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Unilateral problem for some non-coercive Neumann elliptic equation in anisotropic weighted Sobolev spaces

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Abstract In this paper, we establish some definitions and properties in anisotropic weighted Sobolev spaces. As an application, we study the existence and regularity of entropy solutions for the unilateral problem associated to the strongly nonlinear and non-coercive Neumann problem

$$\begin{cases} Au + g(x,u,\nabla u) + |u|^{p_0-2}u\,\omega_0 = f & \text{in } \Omega,\\ \displaystyle\sum_{i=1}^{N} a_i(x,u_n,\nabla u_n).n_i = 0 & \text{on } \partial\Omega, \end{cases}$$

in anisotropic weighted Sobolev space $W_0^{1,\vec{p}}(\Omega,\vec{\omega})$, where A is a degenerated Leray-Lions operator, and the Carathéodory function $g(x,s,\xi)$ verifying only some growth conditions, with the data $f(x)$ is assumed to be in $L^1(\Omega)$.

1 Introduction

Let Ω be a bounded open subset of $I\!R^N$, $(N \geq 2)$, with Lipschitz boundary $\partial\Omega$. Boccardo and Gallouët have considered in [15] the elliptic Dirichlet problem

$$\begin{cases} -\text{div}\, a(x,u,\nabla u) = f & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f is a bounded radon measure, they have proved the existence and regularity of solutions, for more details we refer the reader to [11, 33].

Betta et al. have studied in [12] the nonlinear elliptic Neumann problem of the form

$$\begin{cases} -\triangle_p u - \text{div}\,(c(x)|u|^{p-2}u) = f & \text{in } \Omega,\\ (|\nabla u|^{p-2}\nabla u + c(x)|u|^{p-2}u).\vec{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where f belongs to $L^1(\Omega)$. The authors have demonstrated the existence and regularity of renormalized solutions i n the Sobolev space $W^{1,p}(\Omega)$. For more details, we refer the reader to [7, 17].

In [2], Akdim et al. studied the unilateral problem associated with the elliptic equation of the form

$$\begin{cases} Au - \text{div}\,\phi(u) = f & \text{in} \quad \Omega,\\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $\phi = (\phi_1, \cdots, \phi_N)$ belongs to $C^0(I\!R, I\!R)^N$ and the data f assumed to be in $L^1(\Omega)$. They have proved the existence of entropy solutions for the unilateral anisotropic problem (see also [18, 22]).

In [4], Akdim et al. have considered the unilateral problems associated to quasilinear degenerated elliptic equation of the form

$$\begin{cases} Au + g(x,u,\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

where $f \in L^1(\Omega)$. They have established the existence of solutions to the elliptic equation (1.3) in the framework of a weighted Sobolev space $W^{1,p}(\Omega,\vec{\omega})$, we refer the reader also to [1, 10]. For more information on nonlinear elliptic problems with degenerate coercivity, we refer the reader to [5, 14, 24, 29, 30, 31]. Also, the references [13, 19, 37] and [38] are recommended.

Recently, Azroul et al. have proved in [9] the existence of entropy solutions for the anisotropic quasilinear elliptic equation of the form

$$\begin{cases} Au + |u|^{s-1}u = f + \lambda \dfrac{|u|^{p_0-2}u}{|x|^{p_0}} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

where, the data f is assumed to be in $L^1(\Omega)$. The operator A is a Leray-Lions operator acted from $W_0^{1,\vec{p}}(\Omega,\sigma)$ into its dual, and λ is a positive constant. We refer also to [16], [21] and [10] for more details.

The focus of this paper is to establish some definitions and properties concerning the anisotropic weighted Sobolev spaces. Moreover, we will study the existence of entropy solutions for the unilateral problem associated to the strongly nonlinear and non-coercive elliptic equation given by

$$\begin{cases} Au + g(x,u,\nabla u) + |u|^{p_0-2}u\,\omega_0 = f & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,u_n,\nabla u_n).n_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded open subset of $I\!R^N$ $(N \geq 2)$, and $p_1,\cdots,p_N$ are N real constants numbers with $1 < p_i < \infty$ for $i = 1,\cdots,N$. The Leray-Lions operator A satisfies the growth, the degenerate coercivity and the strict monotone conditions. The strongly nonlinear term $g(x,s,\xi)$ is a Carathéodory function that satisfying only some growth condition, where the right-hand side $f(x)$ belongs to $L^1(\Omega)$.

The new result of this paper concern the study of existence and regularity of entropy solutions for the unilateral problem associated to the strongly nonlinear elliptic equation with Neumann bounded condition in the anisotropic weighted Sobolev spaces $W^{1,\vec{p}}(\Omega,\vec{\omega})$ (for case of p is a constant, we refer the readers to [28] and [35] for more details).

This paper is organized as follows : We introduce in the section 2 some definitions and properties concerning the weighted anisotropic Sobolev spaces. In the section 3, we present the essential assumptions under which our problem has at least one solution. The section 4 is devoted to proved the existence of weak solutions in the case of $f(x)$ belongs to $L^\infty(\Omega)$. In the last section, we study the existence of entropy solutions for the unilateral problem associated to the strongly nonlinear and non-coercive elliptic equation (1.5), where the data $f(x)$ belonging to $L^1(\Omega)$.

2 Preliminaries

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$.
Let $p_1,\ldots,p_N$ be N exponents, such that $1 < p_i < \infty$ for $i = 1,\ldots,N$, we define

$$\vec{p} = (p_1,\ldots,p_N) \qquad \text{and} \qquad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1,\ldots,N,$$

and we set

$$\underline{p} = \min\{p_1,p_2,\ldots,p_N\} \qquad \text{and} \qquad p_m = \max\{p_1,p_2,\ldots,p_N\}.$$

Let us define $\vec{\omega}(x) = (\omega_0(x), \omega_1(x), \ldots, \omega_N(x))$ as a vector of weight functions, such that $\omega_0(x)$ and $\omega_i(x)$ are positive and measurable functions $a.e.$ in Ω, with $\omega_0 \leq \omega_i$ for $i = 1, \ldots, N$. Also, we assume that

$$\omega_0 \in L^1_{loc}(\Omega) \qquad \text{with} \qquad \omega_0^* = \omega_0^{-\frac{1}{\underline{p}-1}} \in L^1_{loc}(\Omega), \tag{2.1}$$

and

$$\omega_i \in L^1_{loc}(\Omega) \qquad \text{with} \qquad \omega_i^* = \omega_i^{-\frac{1}{p_i-1}} \in L^1_{loc}(\Omega) \qquad \text{for} \quad i = 1, \ldots, N. \tag{2.2}$$

The weighted Lebesgue space $L^{p_i}(\Omega, \omega_i)$ is defined as

$$L^{p_i}(\Omega, \omega_i) = \left\{ u \text{ measurable function} \quad \text{such that} \quad \int_\Omega |u(x)|^{p_i} \omega_i(x)\, dx < \infty \right\},$$

equipped with the norm

$$\|u\|_{p_i,\omega_i} = \left(\int_\Omega |u(x)|^{p_i} \omega_i(x) dx \right)^{\frac{1}{p_i}} \qquad \text{for} \quad i = 1, \ldots, N. \tag{2.3}$$

The space $(L^{p_i}(\Omega, \omega_i), \| \cdot \|_{p_i,\omega_i})$ is a separable and reflexive Banach space (cf [28]).
Moreover, we define the Hölder's type inequality by

$$\int_\Omega uv\, dx \leq \|u\|_{p_i,\omega_i} \|v\|_{p_i',\omega_i^*}, \tag{2.4}$$

and the Young's inequality is definedd by

$$\int_\Omega uv\, dx \leq \frac{1}{p_i} \int_\Omega |u|^{p_i} \omega_i\, dx + \frac{1}{p_i'} \int_\Omega |v|^{p_i'} \omega_i^*\, dx \tag{2.5}$$

for any $u \in L^{p_i}(\Omega, \omega_i)$ and $v \in L^{p_i'}(\Omega, \omega_i^*)$, with $\dfrac{1}{p_i} + \dfrac{1}{p_i'} = 1$.
The anisotropic weighted Sobolev space $W^{1,\vec{p}}(\Omega, \vec{\omega})$ is defined by

$$W^{1,\vec{p}}(\Omega, \vec{\omega}) = \left\{ u \in L^{\underline{p}}(\Omega, \omega_0) \text{ and } D^i u \in L^{p_i}(\Omega, \omega_i) \quad \text{for} \quad i = 1, \ldots, N \right\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,\vec{p},\vec{\omega}} = \|u\|_{\underline{p},\omega_0} + \sum_{i=1}^N \|D^i u\|_{p_i,\omega_i}. \tag{2.6}$$

Thus, under the assumptions (2.1) and (2.2), $C_0^\infty(\Omega)$ is a subspace of $W^{1,\vec{p}}(\Omega, \vec{\omega})$.

Lemma 2.1. *Assuming that $\omega_0 \leq \omega_i$, and (2.1) – (2.2) hold true. Thus, the following continuous and compact embedding are concluded*

(i) *The embedding $L^{p_i}(\Omega, \omega_i) \hookrightarrow L^{\underline{p}}(\Omega, \omega_0)$ is continuous.*

(ii) *The embedding $W^{1,\vec{p}}(\Omega, \vec{\omega}) \hookrightarrow W^{1,\underline{p}}(\Omega, \omega_0)$ is continuous.*

(iii) *The embedding $L^{p_i}(\Omega, \omega_i) \hookrightarrow L^1(\Omega)$ is continuous for $i = 1, \ldots, N$.*

(iv) *The embedding $W^{1,\vec{p}}(\Omega, \vec{\omega}) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega, \omega_0)$ is compact.*

(v) *The embedding $W^{1,\vec{p}}(\Omega, \vec{\omega}) \hookrightarrow W^{1,1}(\Omega)$ is continuous.*

(vi) *The embedding $W^{1,\vec{p}}(\Omega, \vec{\omega}) \hookrightarrow\hookrightarrow L^q(\Omega)$ is compact for any $1 \leq q < \dfrac{N}{N-1}$.*

Proof of the lemma 2.1

Let $(u_n)_n$ be a sequences of measurable functions in $L^{p_i}(\Omega,\omega_i)$ such that $u_n \to u$ strongly in $L^{p_i}(\Omega,\omega_i)$. We have $\omega_0 \le \omega_i$, then

$$\begin{aligned}
\|u_n-u\|_{\underline{p},\omega_0} &= \Big(\int_\Omega |u_n-u|^{\underline{p}}\omega_0\,dx\Big)^{\frac{1}{\underline{p}}}\\
&\le \left(\Big(\int_\Omega |u_n-u|^{p_i}\omega_0\,dx\Big)^{\frac{\underline{p}}{p_i}}\Big(\int_\Omega \omega_0\,dx\Big)^{\frac{p_i-\underline{p}}{p_i}}\right)^{\frac{1}{\underline{p}}}\\
&\le \Big(\int_\Omega |u_n-u|^{p_i}\omega_i\,dx\Big)^{\frac{1}{p_i}}\Big(\int_\Omega \omega_0\,dx\Big)^{\frac{p_i-\underline{p}}{p_i\underline{p}}}\\
&= C_i\|u_n-u\|_{p_i,\omega_i}.
\end{aligned}$$

since $\omega_0 \in L^1_{loc}(\Omega)$ and Ω is a bounded open subset, then $\omega_0 \in L^1(\Omega)$. Thus, the proof of **(i)** is concluded.

