbols and terminology used throughout this study, Table 1 provides a comprehensive list of all symbols and notations associated with the definitions, formulas, and aggregators discussed.

Table 1: List of Notations

$\breve{\mathcal{U}}$	Universal set
$\breve{O}_p$	$\breve{q}$-rung orthopair fuzzy set

ζ_ϱ	$\breve{q}$-rung picture fuzzy set
ζ_j	$\breve{q}$-rung picture fuzzy number
$\breve{q}$	An exponent defining the level of fuzziness
$\mu(e)$	Membership value of $e \in \breve{\mathcal{U}}$
$\phi(e)$	Non-Membership value of $e \in \breve{\mathcal{U}}$
$\psi(e)$	Degree of Neutral Membership value $e \in \breve{\mathcal{U}}$
$\mathcal{F}(s,t)$	Frank t-norm operator on the real numbers $s,t \in [0,1]$
$\mathcal{F}^{'}(s,t)$	Frank t-conorm operator on the real numbers $s,t \in [0,1]$
$\breve{\lambda}$	Positive real number
m	Frank parameter
$\mathcal{J}_a$	Set of Alternatives, where $a = 1,2,\ldots,p$
$\mathcal{H}_b$	Set of Attributes, where $b = 1,2,\ldots,q$
M_d	Set of Decision-makers where $d = 1,2,\ldots,r$
$\breve{\omega}_b$	Weight vectors set pertaining to $\mathcal{H}_b$
$\breve{\omega}_d$	Weight vectors set pertaining to M_d
$\breve{\omega}_t$	Weight vector corresponding to ζ_t
t_1	Class of beneficial Attributes
t_2	Class of cost-related Attributes
M^d	Decision matrix from the d^{th} decision maker M_d
δ^d_{ab}	Entries of the d^{th} decision matrix M^d
$(\delta^d_{ab})^c$	Complement of entries in the d^{th} decision matrix M^d
M^*	Combined Decision matrix
δ_{ab}	Entries of combined decision matrix M^*
δ_a	Combined preference value of an alternative J_a
p	Number of alternatives
q	Number of attributes and associated weight vectors
r	Number of decision-makers
$\mathcal{S}(\zeta)$	Score value of ζ
$\mathcal{A}(\zeta)$	Accuracy value of ζ
$\oplus$	Sum operator
$\otimes$	Product operator
ζ_r	Set of $\breve{q}$-RPFNs where $r = 1,2,\ldots,t$
ζ^t	Aggregation of 't' $\breve{q}$-RPFNs
ζ^-	Minimum of ζ
ζ^+	Maximum of ζ
u,v	Parameters of Bonferroni mean operator

3 $\breve{q}$-rung Picture Fuzzy BM Operators

Liu and Wang [24] extended BM operator to the $\breve{q}$-ROF numbers, thus proposed $\breve{q}$-ROF Archimedean BM operators. This research extends their approach by introducing $\breve{q}$-RPFWFBM aggregators integrating Frank norm. This novel approach aims to offer a more nuanced method for handling uncertainty in MAGDM scenarios.

Definition 3.1 *If $\zeta_r = (\mu_r, \psi_r, \phi_r)$ $(r = 1,2,...,t)$ represents an ensemble of $\breve{q}$-RPF numbers, with $u, v \succ 0$, and the $\breve{q}$-RPF Frank averaging Bonferroni mean ($\breve{q}$-RPFFABM) aggregator is* *specified by $\check{q}$-RPFFABM : $\zeta^t \to \zeta$ then:*

$$\check{q}-RPFFABM^{u,v}(\zeta_1, \zeta_2, ..., \zeta_t) = \left(\frac{1}{t(t-1)} \sum_{\substack{k,l=1 \\ k \neq l}}^{t} ((\zeta_k)^u \otimes (\zeta_l)^v) \right)^{\frac{1}{(u+v)}} \tag{3.1}$$

is known as the $\check{q}$-RPF Frank averaging Bonferroni Mean aggregator.

Theorem 3.1. *Assume that the set of $\check{q}$-RPFNs be $\zeta_r = (\mu_r, \psi_r, \phi_r)$ $(r = 1, 2, ..., t)$ and $u, v \succ 0$, then the aggregation operator obtained using the above equation (3.1) will again be a $\check{q}$-RPFN,*

$$\check{q}-RPFFABM^{u,v}(\zeta_1, \zeta_2, ..., \zeta_t) = \left(log_m \left(1 + \frac{(x^{'} - 1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}} \right) \right)^{1/\check{q}},$$
$$\left(1 - log_m \left(1 + \frac{(y^{'} - 1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}} \right) \right)^{1/\check{q}}, \left(1 - log_m \left(1 + \frac{(z^{'} - 1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}} \right) \right)^{1/\check{q}}. \tag{3.2}$$

where,

$$x^{'} = m^{\left(1 - log_m \left(1 + \left(\prod_{\substack{k,l=1 \\ k\neq l}}^{t} \left(m^{1-(log_m(1+\alpha^{'}_{kl}))} - 1\right)\right)^{\frac{1}{t(t-1)}}\right)\right)},$$

$$y^{'} = m^{1-\left(log_m \left(1 + \left(\prod_{\substack{k,l=1 \\ k\neq l}}^{t} \left(m^{(1-log_m(1+\beta^{'}_{kl}))} - 1\right)\right)^{\frac{1}{t(t-1)}}\right)\right)},$$

$$z^{'} = m^{1-\left(log_m \left(1 + \left(\prod_{\substack{k,l=1 \\ k\neq l}}^{t} \left(m^{(1-log_m(1+\gamma^{'}_{kl}))} - 1\right)\right)^{\frac{1}{t(t-1)}}\right)\right)},$$

$$\alpha^{'}_{kl} = \left(\frac{m^{\mu_k^{\check{q}}} - 1}{m-1} \right)^u \left(\frac{m^{\mu_l^{\check{q}}} - 1}{m-1} \right)^v,$$

$$\beta^{'}_{kl} = \left(\frac{m^{1-\psi_k^{\check{q}}} - 1}{m-1} \right)^u \left(\frac{m^{1-\psi_l^{\check{q}}} - 1}{m-1} \right)^v \quad and$$

$$\gamma^{'}_{kl} = \left(\frac{m^{1-\phi_k^{\check{q}}} - 1}{m-1} \right)^u \left(\frac{m^{1-\phi_l^{\check{q}}} - 1}{m-1} \right)^v$$

Theorem 3.2. ***(Idempotency)*** *Consider the set of $\check{q}$-RPFNs, $\zeta_r = (\mu_r, \psi_r, \phi_r)(r = 1, 2, ..., t)$, where $\zeta_r = r$, $\forall r$, then $\check{q} - RPFFABM(\zeta_1, \zeta_2, ..., \zeta_t) = \zeta$.*

Proof. Suppose, $\zeta_r = r$ $\forall r$, then

$$\check{q}-RPFFABM^{u,v}(\zeta_1, \zeta_2, ..., \zeta_t) = \left(\frac{1}{t(t-1)} \sum_{\substack{k,l=1 \\ k\neq l}}^{t} ((\zeta_k)^u \otimes (\zeta_l)^v) \right)^{\frac{1}{(u+v)}}$$
$$= \left(\frac{1}{t(t-1)} \sum_{\substack{k,l=1 \\ k\neq l}}^{t} ((\zeta)^{u+v}) \right)^{\frac{1}{(u+v)}}$$
$$= \left(\frac{1}{t(t-1)} (t(t-1)(\zeta)^{u+v}) \right)^{\frac{1}{(u+v)}} = ((\zeta)^{u+v})^{\frac{1}{(u+v)}} = \zeta.$$

□

Theorem 3.3. ***(Monotonicity)*** *Consider two sets of $\check{q}$-RPFNs, $\zeta_r = (\mu_{\zeta_r}, \psi_{\zeta_r}, \phi_{\zeta_r})$ $(r = 1, 2, ..., t)$ and $\zeta_r^* = (\mu_{\zeta_r^*}, \psi_{\zeta_r^*}, \phi_{\zeta_r^*})(r = 1, 2, ..., t)$ with $\check{q} \geq 1$. Suppose $\mu_{\zeta_r} \leq \mu_{\zeta_r^*}$, $\psi_{\zeta_r} \geq \psi_{\zeta_r^*}$ and $\phi_{\zeta_r} \geq \phi_{\zeta_r^*}$, $\forall r$, then:*
$\check{q} - RPFFABM(\zeta_1, \zeta_2, ..., \zeta_t) \leq \check{q} - RPFFABM(\zeta_1^*, \zeta_2^*, ..., \zeta_t^*)$.

Proof. Suppose, $\check{q} - RPFFABM(\zeta_1, \zeta_2, ..., \zeta_t) = (\mu_\zeta, \psi_\zeta, \phi_\zeta) = \zeta$ and
$\check{q} - RPFFABM(\zeta_1^*, \zeta_2^*, ..., \zeta_t^*) = (\mu_{\zeta^*}, \phi_{\zeta^*}, \phi_{\zeta^*}) = \zeta^*$
As $\mu_{\zeta_r} \leq \mu_{\zeta_r^*}$ for all r,
$\Rightarrow \mu_{\zeta_r}^{\check{q}} \leq \mu_{\zeta_r^*}^{\check{q}}$
$\Rightarrow m^{\mu_{\zeta_r}^{\check{q}}} - 1 \leq m^{\mu_{\zeta_r^*}^{\check{q}}} - 1$
$\Rightarrow \left(m^{\mu_{\zeta_r}^{\check{q}}} - 1\right)^u \leq \left(m^{\mu_{\zeta_r^*}^{\check{q}}} - 1\right)^u$
$\Rightarrow \frac{\left(m^{\mu_{\zeta_r}^{\check{q}}}-1\right)^u}{m-1} \leq \frac{\left(m^{\mu_{\zeta_r^*}^{\check{q}}}-1\right)^u}{m-1}$
$\Rightarrow \frac{\left(m^{\mu_{\zeta_r}^{\check{q}}}-1\right)^u}{m-1} \frac{\left(m^{\mu_{\zeta_s}^{\check{q}}}-1\right)^v}{m-1} \leq \frac{\left(m^{\mu_{\zeta_r^*}^{\check{q}}}-1\right)^u}{m-1} \frac{\left(m^{\mu_{\zeta_s^*}^{\check{q}}}-1\right)^v}{m-1}$
Let $x = \frac{\left(m^{\mu_{\zeta_r}^{\check{q}}}-1\right)^u}{m-1} \frac{\left(m^{\mu_{\zeta_s}^{\check{q}}}-1\right)^v}{m-1}$ and $x^* = \frac{\left(m^{\mu_{\zeta_r^*}^{\check{q}}}-1\right)^u}{m-1} \frac{\left(m^{\mu_{\zeta_s^*}^{\check{q}}}-1\right)^v}{m-1}$
Therefore, $(1 + x) \leq (1 + x^*)$
$log_m(1 + x) \leq log_m(1 + x^*)$
Subsequently, $1 - log_m(1 + x) \geq 1 - log_m(1 + x^*)$
It follows that, $(m^{1-log_m(1+x)} - 1) \geq (m^{1-log_m(1+x^*)} - 1)$
$\Rightarrow \left(\prod_{\substack{r,s=1 \\ r\neq s}}^t (m^{1-log_m(1+x)} - 1)\right) \geq \left(\prod_{\substack{r,s=1 \\ r\neq s}}^t (m^{1-log_m(1+x^*)} - 1)\right)$
$\left(\prod_{\substack{r,s=1 \\ r\neq s}}^t (m^{1-log_m(1+x)} - 1)\right)^{\frac{1}{t(t-1)}} \geq \left(\prod_{\substack{r,s=1 \\ r\neq s}}^t (m^{1-log_m(1+x^*)} - 1)\right)^{\frac{1}{t(t-1)}}$
$1 - log_m \left(\prod_{\substack{r,s=1 \\ r\neq s}}^t (m^{1-log_m(1+x)} - 1)\right)^{\frac{1}{t(t-1)}} \leq 1 - log_m \left(\prod_{\substack{r,s=1 \\ r\neq s}}^t (m^{1-log_m(1+x^*)} - 1)\right)^{\frac{1}{t(t-1)}}$
Let $a = 1 - log_m \left(\prod_{\substack{r,s=1 \\ r\neq s}}^t (m^{1-log_m(1+x)} - 1)\right)^{\frac{1}{t(t-1)}}$ and
$a^* = 1 - log_m \left(\prod_{\substack{r,s=1 \\ r\neq s}}^t (m^{1-log_m(1+x^*)} - 1)\right)^{\frac{1}{t(t-1)}}$
$\Rightarrow m^a - 1 \leq m^{a^*} - 1$
$\Rightarrow (m^a - 1)^{\frac{1}{u+v}} \leq (m^{a^*} - 1)^{\frac{1}{u+v}} \Rightarrow \frac{(m^a-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}} \leq \frac{(m^{a^*}-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}}$
$\Rightarrow 1 + \frac{(m^a-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}} \leq 1 + \frac{(m^{a^*}-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}}$
$\Rightarrow log_m \left(1 + \frac{(m^a-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}}\right) \leq log_m \left(1 + \frac{(m^{a^*}-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}}\right)$
$\Rightarrow log_m \left(1 + \frac{(m^a-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}}\right)^{\frac{1}{\check{q}}} \leq log_m \left(1 + \frac{(m^{a^*}-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}}\right)^{\frac{1}{\check{q}}}$
$\Rightarrow \mu_{\zeta_r} \leq \mu_{\zeta_r^*}$.
In a similar manner, also we can prove $\psi_{\zeta_r} \geq \psi_{\zeta_r^*}$ and $\phi_{\zeta_r} \geq \phi_{\zeta_r^*}, \forall r$.
$\Rightarrow \zeta_r \leq \zeta_r^*$, $\forall r \Rightarrow \zeta \leq \zeta^*$.
Moreover, from equation (2.1) we will have, $\check{S}(\zeta) \leq \check{S}(\zeta^*)$
$\Rightarrow \check{q} - RPFFABM(\zeta_1, \zeta_2, ..., \zeta_t) \leq \check{q} - RPFFABM(\zeta_1^*, \zeta_2^*, ..., \zeta_t^*)$.
Hence the proof. □

Theorem 3.4. ***(Boundedness)*** *Consider the set of $\check{q}$-RPFNs, $\zeta_r = (\mu_{\zeta_r}, \psi_{\zeta_r}, \phi_{\zeta_r})(r = 1, 2, ..., t)$ and suppose $\zeta^- = (\mu^-, \psi^-, \phi^-) = (min_r(\mu_r), max_r(\psi_r), max_r(\phi_r)), \zeta^+ = (\mu^+, \psi^+, \phi^+) = (max_r(\mu_r), min_r(\psi_r), min_r(\phi_r))$, for every r, then*
$\zeta^- \leq \check{q} - RPFFABM(\zeta_1, \zeta_2, ..., \zeta_t) \leq \zeta^+$.