For **(ii)**, let $(u_n)_n$ be a sequences of measurable function in $W^{1,\vec{p}}(\Omega,\vec{\omega})$ such that $u_n \to u$ strongly in $W^{1,\vec{p}}(\Omega,\vec{\omega})$, using (i), it's clear that

$$\begin{aligned}
\|u_n-u\|_{1,\underline{p},\omega_0} &= \|u_n-u\|_{\underline{p},\omega_0} + \sum_{i=1}^N \|D^iu_n - D^iu\|_{\underline{p},\omega_0}\\
&\le \|u_n-u\|_{\underline{p},\omega_0} + \sum_{i=1}^N C_i\|D^iu_n - D^iu\|_{p_i,\omega_i}\\
&\le C_0\|u_n-u\|_{1,\vec{p},\vec{\omega}},
\end{aligned}$$

with $C_0 = \max(1, C_i)$.

Concerning **(iii)**, in view of (2.1) – (2.2), we have

$$\begin{aligned}
\|u_n-u\|_1 &= \int_\Omega |u_n-u|\,dx = \int_\Omega |u_n-u|\omega_i^{\frac{1}{p_i}}\omega_i^{-\frac{1}{p_i}}\,dx\\
&\le \Big(\int_\Omega |u_n-u|^{p_i}\omega_i\,dx\Big)^{\frac{1}{p_i}}\Big(\int_\Omega \omega_i^{-\frac{p_i'}{p_i}}\,dx\Big)^{\frac{1}{p_i'}}\\
&= \|u_n-u\|_{p_i,\omega_i}\Big(\int_\Omega \omega_i^{1-p_i'}\,dx\Big)^{\frac{1}{p_i'}}\\
&= C_i^*\|u_n-u\|_{p_i,\omega_i}.
\end{aligned}$$

Similarly, we can show that

$$\|u_n-u\|_1 \le C_0^*\|u_n-u\|_{\underline{p},\omega_0}.$$

Now, thanks to **(ii)** we have $W^{1,\vec{p}}(\Omega,\vec{\omega}) \hookrightarrow W^{1,\underline{p}}(\Omega,\omega_0)$ is a continuous embedding, and the compact embedding $W^{1,\underline{p}}(\Omega,\omega_0) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega,\omega_0)$ (see. [23, 28]), thus **(iv)** is proved.

Finally, in view of (iii) we can show that the embedding $W^{1,\vec{p}}(\Omega,\vec{\omega}) \hookrightarrow W^{1,1}(\Omega)$ is continuous, and by the compact embedding $W^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for any $1 \le q < \dfrac{N}{N-1}$. We deduce the proof of **(v)** and **(vi)**.

Definition 2.2. Let $k > 0$, we consider the truncation function $T_k(\cdot) : I\!R \longmapsto I\!R$, giving by

$$T_k(s) = \begin{cases} s & \text{if } \ |s| \le k,\\ k\dfrac{s}{|s|} & \text{if } \ |s| > k,\end{cases}$$

and we define

$$\mathcal{T}^{1,\vec{p}}(\Omega,\vec{\omega}) := \big\{u : \Omega \longmapsto I\!R \text{ measurable, such that } T_k(u) \in W^{1,\vec{p}}(\Omega,\vec{\omega}) \quad \text{for any} \quad k > 0\big\}.$$

Proposition 2.3. *Let $u \in \mathcal{T}^{1,\vec{p}}(\Omega,\vec{\omega})$. For any $i \in \{1,\dots,N\}$, there exists a unique measurable function $v_i : \Omega \longmapsto I\!R$ such that*

$$\forall k > 0 \qquad D^i T_k(u) = v_i \cdot \chi_{\{|u|<k\}} \quad \text{a.e. in } \Omega,$$

where χ_E denotes the characteristic function of a measurable set E. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W^{1,1}(\Omega)$, then v_i coincide with the standard distributional of u, that's mean $v_i = D^i u$.

The proof of the Proposition 2.3 follows the usual techniques developed in [11] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [8, 10, 25, 26].
As in [7] for the case of constant exponent, we introduce the set $\mathcal{T}^{1,\vec{p}}_{tr}(\Omega,\vec{\omega})$ as a subset of $\mathcal{T}^{1,\vec{p}}(\Omega,\vec{\omega})$ that satisfying the following conditions: there exists a sequence $(u_n)_n$ in $W^{1,\vec{p}}(\Omega,\vec{\omega})$ such that

(a) $u_n \to u$ a.e. in Ω,

(b) $D^i T_k(u_n) \to D^i T_k(u)$ strongly in $L^1(\Omega)$ for all $k > 0$, and for any $i = 1, 2, \dots, N$,

(c) there exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. in $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [7]. The proof is based on the continuous embedding $W^{1,\vec{p}}(\Omega,\vec{\omega}) \hookrightarrow W^{1,1}(\Omega)$ and the same argument as in [7] for the case of classical Sobolev spaces. for more details, we refer the reader to [27, 32].
Let $u \in W^{1,\vec{p}}(\Omega,\vec{\omega})$, we denote $\tau(u)$ the trace of u on $\partial\Omega$. Also, for any $u \in \mathcal{T}^{1,\vec{p}}_{tr}(\Omega,\vec{\omega})$, we denote either $tr(u)$ or simply u as the trace of u on $\partial\Omega$. The operator $tr(\cdot)$ satisfying the following properties

(i) if $u \in \mathcal{T}^{1,\vec{p}}_{tr}(\Omega,\vec{\omega})$, then $\tau(T_k(u)) = T_k(tr(u))$ for any $k > 0$,

(ii) if $\varphi \in W^{1,\vec{p}}(\Omega,\vec{\omega})$, then for any $u \in \mathcal{T}^{1,\vec{p}}_{tr}(\Omega,\vec{\omega})$, we have $u-\varphi \in \mathcal{T}^{1,\vec{p}}_{tr}(\Omega,\vec{\omega})$ and $tr(u-\varphi) = tr(u) - \tau(\varphi)$.

Indeed, if $u \in W^{1,\vec{p}}(\Omega,\vec{\omega})$, we have $tr(u)$ coincides with $\tau(u)$, and it's clear that

$$W^{1,\vec{p}}(\Omega,\vec{\omega}) \subset \mathcal{T}^{1,\vec{p}}_{tr}(\Omega,\vec{\omega}) \subset \mathcal{T}^{1,\vec{p}}(\Omega,\vec{\omega}).$$

Lemma 2.4. *(Cf. [3]) Let $1 < p_i < \infty$, we consider the measurable function $g \in L^{p_i}(\Omega,\omega_i)$ and $(g_n)_n$ be a sequence in $L^{p_i}(\Omega,\omega_i)$ with $\|g_n\|_{p_i,\omega_i} \le C$.*
If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ weakly in $L^{p_i}(\Omega,\omega_i)$.

Lemma 2.5. *(See [34], Theorem 13.47) Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that*

(i) *$u_n \to u$ a.e. in Ω,*

(ii) *$u_n \ge 0$ and $u \ge 0$ a.e. in Ω,*

(iii) $\displaystyle\int_\Omega u_n\,dx \to \int_\Omega u\,dx,$

then $u_n \to u$ strongly in $L^1(\Omega)$.

3 Essential assumptions

Let us assuming that $p_0 > \underline{p}$, and let A be a Leray-Lions operator acted from $W^{1,\vec{p}}(\Omega,\vec{\omega})$ into its dual $(W^{1,\vec{p}}(\Omega,\vec{\omega}))'$, defined by the formula

$$Au = -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u),$$

where $a_i(x,s,\xi):\Omega\times I\!R\times I\!R^N\longmapsto I\!R^N$ are Carathéodory functions (measurable with respect to x in Ω for any (s,ξ) in $I\!R\times I\!R^N$, and continuous with respect to (s,ξ) in $I\!R\times I\!R^N$ for almost every x in Ω) that satisfy the following conditions:

$$|a_i(x,s,\xi)|\leq\beta\left(K_i(x)+|s|^{\frac{\underline{p}}{p_i'}}\omega_0^{\frac{1}{p_i'}}\omega_i^{\frac{1}{p_i}}+|\xi_i|^{p_i-1}\omega_i\right),\quad\text{for}\quad i=1,\dots,N, \tag{3.1}$$

for a.e. $x\in\Omega$ and all $(s,\xi)\in I\!R\times I\!R^N$, where $K_i(\cdot)$ are non-negative functions lying in $L^{p_i'}(\Omega,\omega_i^*)$ and $\beta>0$.

$$a_i(x,s,\xi)\xi_i\geq b(|s|)|\xi_i|^{p_i}\omega_i(x)\qquad\text{with}\quad b(|s|)\geq\frac{b_0}{(1+|s|)^\lambda}, \tag{3.2}$$

with b_0 is a positive constant and $0\leq\lambda<\min(\underline{p}-1,p_m-1)$ such that $b(|\cdot|):I\!R^+\mapsto I\!R^+$ is a decreasing function.

$$(a_i(x,s,\xi)-a_i(x,s,\xi'))(\xi_i-\xi_i')>0\qquad\text{for}\quad\xi_i\neq\xi_i'. \tag{3.3}$$

As a consequence of (3.2) and the continuity of the function $a_i(x,s,.)$ with respect to ξ, we have

$$a_i(x,s,0)=0.$$

The lower order term $g(x,s,\xi):\Omega\times I\!R\times I\!R^N\longmapsto I\!R^N$ is a Carathéodory function that satisfies the following growth condition:

$$|g(x,s,\xi)|\leq g_0(x)+d(|s|)\sum_{i=1}^N|\xi_i|^{p_i}\omega_i, \tag{3.4}$$

where $g_0(x)$ is assumed to be a positive measurable function in $L^1(\Omega)$, and $d(.):I\!R^+\mapsto I\!R^+$ is a continuous function such that $\dfrac{d(|\cdot|)}{b(|\cdot|)}$ belongs to $L^1(I\!R)\cap L^\infty(I\!R)$.
Finally, let ψ be a measurable function on Ω with values in $\overline{I\!R}$, such that

$$\psi^+\in W^{1,\vec{p}}(\Omega,\vec{\omega})\cap L^\infty(\Omega), \tag{3.5}$$

and we consider the closed convex set

$$K_\psi=\{u\in W^{1,\vec{p}}(\Omega,\vec{\omega}),\quad u\geq\psi\ \text{ a.e. in }\ \Omega\} \tag{3.6}$$

has a non-empty intersection with $L^\infty(\Omega)$, (since $\psi^+\in K_\psi\cap L^\infty(\Omega)$).

Next, let us present the following Lemma essential to proved our main result.

Lemma 3.1. *Let $k>0$, assuming that (2.1), (2.2), and (3.2) – (3.3) hold true, and let $(u_n)_n$ be a bounded sequence in $W^{1,\vec{p}}(\Omega,\vec{\omega})$, such that $u_n\rightharpoonup u$ weakly in $W^{1,\vec{p}}(\Omega,\vec{\omega})$ and*

$$\sum_{i=1}^N\int_\Omega(a_i(x,T_k(u_n),\nabla u_n)-a_i(x,T_k(u_n),\nabla u))(D^iu_n-D^iu)\,dx\longrightarrow0\quad as\quad n\to\infty, \tag{3.7}$$

then $u_n\longrightarrow u$ strongly in $W^{1,\vec{p}}(\Omega,\vec{\omega})$ for a subsequence.

The proof of this Lemma 3.1 is based essentially on the results of [18] and [20].