Proof. Because $\zeta^{-} \leq \zeta_r \leq \zeta^{+}$, as per Theorem 3.2 and Theorem 3.3, we obtain,
$\zeta^{-} = \breve{q} - RPFFABM(\zeta^{-}, \zeta^{-}, ..., \zeta^{-}) \leq \breve{q} - RPFFBM(\zeta_1, \zeta_2, ..., \zeta_t)$ and
$\breve{q} - RPFFABM(\zeta_1, \zeta_2, ..., \zeta_t) \leq \breve{q} - RPFFBM(\zeta^{+}, \zeta^{+}, ..., \zeta^{+}) = \zeta^{+}$
$\Rightarrow \zeta^{-} \leq \breve{q} - RPFFABM(\zeta_1, \zeta_2, ..., \zeta_t) \leq \zeta^{+}$. □

3.1 $\breve{q}$-rung picture fuzzy weighted BM operators

The relationships between various characteristics(attributes) can be explored by the $\breve{q}$-RPFBM operator. Therefore, in this segment the $\breve{q}$-RPF weighted Frank averaging Bonferroni mean($\breve{q}$-RPFWFABM) and the $\breve{q}$-RPF weighted Frank geometric Bonferroni mean($\breve{q}$-RPFWFGBM) aggregators are developed in accordance with the Frank operating principles of $\breve{q}$-RPFNs. We then describe some of the characteristics of these new operators.

Definition 3.5 ($\breve{q}$-RPFWFABM Operator). Consider the set of $\breve{q}$-rung picture fuzzy numbers, $\zeta_r = (\mu_r, \psi_r, \phi_r)(r = 1, 2, ..., t)$, with $u, v \succ 0$ and
$\breve{q}$-RPFWFABM : $\zeta^t \rightarrow \zeta$ then:

$$\breve{q}{-}RPFWFABM^{u,v}(\zeta_1, \zeta_2, ..., \zeta_t) = \left(\frac{1}{t(t-1)} \bigoplus_{\substack{k,l=1 \\ k \neq l}}^{t} \left((t\breve{\omega}_k \zeta_k)^u \otimes (t\breve{\omega}_l \zeta_l)^v \right) \right)^{\frac{1}{(u+v)}}. \quad (3.3)$$

where $\breve{w}_t$ indicates the vector of weights associated with ζ_t so that the total sums to one.

The aggregation result presented as Theorem 3.6 can be obtained from equation (3.3), based upon the operating principles of the $\breve{q}$-RPFNs specified in the equations (2.6) - (2.9).

Theorem 3.6. *Assume that the set of $\breve{q}$-RPFNs be $\zeta_r = (\mu_r, \psi_r, \phi_r)(r = 1, 2, ..., t)$ and $u, v \succ 0$, then the aggregation operator obtained using the above equation (3.3) will again be a $\breve{q}$-RPFN,*

$$\breve{q}{-}RPFWFABM^{u,v}(\zeta_1, \zeta_2, ..., \zeta_t) = \left(log_m \left(1 + \frac{(m^x - 1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}} \right) \right)^{1/\breve{q}},$$

$$\left(1 - log_m \left(1 + \frac{(m^{1-y} - 1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}} \right) \right)^{1/\breve{q}}, \left(1 - log_m \left(1 + \frac{(m^{1-z} - 1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}} \right) \right)^{1/\breve{q}}. \quad (3.4)$$

where,

$$x = 1 - log_m \left(1 + \left(\prod_{\substack{k,l=1 \\ k \neq l}}^{t} \left(m^{1-(log_m(1+\alpha_{kl}))} - 1 \right) \right)^{\frac{1}{t(t-1)}} \right),$$

$$y = log_m \left(1 + \left(\prod_{\substack{k,l=1 \\ k \neq l}}^{t} \left(m^{1-log_m(1+\beta_{kl})} - 1 \right) \right)^{\frac{1}{t(t-1)}} \right),$$

$$z = log_m \left(1 + \left(\prod_{\substack{k,l=1 \\ k \neq l}}^{t} \left(m^{1-log_m(1+\gamma_{kl})} - 1 \right) \right)^{\frac{1}{t(t-1)}} \right),$$

$$\alpha_{kl} = \frac{\left[m^{\left(1 - log_m \left(1 + \frac{\left(m^{1-\mu_k^{\breve{q}}} - 1 \right)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k - 1}} \right) \right)} - 1 \right]^u \left[m^{\left(1 - log_m \left(1 + \frac{\left(m^{1-\mu_l^{\breve{q}}} - 1 \right)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l - 1}} \right) \right)} - 1 \right]^v}{(m-1)^{u+v-1}},$$

$$\beta_{kl}=\frac{\left[m^{1-\left(log_m\left(1+\frac{\left(m^{\psi_k^{\check{q}}}-1\right)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}}\right)\right)}-1\right]^u\left[m^{1-\left(log_m\left(1+\frac{\left(m^{\psi_l^{\check{q}}}-1\right)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}}\right)\right)}-1\right]^v}{(m-1)^{u+v-1}}$$

and

$$\gamma_{kl}=\frac{\left[m^{1-\left(log_m\left(1+\frac{\left(m^{\phi_k^{\check{q}}}-1\right)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}}\right)\right)}-1\right]^u\left[m^{1-\left(log_m\left(1+\frac{\left(m^{\phi_l^{\check{q}}}-1\right)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}}\right)\right)}-1\right]^v}{(m-1)^{u+v-1}}.$$

Proof. Because

$$t\breve{w}_k\zeta_k=\left\{\left(1-log_m(1+\frac{(m^{1-\mu_k^{\check{q}}}-1)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}})\right)^{1/\check{q}},\left(log_m(1+\frac{(m^{\psi_k^{\check{q}}}-1)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}})\right)^{1/\check{q}},\left(log_m(1+\frac{(m^{\phi_k^{\check{q}}}-1)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}})\right)^{1/\check{q}}\right\},$$

$$t\breve{w}_l\zeta_l=\left\{\left(1-log_m(1+\frac{(m^{1-\mu_l^{\check{q}}}-1)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}})\right)^{1/\check{q}},\left(log_m(1+\frac{(m^{\psi_l^{\check{q}}}-1)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}})\right)^{1/\check{q}},\left(log_m(1+\frac{(m^{\phi_l^{\check{q}}}-1)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}})\right)^{1/\check{q}}\right\}.$$

So,

$$(t\breve{w}_k\zeta_k)^u=\left(log_m\left(1+\frac{[m^{1-log_m\left(1+\frac{(m^{1-\mu_k^{\check{q}}}-1)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}}\right)}-1]^u}{(m-1)^{u-1}}\right)\right)^{1/\check{q}},$$

$$\left(1-log_m\left(1+\frac{[m^{1-log_m\left(1+\frac{(m^{\psi_k^{\check{q}}}-1)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}}\right)}-1]^u}{(m-1)^{u-1}}\right)\right)^{1/\check{q}},\left(1-log_m\left(1+\frac{[m^{1-log_m\left(1+\frac{(m^{\phi_k^{\check{q}}}-1)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}}\right)}-1]^u}{(m-1)^{u-1}}\right)\right)^{1/\check{q}}.$$

$$(t\breve{w}_l\zeta_l)^v=\left(log_m\left(1+\frac{[m^{1-log_m\left(1+\frac{(m^{1-\mu_l^{\check{q}}}-1)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}}\right)}-1]^v}{(m-1)^{v-1}}\right)\right)^{1/\check{q}},$$

$$\left(1-log_m\left(1+\frac{[m^{1-log_m\left(1+\frac{(m^{\psi_l^{\check{q}}}-1)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}}\right)}-1]^v}{(m-1)^{v-1}}\right)\right)^{1/\check{q}},\left(1-log_m\left(1+\frac{[m^{1-log_m\left(1+\frac{(m^{\phi_l^{\check{q}}}-1)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}}\right)}-1]^v}{(m-1)^{v-1}}\right)\right)^{1/\check{q}}.$$

Furthermore,

$$(t\breve{w}_k\zeta_k)^u\otimes(t\breve{w}_l\zeta_l)^v=$$

$$\left(log_m\left(1+\frac{[m^{1-log_m\left(1+\frac{(m^{1-\mu_k^{\check{q}}}-1)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}}\right)}-1]^u[m^{1-log_m\left(1+\frac{(m^{1-\mu_l^{\check{q}}}-1)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}}\right)}-1]^v}{(m-1)^{u+v-1}}\right)\right)^{1/\check{q}},$$

$$\left(1-log_m\left(1+\frac{[m^{1-log_m\left(1+\frac{(m^{\psi_k^{\check{q}}}-1)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}}\right)}-1]^u[m^{1-log_m\left(1+\frac{(m^{\psi_l^{\check{q}}}-1)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}}\right)}-1]^v}{(m-1)^{u+v-1}}\right)\right)^{1/\check{q}},$$

$$\left(1-log_m\left(1+\frac{[m^{1-log_m\left(1+\frac{(m^{\phi_k^{\check{q}}}-1)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}}\right)}-1]^u[m^{1-log_m\left(1+\frac{(m^{\phi_l^{\check{q}}}-1)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}}\right)}-1]^v}{(m-1)^{u+v-1}}\right)\right)^{1/\check{q}}.$$

Also,

$$\bigoplus_{\substack{k,l=1\\k\neq l}}^{t}\left((t\check{w}_k\zeta_k)^u\otimes(t\check{w}_l\zeta_l)^v\right)=$$

$$\left(1-log_m\left(1+\prod_{\substack{k,l=1\\k\neq l}}^{t}\left(m^{1-\left(log_m\left(1+\frac{\left[m^{1-log_m\left(1+\frac{(m^{1-\mu_k^{\check{q}}}-1)^{t\check{w}_k}}{(m-1)^{t\check{w}_k-1}}\right)}-1\right]^u\left[m^{1-log_m\left(1+\frac{(m^{1-\mu_l^{\check{q}}}-1)^{t\check{w}_l}}{(m-1)^{t\check{w}_l-1}}\right)}-1\right]^v}{(m-1)^{u+v-1}}\right)\right)}-1\right)\right)\right)^{1/\check{q}},$$

$$\left(log_m\left(1+\prod_{\substack{k,l=1\\k\neq l}}^{t}\left(m^{1-log_m\left(1+\frac{\left[m^{1-log_m\left(1+\frac{(m^{\psi_k^{\check{q}}}-1)^{t\check{w}_k}}{(m-1)^{t\check{w}_k-1}}\right)}-1\right]^u\left[m^{1-log_m\left(1+\frac{(m^{\psi_l^{\check{q}}}-1)^{t\check{w}_l}}{(m-1)^{t\check{w}_l-1}}\right)}-1\right]^v}{(m-1)^{u+v-1}}\right)}-1\right)\right)\right)^{1/\check{q}},$$

$$\left(log_m\left(1+\prod_{\substack{k,l=1\\k\neq l}}^{t}\left(m^{1-log_m\left(1+\frac{\left[m^{1-log_m\left(1+\frac{(m^{\phi_k^{\check{q}}}-1)^{t\check{w}_k}}{(m-1)^{t\check{w}_k-1}}\right)}-1\right]^u\left[m^{1-log_m\left(1+\frac{(m^{\phi_l^{\check{q}}}-1)^{t\check{w}_l}}{(m-1)^{t\check{w}_l-1}}\right)}-1\right]^v}{(m-1)^{u+v-1}}\right)}-1\right)\right)\right)^{1/\check{q}}.$$

Hence we obtain,

$$\frac{1}{t(t-1)}\bigoplus_{\substack{k,l=1\\k\neq l}}^{t}\left((t\check{w}_k\zeta_k)^u\otimes(t\check{w}_l\zeta_l)^v\right)=$$

$$\left(1-log_m\left(1+\left(\prod_{\substack{k,l=1\\k\neq l}}^{t}\left(m^{1-\left(log_m\left(1+\frac{\left[m^{1-log_m\left(1+\frac{(m^{1-\mu_k^{\check{q}}}-1)^{t\check{w}_k}}{(m-1)^{t\check{w}_k-1}}\right)}-1\right]^u\left[m^{1-log_m\left(1+\frac{(m^{1-\mu_l^{\check{q}}}-1)^{t\check{w}_l}}{(m-1)^{t\check{w}_l-1}}\right)}-1\right]^v}{(m-1)^{u+v-1}}\right)\right)}-1\right)\right)^{\frac{1}{t(t-1)}}\right)\right)^{1/\check{q}},$$

$$\left(log_m\left(1+\left(\prod_{\substack{k,l=1\\k\neq l}}^{t}\left(m^{1-log_m\left(1+\frac{\left[m^{1-log_m\left(1+\frac{(m^{\psi_k^{\check{q}}}-1)^{t\check{w}_k}}{(m-1)^{t\check{w}_k-1}}\right)}-1\right]^u\left[m^{1-log_m\left(1+\frac{(m^{\psi_l^{\check{q}}}-1)^{t\check{w}_l}}{(m-1)^{t\check{w}_l-1}}\right)}-1\right]^v}{(m-1)^{u+v-1}}\right)}-1\right)\right)^{\frac{1}{t(t-1)}}\right)\right)^{1/\check{q}},$$

$$\left(log_m\left(1+\left(\prod_{\substack{k,l=1\\k\neq l}}^{t}\left(m^{1-log_m\left(1+\frac{\left[m^{1-log_m\left(1+\frac{\left(m^{\phi_k^{\check{q}}}-1\right)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}}\right)}-1\right]^u\left[m^{1-log_m\left(1+\frac{\left(m^{\phi_l^{\check{q}}}-1\right)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}}\right)}-1\right]^v}{(m-1)^{u+v-1}}\right)}-1\right)\right)^{\frac{1}{t(t-1)}}\right)\right)^{1/\check{q}}.$$

Eventually,

$$\left(\frac{1}{t(t-1)}\bigoplus_{\substack{k,l=1\\k\neq l}}^{t}\left((t\breve{w}_k\zeta_k)^u\otimes(t\breve{w}_l\zeta_l)^v\right)\right)^{\frac{1}{u+v}} =$$

$$\left\{\left(log_m\left(1+\frac{(m^x-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}}\right)\right)^{1/\check{q}},\left(1-log_m\left(1+\frac{(m^{1-y}-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}}\right)\right)^{1/\check{q}},\left(1-log_m\left(1+\frac{(m^{1-z}-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}}\right)\right)^{1/\check{q}}\right\}.$$

$$where,\ x=1-log_m\left(1+\left(\prod_{\substack{k,l=1\\k\neq l}}^{t}\left(m^{1-(log_m(1+\alpha_{kl}))}-1\right)\right)^{\frac{1}{t(t-1)}}\right),$$

$$y=log_m\left(1+\left(\prod_{\substack{k,l=1\\k\neq l}}^{t}\left(m^{1-log_m(1+\beta_{kl})}-1\right)\right)^{\frac{1}{t(t-1)}}\right),$$

$$z=log_m\left(1+\left(\prod_{\substack{k,l=1\\k\neq l}}^{t}\left(m^{1-log_m(1+\gamma_{kl})}-1\right)\right)^{\frac{1}{t(t-1)}}\right),$$

$$\alpha_{kl}=\frac{\left[m^{\left(1-log_m\left(1+\frac{\left(m^{1-\mu_k^{\check{q}}}-1\right)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}}\right)\right)}-1\right]^u\left[m^{\left(1-log_m\left(1+\frac{\left(m^{1-\mu_l^{\check{q}}}-1\right)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}}\right)\right)}-1\right]^v}{(m-1)^{u+v-1}},$$

$$\beta_{kl}=\frac{\left[m^{1-\left(log_m\left(1+\frac{\left(m^{\psi_k^{\check{q}}}-1\right)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}}\right)\right)}-1\right]^u\left[m^{1-\left(log_m\left(1+\frac{\left(m^{\psi_l^{\check{q}}}-1\right)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}}\right)\right)}-1\right]^v}{(m-1)^{u+v-1}}$$

and

$$\gamma_{kl}=\frac{\left[m^{1-\left(log_m\left(1+\frac{\left(m^{\phi_k^{\check{q}}}-1\right)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k-1}}\right)\right)}-1\right]^u\left[m^{1-\left(log_m\left(1+\frac{\left(m^{\phi_l^{\check{q}}}-1\right)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l-1}}\right)\right)}-1\right]^v}{(m-1)^{u+v-1}}.$$

Thus, the $\check{q}$-RPFWFABM Operator has been obtained in this proof.