Proof of lemma 3.1

We set $D_n=\sum_{i=1}^N(a_i(x,T_k(u_n),\nabla u_n)-a_i(x,T_k(u_n),\nabla u))(D^iu_n-D^iu)$, thanks to (3.3) we have D_n is a positive function, and by (3.7), we deduce that $D_n\to0$ strongly in $L^1(\Omega)$ and a.e. in Ω as $n\to\infty$.
We have $u_n\rightharpoonup u$ weakly in $W^{1,\vec{p}}(\Omega,\vec{\omega})$, then $u_n\to u$ strongly in $L^1(\Omega)$ and a.e in Ω, and

since $D_n \to 0$ a.e in Ω, there exists a subset B in Ω with measure zero such that $\forall x \in \Omega \backslash B$ we have : $\omega_0(x) < \infty$ and $\omega_i(x) < \infty$, with

$$|u(x)| < \infty, \quad |u_n(x)| < \infty, \quad |D^i u(x)| < \infty, \quad K_i(x) < \infty, \quad u_n \to u \quad \text{and} \quad D_n \to 0.$$

We set $0 < \alpha = \min_{|s| \leq k} b(|s|)$. For a fixed $x \in \Omega \setminus B$, we define $\xi_n = \nabla u_n(x) \in I\!R^N$ and $\xi = \nabla u(x) \in I\!R^N$ such that $\xi_n = (\xi_{1,n}, \ldots, \xi_{N,n})$ and $\xi = (\xi_1, \ldots, \xi_N)$, then

$$\begin{aligned}
D_n(x) &= \sum_{i=1}^{N} (a_i(x, T_k(u_n), \nabla u_n) - a_i(x, T_k(u_n), \nabla u))(D^i u_n - D^i u) \\
&= \sum_{i=1}^{N} a_i(x, T_k(u_n), \xi_n)\xi_{i,n} + \sum_{i=1}^{N} a_i(x, T_k(u_n), \xi)\xi_i - \sum_{i=1}^{N} a_i(x, T_k(u_n), \xi_n)\xi_i \\
&\quad - \sum_{i=1}^{N} a_i(x, T_k(u_n), \xi)\xi_{i,n} \\
&\geq \alpha \sum_{i=1}^{N} |\xi_{i,n}|^{p_i}\omega_i + \alpha \sum_{i=1}^{N} |\xi_i|^{p_i}\omega_i - \beta \sum_{i=1}^{N} \left(K_i(x) + |T_k(u_n)|^{\frac{p}{p_i'}} \omega_0^{\frac{1}{p_i'}} \omega_i^{\frac{1}{p_i}} + |\xi_{i,n}|^{p_i - 1}\omega_i \right) |\xi_i| \\
&\quad - \beta \sum_{i=1}^{N} \left(K_i(x) + |T_k(u_n)|^{\frac{p}{p_i'}} \omega_0^{\frac{1}{p_i'}} \omega_i^{\frac{1}{p_i}} + |\xi_i|^{p_i - 1}\omega_i \right) |\xi_{i,n}| \\
&\geq \alpha \sum_{i=1}^{N} |\xi_{i,n}|^{p_i}\omega_i - C_x \left(1 + \sum_{i=1}^{N} |\xi_{i,n}|^{p_i - 1} + \sum_{i=1}^{N} |\xi_{i,n}| \right),
\end{aligned}$$

where C_x depends on x, but is independent of n. (Since $u_n(x) \to u(x)$, the sequence $(u_n)_n$ is bounded).Thus, we obtain

$$D_n(x) \geq \sum_{i=1}^{N} |\xi_{i,n}|^{p_i} \left(\alpha\omega_i - \frac{C_x}{|\xi_{i,n}|^{p_i}} - \frac{C_x}{|\xi_{i,n}|} - \frac{C_x}{|\xi_{i,n}|^{p_i - 1}} \right).$$

By the standard argument, we have $(\xi_{i,n})_n$ is bounded almost everywhere in Ω. (Indeed, if $|\xi_{i_0,n}| \to \infty$ in a measurable subset $E \in \Omega$ for $i = i_0$, then

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega D_n(x)\, dx &\geq \lim_{n\to\infty} \int_E (a_{i_0}(x, T_k(u_n), \nabla u_n) - a_{i_0}(x, T_k(u_n), \nabla u))(D^{i_0} u_n - D^{i_0} u)\, dx \\
&\geq \lim_{n\to\infty} \int_E |\xi_{i_0,n}|^{p_{i_0}} \left(\alpha\omega_{i_0} - \frac{C_x}{|\xi_{i_0,n}|^{p_{i_0}}} - \frac{C_x}{|\xi_{i_0,n}|} - \frac{C_x}{|\xi_{i_0,n}|^{p_{i_0} - 1}} \right) dx = \infty,
\end{aligned}$$

which is absurd since $D_n \to 0$ in $L^1(\Omega)$).
Let ξ_i^* be an accumulation point of the sequence $(\xi_{i,n})_n$, we have $|\xi_i^*| < \infty$ and using the continuity of the Carathéodory functions $a_i(x, \cdot, \cdot)$, we obtain

$$\Big(a_i(x, T_k(u(x)), \xi^*) - a_i(x, T_k(u(x)), \xi)\Big)(\xi_i^* - \xi_i) = 0 \qquad \text{for} \quad i = 1, \ldots, N.$$

thanks to (3.3) we have $\xi_i^* = \xi_i$. The uniqueness of the accumulation point implies that $D^i u_n \to D^i u$ almost everywhere in Ω for $i = 1, \ldots, N$.
Since $(a_i(x, T_k(u_n), \nabla u_n))_n$ is bounded in $L^{p_i'}(\Omega, \omega_i^*)$ and $a_i(x, T_k(u_n), \nabla u_n) \to a_i(x, u, \nabla u)$ a.e in Ω, by the Lemma 2.4, we conclude that

$$a_i(x, T_k(u_n), \nabla u_n) \rightharpoonup a_i(x, T_k(u), \nabla u) \qquad \text{weakly in} \quad L^{p_i'}(\Omega, \omega_i^*). \tag{3.8}$$

Now, taking $\bar{y}_{i,n} = a_i(x, u_n, \nabla u_n) D^i u_n$ and $\bar{y}_i = a_i(x, u, \nabla u) D^i u$, in view of (3.7) and (3.8), we obtain $\bar{y}_{i,n} \to \bar{y}_i$ strongly in $L^1(\Omega)$. On the other hand, according to (3.2), we have

$$\alpha |D^i u_n|^{p_i} \omega_i \;\leq\; a_i(x, T_k(u_n), \nabla u_n) D^i u_n.$$

Let $z_{i,n} = D^i u_n$, $z_i = D^i u$ and $y_{i,n} = \dfrac{\bar{y}_{i,n}}{\alpha}$, $y_i = \dfrac{\bar{y}_i}{\alpha}$, in view of Fatou's Lemma, we get

$$\int_\Omega 2y_i\, dx \leq \liminf_{n\to\infty} \int_\Omega \left(y_{i,n} + y_i - \frac{\alpha}{2^{p_i - 1}} |z_{i,n} - z_i|^{p_i} \omega_i \right) dx,$$

then $0 \leq -\limsup_{n\to\infty} \int_\Omega |z_{i,n} - z_i|^{p_i} \omega_i \, dx$, and since

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_{i,n} - z_i|^{p_i} \omega_i \, dx \leq \limsup_{n\to\infty} \int_\Omega |z_{i,n} - z_i|^{p_i} \omega_i \, dx \leq 0,$$

it follows that $\int_\Omega |D^i u_n - D^i u|^{p_i} \omega_i \, dx \longrightarrow 0$ as $n \to \infty$, i.e.

$$D^i u_n \longrightarrow D^i u \qquad \text{strongly in} \quad L^{p_i}(\Omega, \omega_i) \qquad \text{for} \quad i = 1, \ldots, N.$$

Moreover, by the compact embedding $W^{1,\vec{p}}(\Omega, \vec{\omega}) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega, \omega_0)$ we have $u_n \to u$ strongly in $L^{\underline{p}}(\Omega, \omega_0)$. Therefore, we conclude that

$$u_n \longrightarrow u \qquad \text{strongly in} \quad W^{1,\vec{p}}(\Omega, \vec{\omega}),$$

which conclude the proof of the Lemma 3.1.

4 Existence of weak solutions for L^∞-data

We consider the nonlinear and non-coercive elliptic problem

$$\begin{cases} -\sum_{i=1}^N D^i a_i(x, T_n(u), \nabla u) + g_n(x, u, \nabla u) + |u|^{p_0 - 2} u \, \omega_0 = F(x) & \text{in } \Omega, \\ \sum_{i=1}^N a_i(x, T_n(u), \nabla u).n_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

such that

$$g_n(x, s, \xi) = T_n(g(x, s, \xi)) \qquad \text{and} \qquad F(x) \in L^\infty(\Omega). \tag{4.2}$$

Definition 4.1. A measurable function u is called weak solution for the unilateral problem associated to the strongly nonlinear elliptic equation (4.1), if $u \in K_\psi$, $|u|^{p_0} \in L^1(\Omega)$, and u verifies the following equality

$$\begin{aligned} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u), \nabla u)(D^i u - D^i v) \, dx + \int_\Omega g_n(x, u, \nabla u)(u - v) \, dx \\ + \int_\Omega |u|^{p_0 - 2} u(u - v) \omega_0 \, dx \leq \int_\Omega F(x)(u - v) \, dx, \end{aligned} \tag{4.3}$$

for every $v \in K_\psi$.

Theorem 4.2. *Assuming that* (3.1) – (3.4) *and* (4.2) *hold true. Then, there exists at least one weak solution for the strongly nonlinear elliptic problem* (4.1).

Proof of theorem 4.2

Step 1: Approximate problems.

We consider the approximate problem :

$$\begin{cases} -\sum_{i=1}^N D^i a_i(x, T_n(u_m), \nabla u_m) + g_n(x, u_m, \nabla u_m) + |T_m(u_m)|^{p_0 - 2} T_m(u_m) \, \omega_0 \\ \qquad\qquad + \frac{1}{m} |u_m|^{\underline{p} - 2} u_m \, \omega_0 = F(x) & \text{in } \Omega, \\ \sum_{i=1}^N a_i(x, T_n(u_m), \nabla u_m).n_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

We define the operator B_m, acted from $W^{1,\vec{p}}(\Omega,\vec{\omega})$ into its dual $(W^{1,\vec{p}}(\Omega,\vec{\omega}))'$ and defined by

$$\begin{aligned}\langle B_m u, v\rangle &= \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u),\nabla u)D^i v\,dx + \int_\Omega g_n(x,u,\nabla u)v\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}-2}uv\omega_0\,dx \\ &+ \int_\Omega |T_m(u)|^{p_0-2}T_m(u)v\omega_0\,dx, \quad \text{for any} \quad u,v\in W^{1,\vec{p}}(\Omega,\vec{\omega}).\end{aligned}$$

Lemma 4.3. *The operator B_m acts from $W^{1,\vec{p}}(\Omega,\vec{\omega})$ into its dual $(W^{1,\vec{p}}(\Omega,\vec{\omega}))'$ is bounded and pseudo-monotone. Moreover, B_m is coercive in the following sense : there exists $v_0\in K_\psi$ such that*

$$\frac{\langle B_m v, v-v_0\rangle}{\|v\|_{1,\vec{p},\vec{\omega}}} \to +\infty \quad \textit{as} \quad \|v\|_{1,\vec{p},\vec{\omega}}\to+\infty \quad \textit{for} \quad v\in K_\psi.$$