□

Theorem 3.7. ***(Monotonicity)*** *Consider two sets of $\check{q}$-RPFNs, $\zeta_r = (\mu_{\zeta_r}, \psi_{\zeta_r}, \phi_{\zeta_r})$ $(r = 1,2,...,t)$ and $\zeta_r^* = (\mu_{\zeta_r^*}, \psi_{\zeta_r^*}, \phi_{\zeta_r^*})(r = 1,2,...,t)$ with $\check{q} \geq 1$. Suppose $\mu_{\zeta_r} \leq \mu_{\zeta_r^*}$, $\psi_{\zeta_r} \geq \psi_{\zeta_r^*}$ and $\phi_{\zeta_r} \geq \phi_{\zeta_r^*}$, $\forall r$, then*
$\check{q} - RPFWFABM(\zeta_1, \zeta_2, ..., \zeta_t) \leq \check{q} - RPFWFABM(\zeta_1^*, \zeta_2^*, ..., \zeta_t^*)$.

Proof. This theorem's proof is of a similar nature to the proof given in the Theorem 3.3. □

Theorem 3.8. ***(Boundedness)*** *Consider the set of $\check{q}$-RPFNs, $\zeta_r = (\mu_{\zeta_r}, \psi_{\zeta_r}, \phi_{\zeta_r})$ $(r = 1,2,...,t)$ and suppose $\zeta^- = (\mu^-, \psi^-, \phi^-) = (min_r(\mu_r), max_r(\psi_r), max_r(\phi_r))$, $\zeta^+ = (\mu^+, \psi^+, \phi^+) = (max_r(\mu_r), min_r(\psi_r), min_r(\phi_r))$, for every r, then*
$\zeta^- \leq \check{q} - RPFWFABM(\zeta_1, \zeta_2, ..., \zeta_t) \leq \zeta^+$.

Proof. This theorem's proof is of a similar nature to the proof given in the Theorem 3.4. □

Definition 3.9 ($\check{q}$-RPFWFGBM Operator). Consider the set of $\check{q}$-RPFNs, $\zeta_r = (\mu_r, \psi_r, \phi_r)$ $(r = 1,2,...,t)$, with $u, v \succ 0$, and $\check{q}$-RPFWFGBM : $\zeta^t \to \zeta$ if

$$\check{q}{-}RPFWFGBM^{u,v}(\zeta_1, \zeta_2, ..., \zeta_t) = \frac{1}{(u+v)}\left(\bigotimes_{\substack{k,l=1\\k\neq l}}^{t}\left(u(\zeta_k)^{t\check{\omega}_k} \oplus v(\zeta_l)^{t\check{\omega}_l}\right)^{\frac{1}{t(t-1)}}\right) \tag{3.5}$$

Theorem 3.10. *Assume that the set of $\check{q}$-RPFNs be $\zeta_r = (\mu_r, \psi_r, \phi_r)(r = 1,2,...,t)$ and $u, v \succ 0$, then the aggregation operator obtained using the above equation (3.5) will again be a $\check{q}$-RPFN,*

$$\check{q}{-}RPFWFGBM^{u,v}(\zeta_1, \zeta_2, ..., \zeta_t) = \left(1 - log_m\left(1 + \frac{(m^{1-x}-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}}\right)\right)^{1/\check{q}},$$
$$\left(log_m\left(1 + \frac{(m^{y}-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}}\right)\right)^{1/\check{q}}, \left(log_m\left(1 + \frac{(m^{z}-1)^{\frac{1}{u+v}}}{(m-1)^{\frac{1}{u+v}-1}}\right)\right)^{1/\check{q}}. \tag{3.6}$$

where,

$$x = log_m\left(1 + \left(\prod_{\substack{k,l=1\\k\neq l}}^{t}\left(m^{1-log_m(1+\alpha_{kl})} - 1\right)\right)^{\frac{1}{t(t-1)}}\right),$$

$$y = 1 - log_m\left(1 + \left(\prod_{\substack{k,l=1\\k\neq l}}^{t}\left(m^{1-(log_m(1+\beta_{kl}))} - 1\right)\right)^{\frac{1}{t(t-1)}}\right),$$

$$z = 1 - log_m\left(1 + \left(\prod_{\substack{k,l=1\\k\neq l}}^{t}\left(m^{1-(log_m(1+\gamma_{kl}))} - 1\right)\right)^{\frac{1}{t(t-1)}}\right),$$

$$\alpha_{kl} = \frac{\left[m^{1-\left(log_m\left(1+\frac{\left(m^{\mu_k^{\check{q}}}-1\right)^{t\check{w}_k}}{(m-1)^{t\check{w}_k-1}}\right)\right)} - 1\right]^{u}\left[m^{1-\left(log_m\left(1+\frac{\left(m^{\mu_l^{\check{q}}}-1\right)^{t\check{w}_l}}{(m-1)^{t\check{w}_l-1}}\right)\right)} - 1\right]^{v}}{(m-1)^{u+v-1}},$$

$$\beta_{kl} = \frac{\left[m^{\left(1 - log_m \left(1 + \frac{\left(m^{1-\psi_k^{\check{q}}} - 1\right)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k - 1}}\right)\right)} - 1 \right]^u \left[m^{\left(1 - log_m \left(1 + \frac{\left(m^{1-\psi_l^{\check{q}}} - 1\right)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l - 1}}\right)\right)} - 1 \right]^v}{(m-1)^{u+v-1}}$$

$$\gamma_{kl} = \frac{\left[m^{\left(1 - log_m \left(1 + \frac{\left(m^{1-\phi_k^{\check{q}}} - 1\right)^{t\breve{w}_k}}{(m-1)^{t\breve{w}_k - 1}}\right)\right)} - 1 \right]^u \left[m^{\left(1 - log_m \left(1 + \frac{\left(m^{1-\phi_l^{\check{q}}} - 1\right)^{t\breve{w}_l}}{(m-1)^{t\breve{w}_l - 1}}\right)\right)} - 1 \right]^v}{(m-1)^{u+v-1}}.$$

Proof. This theorem's proof is of a similar nature to the proof of Theorem 3.6. □

Figure 1 depicts the graphical representation for the proposed framework.