Proof of Lemma 4.3

According to (3.1) and the Hölder's type inequality, we have : for any $u,v\in W^{1,\vec{p}}(\Omega,\vec{\omega})$

$$\begin{aligned}|\langle B_m u, v\rangle| &\le \sum_{i=1}^{N}\int_\Omega |a_i(x,T_n(u),\nabla u)|\,|D^i v|\,dx + \int_\Omega |g_n(x,u,\nabla u)|\,|v|\,dx \\ &+\frac{1}{m}\int_\Omega |u|^{\underline{p}-1}|v|\omega_0\,dx + \int_\Omega |T_m(u)|^{p_0-1}|v|\omega_0\,dx \\ &\le \beta\sum_{i=1}^{N}\int_\Omega \Big(K_i(x) + n^{\frac{\underline{p}}{p_i'}}\omega_0^{\frac{1}{p_i'}}\omega_i^{\frac{1}{p_i}} + |D^i u|^{p_i-1}\omega_i\Big)|D^i v|\,dx + n\int_\Omega |v|\,dx \\ &+\frac{1}{2m}\|u\|^{\underline{p}}_{\underline{p},\omega_0} + \frac{C_1}{m}\|v\|^{\underline{p}}_{\underline{p},\omega_0} + m^{p_0-1}\int_\Omega (|v|^{\underline{p}}+1)\omega_0\,dx \\ &\le \beta\sum_{i=1}^{N}\int_\Omega |K_i(x)|^{p_i'}\omega_i^{1-p_i'}\,dx + \beta\sum_{i=1}^{N}\int_\Omega n^{\underline{p}}\omega_0\,dx + \frac{b_0}{2(1+n)^\lambda}\sum_{i=1}^{N}\int_\Omega |D^i u|^{p_i}\omega_i\,dx \\ &+C_2\sum_{i=1}^{N}\int_\Omega |D^i v|^{p_i}\omega_i\,dx + C_3\|v\|_{\underline{p},\omega_0} + \frac{1}{2m}\|u\|^{\underline{p}}_{\underline{p},\omega_0} + \frac{C_1}{m}\|v\|^{\underline{p}}_{\underline{p},\omega_0} \\ &+m^{p_0-1}(\|v\|^{\underline{p}}_{\underline{p},\omega_0} + C_4) \\ &\le C_5 + \frac{b_0}{2(1+n)^\lambda}\sum_{i=1}^{N}\int_\Omega |D^i u|^{p_i}\omega_i\,dx + C_6\|v\|^{p_m}_{1,\vec{p},\vec{\omega}} + \frac{1}{2m}\|u\|^{\underline{p}}_{\underline{p},\omega_0},\end{aligned} \tag{4.5}$$

with C_5 and C_6 are positive constants depending only on n and m. Thus, the operator B_m is bounded in $W^{1,\vec{p}}(\Omega,\vec{\omega})$. For coercivity, we have

$$\begin{aligned}\langle B_m u, u\rangle &= \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u),\nabla u)D^i u\,dx + \int_\Omega g_n(x,u,\nabla u)u\,dx \\ &+\int_\Omega |T_m(u)|^{p_0-1}|u|\omega_0\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\omega_0\,dx \\ &\ge \sum_{i=1}^{N}\int_\Omega b(|T_n(u)|)|D^i u|^{p_i}\omega_i\,dx + \frac{1}{m}\|u\|^{\underline{p}}_{\underline{p},\omega_0} - n\int_\Omega |u|\,dx \\ &\ge \frac{b_0}{(1+n)^\lambda}\sum_{i=1}^{N}\|D^i u\|^{p_i}_{p_i,\omega_i} + \frac{1}{m}\|u\|^{\underline{p}}_{\underline{p},\omega_0} - C_7\|u\|_{1,\vec{p},\vec{\omega}} \\ &\ge C_8\|u\|^{\underline{p}}_{1,\vec{p},\vec{\omega}} + \frac{b_0}{2(1+n)^\lambda}\sum_{i=1}^{N}\|D^i u\|^{p_i}_{p_i,\omega_i} + \frac{1}{2m}\|u\|^{\underline{p}}_{\underline{p},\omega_0} - C_7\|u\|_{1,\vec{p},\vec{\omega}}.\end{aligned} \tag{4.6}$$

with $C_8=\min\{\frac{b_0}{2(1+n)^\lambda},\frac{1}{2m}\}$. Thus, by combining (4.5) and (4.6), we conclude that

$$\begin{aligned}\frac{\langle B_m u, u-u_0\rangle}{\|u\|_{1,\vec{p},\vec{\omega}}} &\ge \frac{\langle B_m u, u\rangle}{\|u\|_{1,\vec{p},\vec{\omega}}} - \frac{\langle B_m u, u_0\rangle}{\|u\|_{1,\vec{p},\vec{\omega}}} \\ &\ge \frac{C_8\|u\|^{\underline{p}}_{1,\vec{p},\vec{\omega}} - C_7\|u\|_{1,\vec{p},\vec{\omega}} - C_5 - C_6\|v\|^{p_m}_{1,\vec{p},\vec{\omega}}}{\|u\|_{1,\vec{p},\vec{\omega}}} \longrightarrow \infty \ \text{ as } \ \|u\|_{1,\vec{p},\vec{\omega}}\to+\infty.\end{aligned}$$

Now, we establish the pseudo-monotonicity of the operator $B_m u$.
Let $(u_k)_k$ be a sequence in $W^{1,\vec{p}}(\Omega,\vec{\omega})$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in} \quad W^{1,\vec{p}}(\Omega,\vec{\omega}), \\ B_m u_k \rightharpoonup \chi_m & \text{weakly in} \quad (W^{1,\vec{p}}(\Omega,\vec{\omega}))', \\ \limsup\limits_{k\to\infty} \langle B_m u_k, u_k\rangle \leq \langle \chi_m, u\rangle. \end{cases} \tag{4.7}$$

We show that

$$\chi_m = B_m u \quad \text{and} \quad \langle B_m u_k, u_k\rangle \longrightarrow \langle \chi_m, u\rangle \quad \text{as} \quad k\to+\infty.$$

We have $W^{1,\vec{p}}(\Omega,\vec{\omega}) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega,\omega_0)$, then $u_k \to u$ strongly in $L^{\underline{p}}(\Omega,\omega_0)$ for a subsequence still denoted $(u_k)_k$. The sequence $(u_k)_k$ is bounded in $W^{1,\vec{p}}(\Omega,\vec{\omega})$, and thanks to (3.1), the Carathéodory function $(a_i(x,T_n(u_k),\nabla u_k))_k$ is uniformly bounded in $L^{p_i'}(\Omega,\omega_i^*)$. Thus, there exists a measurable function $\varphi_i \in L^{p_i'}(\Omega,\omega_i^*)$ such that

$$a_i(x,T_n(u_k),\nabla u_k) \rightharpoonup \varphi_i \quad \text{weakly in} \quad L^{p_i'}(\Omega,\omega_i^*) \quad \text{as} \quad k\to+\infty. \tag{4.8}$$

Similarly, we have $(g_n(x,u_k,\nabla u_k))_k$ is uniformly bounded in $L^{\underline{p}'}(\Omega,\omega_0^*)$, then there exists a measurable function $\psi_n \in L^{\underline{p}'}(\Omega,\omega_0^*)$ such that

$$g_n(x,u_k,\nabla u_k) \rightharpoonup \psi_n \quad \text{weakly in} \quad L^{\underline{p}'}(\Omega,\omega_0^*) \quad \text{as} \quad k\to+\infty. \tag{4.9}$$

Moreover, we have $u_k \to u$ a.e. in Ω, in view of Lebesgue dominated convergence theorem, we conclude that

$$|T_m(u_k)|^{p_0-2}T_m(u_k)\omega_0 \longrightarrow |T_m(u)|^{p_0-2}T_m(u)\omega_0 \quad \text{strongly in} \quad L^{\underline{p}'}(\Omega,\omega_0^*), \tag{4.10}$$

and since $u_k \to u$ strongly in $L^{\underline{p}}(\Omega,\omega_0)$, then

$$\frac{1}{m}|u_k|^{\underline{p}-2}u_k\omega_0 \longrightarrow \frac{1}{m}|u|^{\underline{p}-2}u\omega_0 \quad \text{strongly in} \quad L^{\underline{p}'}(\Omega,\omega_0^*). \tag{4.11}$$

Then, for all $v \in W^{1,\vec{p}}(\Omega,\vec{\omega})$ we have

$$\begin{aligned} \langle \chi_m, v\rangle &= \lim_{k\to\infty} \langle B_m u_k, v\rangle \\ &= \lim_{k\to\infty} \sum_{i=1}^N \int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v\,dx + \lim_{k\to\infty}\int_\Omega g_n(x,u_k,\nabla u_k)v\,dx \\ &\quad + \lim_{k\to\infty}\int_\Omega |T_m(u_k)|^{p_0-2}T_m(u_k)v\omega_0\,dx + \frac{1}{m}\lim_{k\to\infty}\int_\Omega |u_k|^{\underline{p}-2}u_k v\omega_0\,dx \\ &= \sum_{i=1}^N\int_\Omega \varphi_i D^i v\,dx + \int_\Omega \psi_n v\,dx + \int_\Omega |T_m(u)|^{p_0-2}T_m(u)v\omega_0\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}-2}uv\omega_0\,dx. \end{aligned} \tag{4.12}$$

According to (4.7) and (4.12), we have

$$\begin{aligned} \limsup_{k\to\infty}\langle B_m u_k, u_k\rangle &= \limsup_{k\to\infty}\Bigg(\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx + \int_\Omega g_n(x,u_k,\nabla u_k)u_k\,dx \\ &\quad + \int_\Omega |T_m(u_k)|^{p_0-1}|u_k|\omega_0\,dx + \frac{1}{m}\int_\Omega |u_k|^{\underline{p}}\omega_0\,dx\Bigg) \\ &\leq \sum_{i=1}^N\int_\Omega \varphi_i D^i u\,dx + \int_\Omega \psi_n u\,dx + \int_\Omega |T_m(u)|^{p_0-1}|u|\omega_0\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\omega_0\,dx. \end{aligned} \tag{4.13}$$

In view of (4.9), and since $u_k \to u$ strongly in $L^{\underline{p}}(\Omega,\omega_0)$ it follows that

$$\int_\Omega g_n(x,u_k,\nabla u_k)u_k\,dx \longrightarrow \int_\Omega \psi_n u\,dx, \tag{4.14}$$

Moreover, thanks to (4.10) and (4.11) we obtain

$$\int_\Omega |T_m(u_k)|^{p_0-1}|u_k|\omega_0\,dx + \frac{1}{m}\int_\Omega |u_k|^{\underline{p}}\omega_0\,dx \longrightarrow \int_\Omega |T_m(u)|^{p_0-1}|u|\omega_0\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\omega_0\,dx. \tag{4.15}$$

It follows that

$$\limsup_{k\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx \le \sum_{i=1}^N\int_\Omega \varphi_i D^iu\,dx. \tag{4.16}$$

On the other hand, in view of (3.3), we have

$$\sum_{i=1}^N\int_\Omega \left(a_i(x,T_n(u_k),\nabla u_k) - a_i(x,T_n(u_k),\nabla u)\right)\left(D^iu_k - D^iu\right)\,dx \ge 0, \tag{4.17}$$

then

$$\begin{aligned}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx \quad &\ge \sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu\,dx\\ &\quad+\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u)\left(D^iu_k - D^iu\right)\,dx.\end{aligned}$$