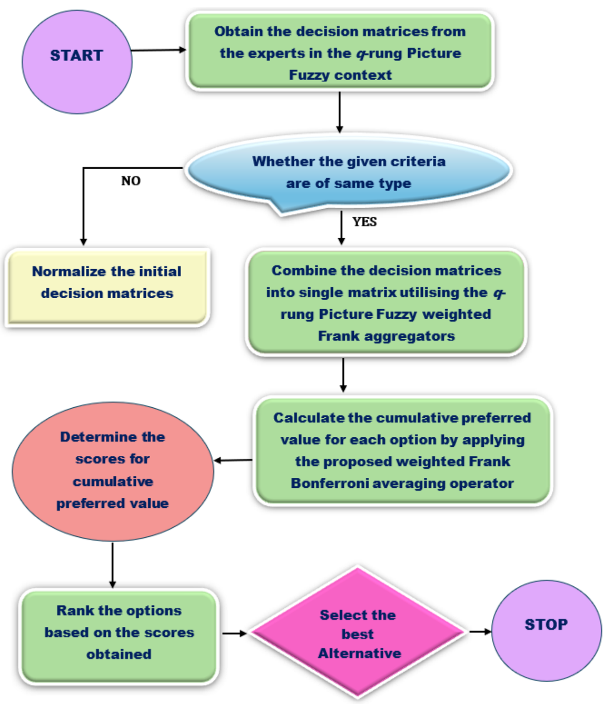

Figure 1. Workflow for the Proposed Decision-Making Framework

4 Framework for Group Decision-Making using Proposed Operator

In a MAGDM issue, let $\{\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_p\}$ indicates a finite collection of 'p' options, $\{\mathcal{H}_1, H_2, ..., \mathcal{H}_q\}$ are any finite collection of 'q' factors, and $\{M_1, M_2, ..., M_r\}$ represents the finite number of 'r' decision-makers. Then, the decision-maker M_d $(d = 1, 2, ..., r)$ is expected to use a $\check{q}$-RPFN to describe their assessment value, which may be represented as δ^d_{ab}, for each alternative $\mathcal{J}_a$ $(a = 1, 2, .., p)$ on attribute $\mathcal{H}_b$ $(b = 1, 2, ..., q)$. Thus, the obtained information could be expressed as decision matrices denoted as $M^d = (\delta^d_{ab})_{p\times q} = (\mu^d_{ab}, \psi^d_{ab}, \phi^d_{ab})$. The factors and experts are assigned weights as $\{\breve{w}_1, \breve{w}_2, ..., \breve{w}_q\}$ and $\{\breve{w}_1, \breve{w}_2, ..., \breve{w}_r\}$ respectively, ensuring that their total sums to one, where $\breve{w}_b \in [0,1]$ for $(b = 1, 2, ..., q)$, $\breve{w}_d \in [0,1]$ for $(d = 1, 2, ..., r)$. The key steps in the MAGDM utilizing the suggested aggregators are outlined below:

Step 1: Factors may be categorized into two primary types: “beneficial"(t_1) type factor and the “cost-type"(t_2). In the provided scenario, if the factors vary, the initial decision matrices may be standardized (normalized) using:

$$\delta^d_{ab} = \begin{cases} \delta^d_{ab} & \text{if } b \in t_1 \\ (\delta^d_{ab})^c & \text{if } b \in t_2 \end{cases} \tag{4.1}$$

where $(\delta^d_{ab})^c = (\phi^d_{ab}, \psi^d_{ab}, \mu^d_{ab})$.

Step 2: Combine the assessment values from each decision-maker M_d for each option to get a combined decision matrix M^* by employing the $\breve{q}$-RPF Frank weighted aggregators as specified in the equations (4.2) and (4.3).

$$\breve{q}\text{-}RPFFWA(\zeta_1, \zeta_2, ..., \zeta_t) = \bigoplus_{r=1}^{t} \breve{w}_r \zeta_r = \{(1 - log_m(1 + \prod_{r=1}^{t}(m^{1-\mu^{\breve{q}}_{\zeta_r}} - 1)^{\breve{w}_r}))^{1/\breve{q}},$$
$$(log_m(1 + \prod_{r=1}^{t}(m^{\psi^{\breve{q}}_{\zeta_r}} - 1)^{\breve{w}_r}))^{1/\breve{q}}, (log_m(1 + \prod_{r=1}^{t}(m^{\phi^{\breve{q}}_{\zeta_r}} - 1)^{\breve{w}_r}))^{1/\breve{q}}\}. \tag{4.2}$$

$$\breve{q}\text{-}RPFFWG(\zeta_1, \zeta_2, ..., \zeta_t) = \bigotimes_{r=1}^{t} (\zeta_r)^{\breve{w}_r} = \{(log_m(1 + \prod_{r=1}^{t}(m^{\mu^{\breve{q}}_{\zeta_r}} - 1)^{\breve{w}_r}))^{1/\breve{q}},$$
$$(1 - log_m(1 + \prod_{r=1}^{t}(m^{1-\psi^{\breve{q}}_{\zeta_r}} - 1)^{\breve{w}_r}))^{1/\breve{q}}, (1 - log_m(1 + \prod_{r=1}^{t}(m^{1-\phi^{\breve{q}}_{\zeta_r}} - 1)^{\breve{w}_r}))^{1/\breve{q}}\}. \tag{4.3}$$

Step 3: Calculate the cumulative preferred value δ_a for each option by applying the $\breve{q}$-RPFWFABM or $\breve{q}$-RPFWFGBM operator as specified in the equations (3.4) and (3.6) respectively.

Step 4: Determine the scores for the cumulative preferred value (δ_a) $(a = 1, 2, ..., p)$ of each option using equation (2.1).

Step 5: Sort the options by scores from highest to lowest to identify the top choice.

5 Numerical Instance

This section illustrates the decision-making process using a numerical example of a project plan evaluation, applying the proposed MAGDM technique.

[13] Consider there are a total of five project plans $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4, \mathcal{J}_5$ and that three experts M_1, M_2, M_3 are needed to assess the project plan's accomplishments in terms of the economic $(\mathcal{H}_1)$, social $(\mathcal{H}_2)$, sustainable $(\mathcal{H}_3)$, and ecological $(\mathcal{H}_4)$ beneficial impacts respectively. The attributes weights are (0.4, 0.2, 0.3, 0.1), while the weights of the experts are (0.35, 0.20, 0.45). Each expert is required to use $\breve{q}$-RPFNs to evaluate five project-plans from four different perspectives of factors resulting in the decision matrices $M^d = (\delta^d_{ab})_{5\times4}$, for d = 1, 2, 3 as shown below:

Matrix M^1 obtained from expertise M_1

$$M^1 = \begin{array}{c} \\ \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \\ \mathcal{J}_4 \\ \mathcal{J}_5 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 & \mathcal{H}_4 \end{array} \\ \begin{pmatrix} (0.5,0.4,0.1) & (0.8,0.1,0.1) & (0.4,0.3,0.2) & (0.1,0.8,0.1) \\ (0.7,0.1,0.1) & (0.1,0.7,0.2) & (0.1,0.7,0.2) & (0.7,0.1,0.1) \\ (0.8,0.1,0.1) & (0.1,0.8,0.1) & (0.1,0.8,0.1) & (0.6,0.2,0.1) \\ (0.7,0.1,0.1) & (0.7,0.2,0.1) & (0.1,0.7,0.1) & (0.1,0.8,0.1) \\ (0.7,0.2,0.1) & (0.6,0.2,0.1) & (0.8,0.1,0.1) & (0.1,0.7,0.1) \end{pmatrix} \end{array}$$

Matrix M^2 obtained from expertise M_2

$$M^2 = \begin{array}{c} \\ \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \\ \mathcal{J}_4 \\ \mathcal{J}_5 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 & \mathcal{H}_4 \end{array} \\ \begin{pmatrix} (0.5,0.3,0.1) & (0.8,0.1,0.1) & (0.5,0.1,0.3) & (0.1,0.8,0.1) \\ (0.6,0.1,0.2) & (0.2,0.5,0.2) & (0.2,0.6,0.1) & (0.6,0.2,0.1) \\ (0.8,0.1,0.1) & (0.1,0.7,0.1) & (0.2,0.6,0.1) & (0.6,0.1,0.2) \\ (0.8,0.1,0.1) & (0.7,0.1,0.2) & (0.2,0.6,0.1) & (0.5,0.3,0.1) \\ (0.6,0.1,0.1) & (0.8,0.1,0.1) & (0.6,0.3,0.1) & (0.1,0.8,0.1) \end{pmatrix} \end{array}$$

Matrix M^3 obtained from expertise M_3

$$M^3 = \begin{array}{c} \\ \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \\ \mathcal{J}_4 \\ \mathcal{J}_5 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 & \mathcal{H}_4 \end{array} \\ \begin{pmatrix} (0.7,0.2,0.1) & (0.3,0.5,0.1) & (0.8,0.1,0.1) & (0.5,0.3,0.1) \\ (0.1,0.7,0.1) & (0.5,0.2,0.2) & (0.8,0.1,0.1) & (0.7,0.1,0.2) \\ (0.5,0.2,0.2) & (0.3,0.5,0.1) & (0.1,0.7,0.1) & (0.6,0.2,0.1) \\ (0.7,0.2,0.1) & (0.2,0.7,0.1) & (0.5,0.2,0.3) & (0.3,0.5,0.1) \\ (0.5,0.2,0.1) & (0.6,0.2,0.1) & (0.7,0.1,0.1) & (0.2,0.7,0.1) \end{pmatrix} \end{array}$$

Step 1: Normalization is not required as the attributes belong to the same category.

Step 2: Next, combine the assessment values from the three decision-makers for each project proposal corresponding to every factor, using the $\breve{q}$-RPFFWA operator as specified in equation (4.2) with $\breve{q} = 3$ and $m = 2$. Thus, the combined decision matrix M^* is shown below:

$$\begin{array}{c} \\ \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \\ \mathcal{J}_4 \\ \mathcal{J}_5 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 & \mathcal{H}_4 \end{array} \\ \begin{pmatrix} (0.612,0.316,0.1) & (0.688,0.388,0.1) & (0.672,0.217,0.206) & (0.388,0.688,0.1) \\ (0.559,0.552,0.134) & (0.391,0.542,0.2) & (0.644,0.559,0.151) & (0.683,0.134,0.161) \\ (0.711,0.161,0.161) & (0.234,0.685,0.1) & (0.134,0.727,0.1) & (0.6,0.188,0.134) \\ (0.724,0.161,0.1) & (0.590,0.554,0.134) & (0.391,0.562,0.234) & (0.337,0.642,0.1) \\ (0.607,0.188,0.1) & (0.658,0.188,0.1) & (0.727,0.184,0.1) & (0.161,0.724,0.1) \end{pmatrix} \end{array}$$

Step 3: To determine the total preferred value (δ_a) of all project proposals, the $\breve{q}$-RPFWFABM operator given in equation (3.4) is employed. The parameter $\breve{q}$ can be chosen in such a way that $\mu^{\breve{q}} + \psi^{\breve{q}} + \phi^{\breve{q}} \leq 1$. Examining the first element in the above combined matrix, $\delta_{11} = (0.612, 0.316, 0.1)$, reveals the sum of three values exceeds one for $\breve{q} = 1$. But, when $\breve{q} \geq 2$, it fulfills the above condition. In this case, let $\breve{q} = 2$; $r = 3$; $u = v = 5$. Thus the total preferred values are,
$\delta_1 = (0.775, 0.060, 0.069)$; $\delta_2 = (0.724, 0.036, 0.063)$; $\delta_3 = (0.623, 0.050, 0.067)$; $\delta_4 = (0.720, 0.052, 0.069)$ and $\delta_5 = (0.795, 0.063, 0.069)$.

Step 4: Utilizing equation 2.1 the scores of the total preferred value (δ_a) $(a = 1, 2, 3, 4, 5)$ are calculated. The scores of the five project proposals are,
$\breve{S}(\delta_1) = 1.5952$; $\breve{S}(\delta_2) = 1.5202$; $\breve{S}(\delta_3) = 1.3830$; $\breve{S}(\delta_4) = 1.5133$; $\breve{S}(\delta_5) = 1.6279$.

Step 5: Arrange the projects in descending order according to the scores achieved in the prior step, i.e., $\mathcal{J}_5 \succ \mathcal{J}_1 \succ \mathcal{J}_2 \succ \mathcal{J}_4 \succ \mathcal{J}_3$. Therefore, the best project plan is obviously $\mathcal{J}_5$.

5.1 Impact of Parameters on Decision Outcomes

This segment analyzes how various parameter values, such as $\breve{q}$, m, u and v impact the decision outcomes for project proposals.

Table 2 displays the evaluation of the outcomes for the order of project proposals across various values of $\breve{q}$ (with $m = 2$ and $u = v = 1$) utilizing the $\breve{q}$-RPFWFABM aggregator. The ideal option is $\mathcal{J}_1$ only when $\breve{q} = 2$, while $\mathcal{J}_5$ is the best option for the other values of $\breve{q}$ ($\breve{q} = 5, 7, 10$). Figure 2 portrays the scores of the five projects against these $\breve{q}$ values.

Table 2. Ranking Outcomes for fixed $m = 2$, $u = v = 1$.

$\breve{q}$	$\breve{S}(\delta_1)$	$\breve{S}(\delta_2)$	$\breve{S}(\delta_3)$	$\breve{S}(\delta_4)$	$\breve{S}(\delta_5)$	**Ranking order**	**Best Project**
2	1.3758	1.3129	1.1793	1.2853	1.3711	$\mathcal{J}_1 > \mathcal{J}_5 > \mathcal{J}_2 > \mathcal{J}_4 > \mathcal{J}_3$	$\mathcal{J}_1$
5	1.1018	1.0675	1.0403	1.0630	1.1085	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_2 > \mathcal{J}_4 > \mathcal{J}_3$	$\mathcal{J}_5$
7	1.0435	1.0257	1.0168	1.0250	1.0484	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_2 > \mathcal{J}_4 > \mathcal{J}_3$	$\mathcal{J}_5$
10	1.0124	1.0064	1.0046	1.0067	1.0146	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$

Table 3 displays the outcomes for the order of project proposals across various values of $\breve{m}$ ($\breve{m} = 2, 3, 5, 7, 10$) with $\breve{q} = 3$ and $u = v = 2$, using the same operator. Figure 3 illustrates the scores of the five projects against the different m values.

Table 3. Ranking outcomes for fixed $\breve{q} = 3$, $u = v = 2$.

m	$\breve{S}(\delta_1)$	$\breve{S}(\delta_2)$	$\breve{S}(\delta_3)$	$\breve{S}(\delta_4)$	$\breve{S}(\delta_5)$	**Ranking order**	**Best Project**
2	1.2772	1.2097	1.1477	1.2123	1.2925	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$
3	1.2809	1.2126	1.1530	1.2194	1.2978	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$
5	1.2854	1.2162	1.1600	1.2286	1.3043	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$
7	1.2882	1.2186	1.1649	1.2348	1.3085	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$
10	1.2911	1.2211	1.1701	1.2413	1.3128	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$

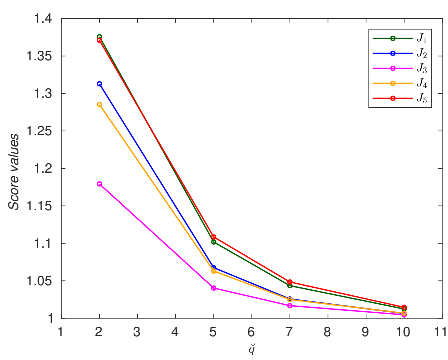

Figure 2. Scores of project plans for distinct $\breve{q}$.