In view of Lebesgue's dominated convergence theorem, we have $T_n(u_k)\to T_n(u)$ strongly in $L^{p_i}(\Omega,\omega_i)$, then $a_i(x,T_n(u_k),\nabla u)\to a_i(x,T_n(u),\nabla u)$ strongly in $L^{p_i'}(\Omega,\omega_i^*)$, and thanks to (4.8), we get

$$\liminf_{k\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx \ge \sum_{i=1}^N\int_\Omega \varphi_i D^iu\,dx.$$

Having in mind that (4.16), we conclude that

$$\lim_{k\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx = \sum_{i=1}^N\int_\Omega \varphi_i D^iu\,dx. \tag{4.18}$$

Thus, by combining (4.12), (4.14) – (4.15) and (4.18), we deduce that

$$\langle B_mu_k,u_k\rangle \longrightarrow \langle \chi_m,u\rangle \quad \text{as} \quad k\to+\infty. \tag{4.19}$$

Moreover, thanks to (4.18), we have

$$\lim_{k\to+\infty}\sum_{i=1}^N\int_\Omega \left(a_i(x,T_n(u_k),\nabla u_k) - a_i(x,T_n(u_k),\nabla u)\right)\left(D^iu_k - D^iu\right)\,dx = 0,$$

In view of (4.7) and Lemma 3.1, we conclude that

$$u_k \longrightarrow u \quad \text{strongly in} \quad W^{1,\vec{p}}(\Omega,\vec{\omega}) \qquad \text{and} \qquad D^iu_k \longrightarrow D^iu \quad \text{a.e. in} \quad \Omega,$$

therefore, $a_i(x,T_n(u_k),\nabla u_k)\to a_i(x,T_n(u),\nabla u)$ and $g_n(x,u_k,\nabla u_k)\to g_n(x,u,\nabla u)$ almost everywhere in Ω, thanks to (3.1) and lemma 2.4, we obtain

$$a_i(x,T_n(u_k),\nabla u_k) \rightharpoonup a_i(x,T_n(u),\nabla u) \quad \text{weakly in} \quad L^{p_i'}(\Omega,\omega_i^*) \quad \text{for} \quad i=1,\dots,N. \tag{4.20}$$

and

$$g_n(x,u_k,\nabla u_k) \rightharpoonup g_n(x,u,\nabla u) \quad \text{weakly in} \quad L^{\underline{p}'}(\Omega,\omega_0^*). \tag{4.21}$$

According to (4.10) – (4.11) and (4.20) – (4.21), we deduce that $\chi_m = B_mu$, which concludes the proof of the Lemma 4.3.

In view of Lemma 4.3 (see [36], Theorem 8.2), there existence of at least one weak solution $u_m \in K_\psi$ for the unilateral problem associated to the approximate problem (4.4), i.e.

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_m),\nabla u_m)(D^iu_m-D^iv)\,dx+\int_\Omega g_n(x,u_m,\nabla u_m)(u_m-v)\,dx\\
&\quad+\int_\Omega |T_m(u_m)|^{p_0-2}T_m(u_m)(u_m-v)\omega_0\,dx+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_m(u_m-v)\omega_0\,dx\\
&\le\int_\Omega F(x)(u_m-v)\,dx,
\end{aligned}\tag{4.22}$$

for any $v\in K_\psi$.

Step 2: A priori estimates.

By choosing $v=u_m-\eta T_m(u_m-\psi^+)$, we have $v\in W^{1,\vec{p}}(\Omega,\vec{\omega})$ and $v\ge\psi$ a.e. in Ω for $\eta>0$ small enough. Thus, $v\in K_\psi$ is an admissible test function for the approximate problem (4.22), and we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_m-\psi^+|\le m\}} a_i(x,T_n(u_m),\nabla u_m)(D^iu_m-D^i\psi^+)\,dx+\int_\Omega g_n(x,u_m,\nabla u_m)T_m(u_m-\psi^+)\,dx\\
&\quad+\int_\Omega |T_m(u_m)|^{p_0-2}T_m(u_m)T_m(u_m-\psi^+)\omega_0\,dx+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_mT_m(u_m-\psi^+)\omega_0\,dx\\
&\le\int_\Omega F(x)T_m(u_m-\psi^+)\,dx.
\end{aligned}\tag{4.23}$$

Since $T_m(u_m-\psi^+)$ have the same sign as u_m, and thanks to (3.2), (4.2) and the Young's inequality, we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_m-\psi^+|\le m\}} b(T_n(|u_m|))|D^iu_m|^{p_i}\omega_i\,dx+\int_\Omega |T_m(u_m)|^{p_0-1}|T_m(u_m-\psi^+)|\omega_0\,dx\\
&\quad+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}|T_m(u_m-\psi^+)|\omega_0\,dx\\
&\le\int_\Omega |F(x)|\,|T_m(u_m-\psi^+)|\,dx+\int_\Omega |g_n(x,u_m,\nabla u_m)|\,|T_m(u_m-\psi^+)|\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{|u_m-\psi^+|\le m\}} |a_i(x,T_n(u_m),\nabla u_m)|\,|D^i\psi^+|\,dx.
\end{aligned}\tag{4.24}$$

For the second term on the left-hand side of (4.24). We have $\{|u_m|\le m\}\subseteq\{|u_m-\psi^+|\le m\}$, and in view of Young's inequality, we obtain

$$\begin{aligned}
&\int_\Omega |T_m(u_m)|^{p_0-1}|T_m(u_m-\psi^+)|\omega_0\,dx\\
&\ge m\int_{\{|u_m-\psi^+|>m\}} |T_m(u_m)|^{p_0-1}\omega_0\,dx+\int_{\{|u_m-\psi^+|\le m\}} |T_m(u_m)|^{p_0-1}|u_m|\omega_0\,dx\\
&\quad-\int_{\{|u_m-\psi^+|\le m\}} |T_m(u_m)|^{p_0-1}|\psi^+|\omega_0\,dx\\
&\ge\int_\Omega |T_m(u_m)|^{p_0}\omega_0\,dx-\int_\Omega |T_m(u_m)|^{p_0-1}|\psi^+|\omega_0\,dx\\
&\ge\int_\Omega |T_m(u_m)|^{p_0}\omega_0\,dx-\frac{1}{2}\int_\Omega |T_m(u_m)|^{p_0}\omega_0\,dx-C_1\int_\Omega |\psi^+|^{p_0}\omega_0\,dx\\
&\ge\frac{1}{2}\int_\Omega |T_m(u_m)|^{p_0}\omega_0\,dx-C_2.
\end{aligned}\tag{4.25}$$

Concerning the first term on the right-hand side of (4.24), by using Young's inequality, we get

$$\begin{aligned}\int_\Omega |F|\,|T_m(u_m-\psi^+)|\,dx &\le \int_{\{|u_m-\psi^+|\le m\}} |F|\,|T_m(u_m)|\,dx + \int_{\{|u_m-\psi^+|\le m\}} |F|\,|\psi^+|\,dx \\ &\quad + m\int_{\{|u_m-\psi^+|>m\}} |F|\,dx \\ &\le \int_\Omega |F|\,|T_m(u_m)|\,dx + \|F\|_{L^\infty(\Omega)}\|\psi^+\|_{L^\infty(\Omega)}\,\text{meas}(\Omega) \\ &\le \frac{1}{4}\int_\Omega |T_m(u_m)|^{p_0}\omega_0\,dx + C_3\int_\Omega |F(x)|^{p_0'}\omega_0^*\,dx + C_4 \\ &\le \frac{1}{4}\int_\Omega |T_m(u_m)|^{p_0}\omega_0\,dx + C_5.\end{aligned} \tag{4.26}$$

Similarly, we show that

$$\int_\Omega |g_n(x,u_m,\nabla u_m)|\,|T_m(u_m-\psi^+)|\,dx \le \frac{1}{8}\int_\Omega |T_m(u_m)|^{p_0}\omega_0\,dx + C_6(n). \tag{4.27}$$

For the last term on the right-hand side of (4.24), thanks to (3.1), we obtain

$$\begin{aligned}&\sum_{i=1}^N \int_{\{|u_m-\psi^+|\le m\}} |a_i(x,T_n(u_m),\nabla u_m)|\,|D^i\psi^+|\,dx \\ &\le \frac{1}{2\beta}\sum_{i=1}^N \int_{\{|u_m-\psi^+|\le m\}} b(|T_n(|u_m|)|)\,|a_i(x,T_n(u_m),\nabla u_m)|^{p_i'}\omega_i^*\,dx \\ &\quad + C_7\sum_{i=1}^N \int_{\{|u_m-\psi^+|\le m\}} \frac{|D^i\psi^+|^{p_i}}{b(|T_n(|u_m|)|)^{\frac{1}{p_i'-1}}}\omega_i\,dx \\ &\le \frac{1}{2}\sum_{i=1}^N \int_{\{|u_m-\psi^+|\le m\}} b(|T_n(|u_m|)|)\left(|K_i(x)|^{p_i'}\omega_i^* + |n|^{\underline{p}}\omega_0 + |D^iu_m|^{p_i}\omega_i\right)\,dx + C_8 \\ &\le \frac{1}{2}\sum_{i=1}^N \int_{\{|u_m-\psi^+|\le m\}} b(|T_n(|u_m|)|)|D^iu_m|^{p_i}\omega_i\,dx + C_9(n).\end{aligned} \tag{4.28}$$

It follows that

$$\begin{aligned}&\frac{b_0}{2(1+n)^\lambda}\sum_{i=1}^N \int_{\{|u_m-\psi^+|\le m\}} |D^iu_m|^{p_i}\omega_i\,dx + \frac{1}{4}\int_\Omega |T_m(u_m)|^{p_0}\omega_0\,dx \\ &\quad + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}|T_m(u_m-\psi^+)|\omega_0\,dx \le C_{10}(n).\end{aligned} \tag{4.29}$$

with $C_{10}(n)$ is a positive constant that does not depend on m. Having in mind that $\{|u_m|\le m\}\subset\{|u_m-\psi^+|\le m\}$, we conclude that

$$\begin{aligned}\|T_m(u_m)\|_{1,\vec{p},\vec{\omega}} &= \int_\Omega |T_m(u_m)|^{\underline{p}}\omega_0\,dx + \sum_{i=1}^N \int_\Omega |D^iT_m(u_m)|^{p_i}\omega_i\,dx \\ &\le \int_\Omega |T_m(u_m)|^{p_0}\omega_0\,dx + \sum_{i=1}^N \int_{\{|u_m-\psi^+|\le m\}} |D^iu_m|^{p_i}\omega_i\,dx \\ &\le C_{11}.\end{aligned} \tag{4.30}$$

As a result, the sequence $(T_m(u_m))_m$ is uniformly bounded in $W^{1,\vec{p}}(\Omega,\vec{\omega})$, and there exists a subsequence still denoted $(T_m(u_m))_m$ such that

$$\begin{cases} T_m(u_m) \rightharpoonup u & \text{weakly in } W^{1,\vec{p}}(\Omega,\vec{\omega}), \\ T_m(u_m) \to u & \text{strongly in } L^{\underline{p}}(\Omega,\omega_0) \quad \text{and a.e in } \Omega. \end{cases} \tag{4.31}$$

It follows that

$$\frac{1}{m}|u_m|^{\underline{p}-1} \longrightarrow 0 \qquad \text{strongly in } L^{\underline{p}'}(\Omega,\omega_0^*). \tag{4.32}$$

Moreover, thanks to (4.29) we have $(T_m(u_m))_m$ is bounded in $L^{p_0}(\Omega, \omega_0)$, and since $T_m(u_m) \to u$ almost everywhere in Ω, we obtain

$$T_m(u_m) \rightharpoonup u \quad \text{weakly in} \quad L^{p_0}(\Omega, \omega_0). \tag{4.33}$$

Step 3 : The convergence almost everywhere of the gradient.

By taking $v = u \in K_\psi$ as a test function for the approximate problem (4.22), we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m) D^i(u_m - u)\, dx + \int_\Omega g_n(x, u_m, \nabla u_m)(u_m - u)\, dx \\ &\quad + \int_\Omega |T_m(u_m)|^{p_0 - 2} T_m(u_m)(u_m - u)\omega_0\, dx + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2} u_m (u_m - u)\omega_0\, dx \\ &\leq \int_\Omega F(u_m - u)\, dx. \end{aligned} \tag{4.34}$$

It implies that

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \Big(a_i(x, T_n(u_m), \nabla u_m) - a_i(x, T_n(u_m), \nabla u)\Big)(D^i u_m - D^i u)\, dx \\ &\quad + \int_\Omega \Big(|T_m(u_m)|^{p_0-2} T_m(u_m) - |T_m(u)|^{p_0-2} T_m(u)\Big)(u_m - u)\omega_0\, dx \\ &\leq \int_\Omega |F(x)|\, |u_m - u|\, dx + \int_\Omega |g_n(x, u_m, \nabla u_m)|\, |u_m - u|\, dx + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1} |u_m - u|\omega_0\, dx \\ &\quad + \int_\Omega |T_m(u)|^{p_0-1} |u_m - u|\omega_0\, dx + \sum_{i=1}^{N}\int_\Omega |a_i(x, T_n(u_m), \nabla u)|\, |D^i u_m - D^i u|\, dx. \end{aligned} \tag{4.35}$$

For the first and second term on the right-hand side of (4.35), we have $F \in L^\infty(\Omega)$ and $|g_n(x, u_m, \nabla u_m)| \leq n$, having in mind $u_m \to u$ strongly in $L^1(\Omega)$, we obtain

$$\int_\Omega |F(x)|\, |u_m - u|\, dx \longrightarrow 0 \quad \text{as} \quad m \to \infty, \tag{4.36}$$

and

$$\int_\Omega |g_n(x, u_m, \nabla u_m)|\, |u_m - u|\, dx \longrightarrow 0 \quad \text{as} \quad m \to \infty. \tag{4.37}$$

Concerning the third and fourth terms on the right-hand side of (4.35), using (4.31), (4.32) and (4.33) to get

$$\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1} |u_m - u|\omega_0\, dx \longrightarrow 0 \quad \text{as} \quad m \to \infty, \tag{4.38}$$

and

$$\int_\Omega |T_m(u)|^{p_0-1} |u_m - u|\omega_0\, dx \longrightarrow 0 \quad \text{as} \quad m \to \infty. \tag{4.39}$$

For the last term on the right-hand side of (4.35), we have $|a_i(x, T_n(u_m), \nabla u)| \to |a_i(x, T_n(u), \nabla u)|$ strongly in $L^{p_i'}(\Omega, \omega_i^*)$ and since $D^i u_m \rightharpoonup D^i u$ weakly in $L^{p_i}(\Omega, \omega_i)$, it follows that

$$\sum_{i=1}^{N}\int_\Omega |a_i(x, T_n(u_m), \nabla u)|\, |D^i u_m - D^i u|\, dx \longrightarrow 0 \quad \text{as} \quad m \to \infty. \tag{4.40}$$

By combining (4.35) and (4.36) – (4.40), we obtain

$$\sum_{i=1}^{N}\int_\Omega [a_i(x, T_n(u_m), \nabla u_m) - a_i(x, T_n(u_m), \nabla u)]\, (D^i u_m - D^i u)\, dx \longrightarrow 0 \ \text{as} \ m \to \infty. \tag{4.41}$$

By applying Lemma 3.1, we conclude that

$$\begin{cases} u_m \to u & \text{strongly in} \quad W^{1,\vec{p}}(\Omega,\vec{\omega}), \\ D^i u_m \to D^i u & \text{a.e in} \quad \Omega \quad \text{for } i = 1,\cdots,N. \end{cases} \tag{4.42}$$

Moreover, we have $a_i(x, T_n(u_m), \nabla u_m) \longrightarrow a_i(x, T_n(u), \nabla u)$ and $g_n(x, u_m, \nabla u_m) \longrightarrow g_n(x, u, \nabla u)$ almost everywhere in Ω, and in view of Lemma 2.4, we conclude that

$$a_i(x, T_n(u_m), \nabla u_m) \rightharpoonup a_i(x, T_n(u), \nabla u) \quad \text{weakly in} \quad L^{p_i'}(\Omega, \omega_i^*) \quad \text{for} \quad i = 1,\cdots,N, \tag{4.43}$$

and

$$g_n(x, u_m, \nabla u_m) \rightharpoonup g_n(x, u, \nabla u) \quad \text{weakly in} \quad L^{\underline{p}'}(\Omega, \omega_0^*). \tag{4.44}$$

Step 4: Passage to the limit.

By taking $v \in K_\psi$ as a test function for the approximate problem (4.4), we have

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_m), \nabla u_m)(D^i u_m - D^i v)\,dx + \int_\Omega g_n(x, u_m, \nabla u_m)(u_m - v)\,dx \\ &\quad + \int_\Omega |T_m(u_m)|^{p_0-2} T_m(u_m)(u_m - v)\omega_0\,dx + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2} u_m(u_m - v)\omega_0\,dx \\ &\leq \int_\Omega F(x)(u_m - v)\,dx. \end{aligned} \tag{4.45}$$

For the first term on the left-hand side of (4.34), we have $D^i u_m \to D^i u$ strongly in $L^{p_i}(\Omega, \omega_i)$, and thanks to (4.43), we obtain

$$\lim_{m\to\infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_m), \nabla u_m)(D^i u_m - D^i v)\,dx = \sum_{i=1}^N \int_\Omega a_i(x, T_n(u), \nabla u)(D^i u - D^i v)\,dx. \tag{4.46}$$

Similarly, we have $u_m \to u$ strongly in $L^{\underline{p}}(\Omega, \omega_0)$, and in view of (4.32) and (4.44), we get

$$\lim_{m\to\infty} \int_\Omega g_n(x, u_m, \nabla u_m)(u_m - v)\,dx = \int_\Omega g_n(x, u, \nabla u)(u - v)\,dx, \tag{4.47}$$

and

$$\lim_{m\to\infty} \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2} u_m(u_m - v)\omega_0\,dx = 0, \tag{4.48}$$

and

$$\lim_{m\to\infty} \int_\Omega F(x)(u_m - v)\,dx = \int_\Omega F(x)(u - v)\,dx. \tag{4.49}$$

Furthermore, in view of (4.33) and Fatou's lemma, we obtain

$$\begin{aligned} &\liminf_{m\to\infty} \int_\Omega |T_m(u_m)|^{p_0-2} T_m(u_m)(u_m - v)\omega_0\,dx \\ &= \liminf_{m\to\infty} \int_\Omega \Big(|T_m(u_m)|^{p_0-2} T_m(u_m) - |T_m(v)|^{p_0-2} T_m(v)\Big)(u_m - v)\omega_0\,dx \\ &\quad + \int_\Omega |v|^{p_0-2} v(u - v)\omega_0\,dx \\ &\geq \int_\Omega |u|^{p_0-2} u(u - v)\omega_0\,dx. \end{aligned} \tag{4.50}$$

By combining (4.45) and (4.46) – (4.50), we deduce that

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u), \nabla u)(D^i u - D^i v)\,dx + \int_\Omega g_n(x, u, \nabla u)(u - v)\,dx \\ &\quad + \int_\Omega |u|^{p_0-2} u(u - v)\omega_0\,dx \leq \int_\Omega F(x)(u - v)\,dx. \end{aligned} \tag{4.51}$$

Consequently, the proof of the theorem 4.2 is concluded.

5 Existence of entropy solutions for L^1-data

Let Ω be a bounded open subset of $I\!R$ $(N \geq 2)$. We consider the strongly nonlinear and non-coercive elliptic equation

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + g(x,u,\nabla u) + |u|^{p_0-2}u\,\omega_0 = f(x) & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x,u,\nabla u).n_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

with the data $f(\cdot)$ is assumed to be in $L^1(\Omega)$.

Definition 5.1. A measurable function u is an entropy solution for the unilateral problem associated to the strongly nonlinear and non-coercive elliptic equation (5.1), if $u \geq \psi$ a.e. in Ω and $T_k(u) \in \mathcal{T}^{1,\vec{p}}(\Omega,\vec{\omega})$ for any $k > 0$, with $g(x,u,\nabla u) \in L^1(\Omega)$, and u satisfies the following inequality

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D^i T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u) T_k(u-v)\,dx \\ + \int_\Omega |u|^{p_0-2} u T_k(u-v)\omega_0\,dx \leq \int_\Omega f(x) T_k(u-v)\,dx, \end{aligned} \tag{5.2}$$

for every $v \in K_\psi \cap L^\infty(\Omega)$.

The aim of this paper is to show the existence of entropy solution for the unilateral problem associated to our Neumann elliptic equation.

Theorem 5.2. *Assuming that* (3.1) $-$ (3.4) *holds true, then there exists at least one entropy solution u for the unilateral problem associated to the strongly nonlinear and non-coercive elliptic equation* (5.1) *in the anisotropic weighted Sobolev spaces.*

Proof of theorem 5.2

Step 1: Approximate problems.

Let $f_n(x) = T_n(f(x))$, we consider the following approximate equation

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,T_n(u_n),\nabla u_n) + g_n(x,u_n,\nabla u_n) + |u_n|^{p_0-2}u_n\,\omega_0 = f_n(x) & \text{in} \quad \Omega, \\ \sum_{i=1}^{N} a_i(x,T_n(u_n),\nabla u_n).n_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5.3}$$

According to the theorem 4.2, there exists at least one weak solution $u_n \in K_\psi$ for the unilateral problem associated to the approximate problem (5.3), i. e.

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x,T_n(u_n),\nabla u_n)(D^i u_n - D^i v)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v)\,dx \\ + \int_\Omega |u_n|^{p_0-2} u_n (u_n - v)\omega_0\,dx \leq \int_\Omega f_n(x)(u_n - v)\,dx, \end{aligned} \tag{5.4}$$

for any $v \in K_\psi$.

Step 2: Weak convergence of truncations.

Let $k \geq \max(1, \|\psi^+\|_\infty)$, and we set $M = k + \|\psi^+\|_\infty$ then $M \leq 2k$.
Taking $B(s) = 2\int_0^s \frac{d(|\tau|)}{b(|\tau|)}\,d\tau$, and having in mind that $\frac{d(|\cdot|)}{b(|\cdot|)} \in L^1(I\!R) \cap L^\infty(I\!R)$, we conclude

that $0 \leq B(s) \leq B(\infty) := \displaystyle\int_0^{\infty} \frac{2d(|\tau|)}{b(|\tau|)}\, d\tau < \infty$ is a finite real number.