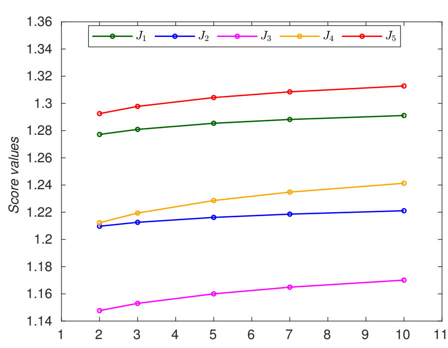

Figure 3. Scores of project plans for distinct m.

Table 4 displays the evaluation regarding the outcomes for the order of project proposals for varied values of u, v obtained using the $\breve{q}$-RPFWFABM operator by keeping $\breve{q}$ = 2 and $m = 2$. We take into account the combinations of different u and v values and the best option happens to be $\mathcal{J}_5$ in all these cases.

Table 4. Ranking outcomes for fixed $\breve{q} = 2$, $m = 2$.

u, v	$\breve{S}(\delta_1)$	$\breve{S}(\delta_2)$	$\breve{S}(\delta_3)$	$\breve{S}(\delta_4)$	$\breve{S}(\delta_5)$	Ranking order	Best Project
$u = 1, v = 2$	1.4054	1.3352	1.2395	1.3338	1.4125	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$
$u = 1, v = 3$	1.4269	1.3579	1.3080	1.3888	1.4403	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$
$u = 1, v = 4$	1.4420	1.3759	1.3633	1.4355	1.4605	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$
$u = 2, v = 2$	1.4202	1.3440	1.2316	1.3351	1.4332	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$
$u = 2, v = 3$	1.4341	1.3594	1.2647	1.3625	1.4507	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$
$u = 2, v = 4$	1.4452	1.3737	1.3072	1.3969	1.4651	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$
$u = 3, v = 4$	1.4512	1.3802	1.2827	1.3819	1.4721	$\mathcal{J}_5 > \mathcal{J}_1 > \mathcal{J}_4 > \mathcal{J}_2 > \mathcal{J}_3$	$\mathcal{J}_5$

5.2 Comparative Analysis

The comparison across multiple operators is presented in Table 5. A key finding is that, while the results of the suggested $\breve{q}$-RPFWFABM aggregator are closely aligned with available approaches, it demonstrates improved scores due to its ability to better capture attribute interdependencies. To further validate the model, parameter analysis is performed, ensuring the stability and reliability of the suggested operator under varying conditions.

To enhance the comparison, additional results and discussions have been incorporated into the experimental setup. This includes testing with varying parameter values ($\breve{q}$, m, u, v) to assess their impact on decision outcomes. For instance, increasing $\breve{q}$ or adjusting m influences the aggregation process, providing deeper insights into how the proposed model outperforms existing methods in specific scenarios. The expanded experimental setup allows for more in-depth discussions about the practical applicability and efficacy of the established model. These comparative findings illustrate the flexibility of the Frank Bonferroni operator, especially in handling attribute interdependencies. However, the study also highlights limitations, such as computational complexity when dealing with larger datasets, which opens up opportunities for future refinement.

Table 5. Proposed Frank Bonferroni operator versus existing operators.

Operators	$\breve{S}(\delta_1)$	$\breve{S}(\delta_2)$	$\breve{S}(\delta_3)$	$\breve{S}(\delta_4)$	$\breve{S}(\delta_5)$	Ranking order	Best Project
PFWA[37]	0.490	0.367	0.365	0.440	0.525	$\mathcal{J}_5 \succ \mathcal{J}_1 \succ \mathcal{J}_4 \succ \mathcal{J}_2 \succ \mathcal{J}_3$	$\mathcal{J}_5$
PFHWA[38]	0.739	0.669	0.668	0.710	0.758	$\mathcal{J}_5 \succ \mathcal{J}_1 \succ \mathcal{J}_4 \succ \mathcal{J}_2 \succ \mathcal{J}_3$	$\mathcal{J}_5$
PFDWA[16]	0.769	0.724	0.727	0.748	0.776	$\mathcal{J}_5 \succ \mathcal{J}_1 \succ \mathcal{J}_4 \succ \mathcal{J}_3 \succ \mathcal{J}_2$	$\mathcal{J}_5$
PFDWHM[47]	0.008	-0.008	-0.052	-0.035	0.039	$\mathcal{J}_5 \succ \mathcal{J}_1 \succ \mathcal{J}_4 \succ \mathcal{J}_3 \succ \mathcal{J}_2$	$\mathcal{J}_5$
SFWA[2]	0.760	0.721	0.702	0.740	0.785	$\mathcal{J}_5 \succ \mathcal{J}_1 \succ \mathcal{J}_4 \succ \mathcal{J}_2 \succ \mathcal{J}_3$	$\mathcal{J}_5$
q-RPFDWHM[13]	1.580	1.556	1.417	1.438	1.531	$\mathcal{J}_1 \succ \mathcal{J}_2 \succ \mathcal{J}_5 \succ \mathcal{J}_4 \succ \mathcal{J}_3$	$\mathcal{J}_1$
q-RPFDWDHM[13]	1.015	0.981	0.993	0.988	01.066	$\mathcal{J}_5 \succ \mathcal{J}_1 \succ \mathcal{J}_3 \succ \mathcal{J}_4 \succ \mathcal{J}_2$	$\mathcal{J}_5$
$\breve{q}$-RPFWFABM (Proposed)	1.5952	1.5202	1.3830	1.5133	1.6279	$\mathcal{J}_5 \succ \mathcal{J}_1 \succ \mathcal{J}_2 \succ \mathcal{J}_4 \succ \mathcal{J}_3$	$\mathcal{J}_5$

6 Conclusion

This study introduced novel $\breve{q}$-RPFWFBM operators by integrating the Bonferroni mean operator with $\breve{q}$-RPF Frank operational rules, enhancing the aggregation of $\breve{q}$-RPFNs. A distinctive contribution of this research lies in its application to MAGDM problems, specifically addressing the interdependencies between criteria, which often have a significant impact on decision outcomes. The proposed operators allow for more accurate and flexible modelling of such interdependencies, a key improvement over traditional methods. Through a detailed numerical example, the viability and robustness of the proposed operator were demonstrated. The results showed its effectiveness in evaluating project plans, outperforming other fuzzy aggregation methods in terms of accuracy and adaptability. Sensitivity analysis further confirmed the operator's robustness, reinforcing its practical value in real-time decision-making contexts.

Limitations

- The parameter's flexibility presents both an advantage and a potential limitation, since improper selection could lead to suboptimal aggregation.
- This increases computational demands, particularly for decision-making problems involving larger datasets or more diverse criteria.
- The Bonferroni mean is particularly suited for situations where attributes are interdependent. However, in scenarios where attributes are more independent, the use of Bonferroni operators may lead to biased results.

Managing Limitations

In the present study, efforts were made to mitigate these limitations through sensitivity analysis to observe the impact of parameter changes, as well as by comparing the proposed operators with existing methods to ensure reliability. However, further enhancements, especially in terms of reducing computational complexity and automating expert input, are suggested for future research.

Future Research Directions

Several potential avenues for future research arise from this study:

- Optimization of computational efficiency: Future research could focus on developing algorithms to reduce the computational complexity of the $\breve{q}$-RPFWFBM operators, making them more scalable for larger datasets.
- Extension to other decision-making contexts: The application of these operators could be expanded to other types of decision-making frameworks.
- Hybrid approaches: Exploring the integration of $\breve{q}$-RPFWFBM operators with machine learning techniques or other optimization methods may lead to even more effective decision-making models.
- Real-world case studies: Future studies could apply the proposed operators to diverse real-world case studies across various industries to validate their practical utility and adaptability.

By addressing the limitations and pursuing the outlined research directions, the suggested model demonstrates promise across a broader spectrum of applications and enhancements, ultimately contributing to the development of more robust and versatile MAGDM frameworks.

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