Let $v = u_n - \eta T_k(u_n - \psi^+)e^{B(|u_n|)}$, we have $v \in W_0^{1,\vec{p}}(\Omega, \vec{\omega})$, and for $\eta > 0$ small enough we have $v \geq \psi$. Thus, v is an admissible test function for the approximate problem (5.4), and we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i \left(T_k(u_n - \psi^+)e^{B(|u_n|)}\right)\, dx \\
&\quad + \int_\Omega |u_n|^{p_0-2} u_n T_k(u_n - \psi^+) e^{B(|u_n|)} \omega_0\, dx + \int_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n - \psi^+) e^{B(|u_n|)}\, dx \\
&\leq \int_\Omega f_n T_k(u_n - \psi^+) e^{B(|u_n|)}\, dx.
\end{aligned} \tag{5.5}$$

Since $T_k(u_n - \psi^+)$ have the same sign as u_n, then by (3.2) and (3.4), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n - \psi^+| \leq k\}} b(|T_n(u_n)|) |D^i u_n|^{p_i} \omega_i\, dx + 2\sum_{i=1}^{N} \int_\Omega d(|u_n|) |D^i u_n|^{p_i} |T_k(u_n - \psi^+)| e^{B(|u_n|)} \omega_i\, dx \\
&\quad + \int_\Omega |u_n|^{p_0-1} |T_k(u_n - \psi^+)| \omega_0\, dx \\
&\leq \int_\Omega (|g_0| + |f_n|)\ |T_k(u_n - \psi^+)| e^{B(|u_n|)}\, dx + \sum_{i=1}^{N} \int_\Omega d(|u_n|) |D^i u_n|^{p_i} |T_k(u_n - \psi^+)| e^{B(|u_n|)} \omega_i\, dx \\
&\quad + \sum_{i=1}^{N} \int_{\{|u_n - \psi^+| \leq k\}} |a_i(x, T_n(u_n), \nabla u_n)|\, |D^i \psi^+| e^{B(|u_n|)}\, dx.
\end{aligned} \tag{5.6}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n - \psi^+| \leq k\}} b(|T_n(u_n)|) |D^i u_n|^{p_i} \omega_i\, dx + \sum_{i=1}^{N} \int_\Omega d(|u_n|) |D^i u_n|^{p_i} |T_k(u_n - \psi^+)| \omega_i\, dx \\
&\qquad + \int_\Omega |u_n|^{p_0-1} |T_k(u_n - \psi^+)| \omega_0\, dx \\
&\quad \leq C_1 k + e^{B(\infty)} \sum_{i=1}^{N} \int_{\{|u_n - \psi^+| \leq k\}} |a_i(x, T_n(u_n), \nabla u_n)|\, |D^i \psi^+|\, dx.
\end{aligned} \tag{5.7}$$

Concerning the third term on the left-hand side of (5.7). In view of Young's inequality, we have

$$\begin{aligned}
&\int_\Omega |u_n|^{p_0-1} |T_k(u_n - \psi^+)| \omega_0\, dx \\
&= \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0-1} |u_n - \psi^+| \omega_0\, dx + k \int_{\{|u_n - \psi^+| > k\}} |u_n|^{p_0-1} \omega_0\, dx \\
&\geq \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0} \omega_0\, dx - \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0-1} |\psi^+| \omega_0\, dx + k \int_{\{|u_n - \psi^+| > k\}} |u_n|^{p_0-1} \omega_0\, dx \\
&\geq \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0} \omega_0\, dx - \frac{1}{2} \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0} \omega_0\, dx - C_2 \int_{\{|u_n - \psi^+| \leq k\}} |\psi^+|^{p_0} \omega_0\, dx \\
&\quad + k \int_{\{|u_n - \psi^+| > k\}} |u_n|^{p_0-1} \omega_0\, dx \\
&\geq \frac{1}{2} \int_{\{|u_n - \psi^+| \leq k\}} |u_n|^{p_0} \omega_0\, dx + k \int_{\{|u_n - \psi^+| > k\}} |u_n|^{p_0-1} \omega_0\, dx - C_3.
\end{aligned} \tag{5.8}$$

For the second term on the right-hand side of (5.7), since $\lambda(p_i-1)<1$, then

$$\begin{aligned}
& e^{B(\infty)}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,dx\\
&\le \beta e^{B(\infty)}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\left(K_i(x)+|u_n|^{\frac{\underline{p}}{p_i'}}\omega_0^{\frac{1}{p_i'}}\omega_i^{\frac{1}{p_i}}+|D^iu_n|^{p_i-1}\omega_i\right)|D^i\psi^+|\,dx\\
&\le \sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|K_i(x)|^{p_i'}\omega_i^*\,dx+\frac{1}{4}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{\underline{p}}\omega_0\,dx\\
&\quad +C_4\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^i\psi^+|^{p_i}\omega_i\,dx+\frac{1}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}b(|T_n(u_n)|)|D^iu_n|^{p_i}\omega_i\,dx\\
&\quad +C_5\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\frac{|D^i\psi^+|^{p_i}}{b(|T_n(u_n)|)^{p_i-1}}\omega_i\,dx+C_6\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^i\psi^+|^{p_i}\omega_i\,dx\\
&\le \frac{1}{4}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{p_0}\omega_0\,dx+\frac{1}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}b(|T_n(u_n)|)|D^iu_n|^{p_i}\omega_i\,dx\\
&\quad +C_5\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^i\psi^+|^{p_i}(1+|u_n|)^{\lambda(p_i-1)}\omega_i\,dx+C_7\\
&\le \frac{1}{4}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{p_0}\omega_0\,dx+\frac{1}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}b(|T_n(u_n)|)|D^iu_n|^{p_i}\omega_i\,dx+C_8k.
\end{aligned}\tag{5.9}$$

According to (5.7), (5.8) and (5.9), we deduce that

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}b(|T_n(u_n)|)|D^iu_n|^{p_i}\omega_i\,dx+\sum_{i=1}^{N}\int_{\Omega}d(|u_n|)|D^iu_n|^{p_i}|T_k(u_n-\psi^+)|\omega_i\,dx\\
&\quad+\frac{1}{4}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{p_0}\omega_0\,dx+k\int_{\{|u_n-\psi^+|>k\}}|u_n|^{p_0-1}\omega_0\,dx\\
&\le C_9k.
\end{aligned}\tag{5.10}$$

Having in mind that $\{|u_n|\le k\}\subset\{|u_n-\psi^+|\le k\}$, we obtain

$$\begin{aligned}
\frac{b_0}{(1+k)^\lambda}\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i}\omega_i\,dx &= \frac{b_0}{(1+k)^\lambda}\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}|D^iu_n|^{p_i}\omega_i\,dx\\
&\le \sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}b(|T_n(u_n)|)|D^iu_n|^{p_i}\omega_i\,dx\\
&\le C_{10}k.
\end{aligned}\tag{5.11}$$

Hence, we get

$$\begin{aligned}
\|T_k(u_n)\|_{1,\vec{p},\vec{\omega}} &\le \int_{\Omega}|T_k(u_n)|^{\underline{p}}\omega_0\,dx+\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i}\omega_i\,dx+N+1\\
&\le \int_{\Omega}|T_k(u_n)|^{p_0}\omega_0\,dx+\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^iu_n|^{p_i}\omega_i\,dx+C_{11}\\
&\le C_{12}k^{\lambda+1}\quad\text{for any}\quad k\ge 1,
\end{aligned}\tag{5.12}$$

where C_{11} is a positive constant that does not depend on k and n. Thus, we conclude that the sequence $(T_k(u_n))_{n\in\mathbb{N}^*}$ is uniformly bounded in $W^{1,\vec{p}}(\Omega,\vec{\omega})$, and there exists a subsequence, still denoted $(T_k(u_n))_{n\in\mathbb{N}^*}$ and a measurable function $\nu_k\in W^{1,\vec{p}}(\Omega,\vec{\omega})$ such that

$$\begin{cases} T_k(u_n)\rightharpoonup\nu_k & \text{weakly in } W^{1,\vec{p}}(\Omega,\vec{\omega}),\\ T_k(u_n)\to\nu_k & \text{strongly in } L^{\underline{p}}(\Omega,\omega_0)\ \text{ and a.e in } \Omega.\end{cases}\tag{5.13}$$

On the other hand, thanks to (5.12) and (iii) in Lemma 2.1, we obtain

$$\begin{aligned} k \text{ meas}(\{|u_n| > k\}) &= \int_{\{|u_n|>k\}} |T_k(u_n)|\,dx \le \int_\Omega |T_k(u_n)|\,dx \\ &\le C_{13}\,\|T_k(u_n)\|_{\underline{p},\omega_0} \\ &\le C_{14} k^{\frac{\lambda+1}{\underline{p}}}. \end{aligned} \tag{5.14}$$

It follows that

$$\text{meas}(\{|u_n| > k\}) \le \frac{C_{14} k^{\frac{\lambda+1}{\underline{p}}}}{k} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{5.15}$$

Now, we will show that the sequence $(u_n)_n$ is a Cauchy sequence in measure.
Let $\rho > 0$, we have

$$\text{meas}\{|u_n - u_m| > \rho\} \le \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \rho\}.$$

For any $\varepsilon > 0$, thanks to (5.15), we choose $k_0(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \le \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m| > k\} \le \frac{\varepsilon}{3} \quad \text{for any} \quad k \ge k_0(\varepsilon). \tag{5.16}$$

On the other hand, thanks to (5.13), we have $T_k(u_n) \to \nu_k$ strongly in $L^{\underline{p}}(\Omega, \omega_0)$ and a.e in Ω. This implies that the sequence $(T_k(u_n))_{n\in I\!N}$ is a Cauchy sequence in measure. Thus, for any $k > 0$ and $\rho, \varepsilon > 0$, there exists $n_0 = n_0(k, \rho, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \rho\} \le \frac{\varepsilon}{3} \quad \text{for all} \quad n, m \ge n_0(k, \rho, \varepsilon). \tag{5.17}$$

By combining (5.16) and (5.17), we conclude that : for all $\rho, \varepsilon > 0$, there exists $n_0 = n_0(\rho, \varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \rho\} \le \varepsilon \quad \text{for any} \quad n, m \ge n_0.$$

Then, the sequence $(u_n)_n$ is a Cauchy sequence in measure, and converges almost everywhere to some measurable function u. Consequently, thanks to (5.13), we obtain

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } \ W^{1,\vec{p}}(\Omega, \vec{\omega}), \\ T_k(u_n) \to T_k(u) & \text{strongly in } \ L^{\underline{p}}(\Omega, \omega_0) \ \text{ and a.e in } \ \Omega, \end{cases} \tag{5.18}$$

Moreover, in view of Lebesgue's dominated convergence theorem, we conclude that

$$T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad L^{p_i}(\Omega, \omega_i) \quad \text{and a.e in} \quad \Omega \quad \text{for} \quad i = 1, \cdots, N. \tag{5.19}$$

Step 3: Some regularity results.

In this sequel, we note by $\varepsilon_i(n)$ for $i = 1, 2, \ldots$, some various functions of real variables that approach zero as n teds infinity. Similarly, we define $\varepsilon_i(n)$ and $\varepsilon_i(n, h)$.
In this step, we will show that :

$$\lim_{h\to\infty} \limsup_{n\to\infty} \frac{1}{h} \sum_{i=1}^N \int_{\{|u_n|\le h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n\,dx = 0. \tag{5.20}$$

Let $h \ge k \ge \max(1, \|\psi^+\|_\infty)$, we set $v = u_n - \eta \dfrac{T_h(u_n - \psi^+)}{h} e^{B(|u_n|)}$.
We have $v \in W_0^{1,\vec{p}}(\Omega, \vec{\omega})$, and $v \ge \psi$ for η small enough. Thus, $v \in K_\psi$ is an admissible test function for the approximate problem (5.3), and we have

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i\Big(\frac{T_h(u_n - \psi^+)}{h} e^{B(|u_n|)}\Big)\,dx \\ &\quad + \int_\Omega g_n(x, u_n, \nabla u_n) \frac{T_h(u_n - \psi^+)}{h} e^{B(|u_n|)}\,dx + \int_\Omega |u_n|^{p_0-2} u_n \frac{T_h(u_n - \psi^+)}{h} e^{B(|u_n|)} \omega_0\,dx \\ &\le \int_\Omega |f_n(x)| \frac{T_h(u_n - \psi^+)}{h} e^{B(|u_n|)}\,dx, \end{aligned} \tag{5.21}$$

since $T_h(u_n - \psi^+)$ have the same sign as u_n, and by (3.2) and (3.4), we get

$$\begin{aligned}
&\frac{1}{2h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,dx + \frac{1}{2h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} b(|u_n|)|D^i u_n|^{p_i}\omega_i\,dx\\
&\quad+\frac{2}{h}\sum_{i=1}^{N}\int_\Omega d(|u_n|)|D^i u_n|^{p_i}|T_h(u_n-\psi^+)|e^{B(|u_n|)}\omega_i\,dx + \frac{1}{h}\int_\Omega |u_n|^{p_0-1}|T_h(u_n-\psi^+)|\omega_0\,dx\\
&\le \frac{e^{B(\infty)}}{h}\int_\Omega \left(|g_0(x)|+|f(x)|\right)|T_h(u_n-\psi^+)|\,dx\\
&\quad+\frac{1}{h}\sum_{i=1}^{N}\int_\Omega d(|u_n|)|D^i u_n|^{p_i}|T_h(u_n-\psi^+)|e^{B(|u_n|)}\omega_i\,dx\\
&\quad+\frac{e^{B(\infty)}}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} |a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,dx.
\end{aligned} \tag{5.22}$$

We have $\psi^+ \in L^\infty(\Omega)$ and meas$\{|u_n| > h\} \to 0$ as h tends to infinity, then $\dfrac{|T_h(u_n-\psi^+)|}{h} \rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$, and since $g_0(x) \in L^1(\Omega)$ and $f(x) \in L^1(\Omega)$, it follows that

$$\varepsilon_1(h) = \frac{e^{B(\infty)}}{h}\int_\Omega \left(|g_0(x)|+|f(x)|\right)|T_h(u_n-\psi^+)|\,dx \longrightarrow 0 \qquad \text{as} \quad h\to\infty. \tag{5.23}$$

Concerning the last term on the right-hand side of (5.22), by using Young's inequality, we get

$$\begin{aligned}
&\frac{e^{B(\infty)}}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} |a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,dx\\
&\le \frac{e^{B(\infty)}\beta}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} \Big(K_i(x) + |u_n|^{\frac{\underline{p}}{p_i'}}\omega_0^{\frac{1}{p_i'}}\omega_i^{\frac{1}{p_i}} + |D^i u_n|^{p_i-1}\omega_i\Big)|D^i\psi^+|\,dx\\
&\le \frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} |K_i(x)|^{p_i'}\omega_i^*\,dx + \frac{1}{4h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} |u_n|^{\underline{p}-1}|u_n-\psi^++\psi^+|\omega_0\,dx\\
&\quad+\frac{C_1}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} |D^i\psi^+|^{p_i}\omega_i\,dx + \frac{1}{4h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} b(|T_n(u_n)|)|D^i u_n|^{p_i}\omega_i\,dx\\
&\quad+\frac{C_2}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} |D^i\psi^+|^{p_i}(1+|u_n|)^{\lambda(p_i-1)}\omega_i\,dx + \frac{C_3}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} |D^i\psi^+|^{p_i}\omega_i\,dx\\
&\le \frac{1}{2h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} |u_n|^{p_0-1}|T_h(u_n-\psi^+)|\omega_0\,dx\\
&\quad+\frac{1}{4h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} b(|T_n(u_n)|)|D^i u_n|^{p_i}\omega_i\,dx\\
&\quad+\frac{C_2}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} |D^i\psi^+|^{p_i}(1+h+\|\psi^+\|_\infty)^{\lambda(p_i-1)}\omega_i\,dx + \frac{C_4}{h}.
\end{aligned} \tag{5.24}$$

Having in mind that $\lambda(p_i-1) < 1$, we have

$$\varepsilon_2(h) = \frac{C_2}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} |D^i\psi^+|^{p_i}(1+h+\|\psi^+\|_\infty)^{\lambda(p_i-1)}\omega_i\,dx + \frac{C_4}{h} \longrightarrow 0 \quad \text{as} \quad h\to\infty. \tag{5.25}$$

According to (5.22) – (5.25), we obtain

$$\begin{aligned}&\frac{1}{2h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx+\frac{1}{4h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} b(|u_n|)|D^iu_n|^{p_i}\omega_i\,dx\\&\quad+\frac{1}{h}\sum_{i=1}^{N}\int_\Omega d(|u_n|)|D^iu_n|^{p_i}|T_h(u_n-\psi^+)|\omega_i\,dx+\frac{1}{2h}\int_\Omega |u_n|^{p_0-1}|T_h(u_n-\psi^+)|\omega_0\,dx\\&\le \varepsilon_3(n,h).\end{aligned}\tag{5.26}$$

By letting h tends to infinity in (5.26), we conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx=0,\tag{5.27}$$

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{|u_n|>h\}} d(|u_n|)|D^iu_n|^{p_i}\omega_i\,dx=0,\tag{5.28}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}}|u_n|^{p_0-1}\omega_0\,dx=0.\tag{5.29}$$

Thus, for any $\varepsilon>0$, there exists $\beta(\varepsilon)>0$ verifying : for any measurable subset E of Ω with meas$(E)<\beta(\varepsilon)$ such that

$$\int_E|u_n|^{p_0-1}\omega_0\,dx\le\int_{\{|u_n|>h\}}|u_n|^{p_0-1}\omega_0\,dx+\int_E|T_h(u_n)|^{p_0-1}\omega_0\,dx\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\le\varepsilon,\tag{5.30}$$

for $h>0$ large enough. Thus, the sequence $(|u_n|^{p_0-2}u_n\omega_0)_n$ is uniformly equi-integrable. In view of Vitali's theorem and the fact that $|u_n|^{p_0-1}\omega_0\longrightarrow|u|^{p_0-1}\omega_0$ almost everywhere in Ω, we conclude that

$$|u_n|^{p_0-1}\omega_0\longrightarrow|u|^{p_0-1}\omega_0\quad\text{strongly in}\quad L^1(\Omega).\tag{5.31}$$

Step 4 : The strong convergence of the gradients.

Let $h>k\ge\max\{1,\|\psi^+\|_\infty\}$, we set

$$S_h(s)=1-\frac{|T_{2h}(s)-T_h(s)|}{h}\qquad\text{and}\qquad B(s)=2\int_0^s\frac{d(|\tau|)}{b(|\tau|)}\,d\tau\quad\text{for any}\quad s\in I\!R.$$

Let $\varphi(s)=s.\exp\left(\frac{1}{2}\gamma^2s^2\right)$, where $\gamma=3\left\|\frac{d(|\cdot|)}{b(|\cdot|)}\right\|_{L^\infty(I\!R)}$. It is obvious that

$$\varphi'(s)-\gamma|\varphi(s)|\ge\frac{1}{2}\quad\text{for all}\quad s\in I\!R.$$

We have $v=u_n-\eta\varphi\left(T_k(u_n)-T_k(u)\right)S_h(u_n)e^{B(|u_n|)}\in W_0^{1,\vec{p}}(\Omega,\vec{\omega})$ and $v\ge\psi$ for η small enough. Thus, v is an admissible test function for the approximate problem (5.3), and we get

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i(\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)})\,dx\\&\quad+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}\,dx\\&\quad+\int_\Omega|u_n|^{p_0-2}u_n\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}\omega_0\,dx\\&\le\int_\Omega f_n(x)\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}\,dx.\end{aligned}\tag{5.32}$$

In view of (3.2) and (3.4), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)(D^iT_k(u_n)-D^iT_k(u))\varphi'\left(T_k(u_n)-T_k(u)\right)S_h(u_n)e^{B(|u_n|)}\,dx\\
&\quad+2\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\varphi\left(T_k(u_n)-T_k(u)\right)\operatorname{sign}(u_n)\frac{d(|u_n|)}{b(|u_n|)}S_h(u_n)e^{B(|u_n|)}\,dx\\
&\quad+\int_{\Omega}|u_n|^{p_0-2}u_n\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}\omega_0\,dx\\
&\le \int_{\Omega}\left(|g_0(x)|+|f(x)|\right)\left|\varphi\left(T_k(u_n)-T_k(u)\right)\right|S_h(u_n)e^{B(|u_n|)}\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{|u_n|\le 2h\}} d(|u_n|)|D^iu_n|^{p_i}\left|\varphi\left(T_k(u_n)-T_k(u)\right)\right|S_h(u_n)e^{B(|u_n|)}\omega_i\,dx\\
&\quad+\frac{1}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\left|\varphi\left(T_k(u_n)-T_k(u)\right)\right|e^{B(|u_n|)}\,dx.
\end{aligned}\tag{5.33}$$

Concerning the first term on the right-hand side of (5.33), since $g_0(x)$ and $f(x)$ belongs to $L^1(\Omega)$, and $\varphi\left(T_k(u_n)-T_k(u)\right)\rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$ as n goes to infinity, we derive

$$\begin{aligned}
\varepsilon_1(n) &= \int_{\Omega}\left(|g_0(x)|+|f(x)|\right)\left|\varphi\left(T_k(u_n)-T_k(u)\right)\right|S_h(u_n)e^{B(|u_n|)}\,dx\\
&\le e^{B(\infty)}\int_{\Omega}\left(|g_0(x)|+|f(x)|\right)\left|\varphi\left(T_k(u_n)-T_k(u)\right)\right|\,dx\longrightarrow 0 \qquad \text{as} \quad n\to\infty,
\end{aligned}\tag{5.34}$$

For the last term on the right-hand side of (5.33), thanks to (5.27), we get

$$\begin{aligned}
\varepsilon_2(h) &= \frac{1}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\left|\varphi\left(T_k(u_n)-T_k(u)\right)\right|e^{B(|u_n|)}\,dx\\
&\le \frac{e^{B(\infty)}\varphi(2k)}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx\longrightarrow 0 \qquad \text{as} \quad h\to\infty.
\end{aligned}\tag{5.35}$$

We have $S_h(u_n)=1$ on $\{|u_n|\le h\}$, and $\varphi\left(T_k(u_n)-T_k(u)\right)$ has the same sign as u_n on the set $\{|u_n|>k\}$. Thus, by combining (5.33) and (5.34) – (5.35), we conclude that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\left(D^iT_k(u_n)-D^iT_k(u)\right)\varphi'\left(T_k(u_n)-T_k(u)\right)S_h(u_n)e^{B(|u_n|)}\,dx\\
&\quad-\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_k(u)\varphi'\left(T_k(u_n)-T_k(u)\right)S_h(u_n)e^{B(|u_n|)}\,dx\\
&\quad+2\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} d(|u_n|)|D^iu_n|^{p_i}|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)e^{B(|u_n|)}\omega_0\,dx\\
&\quad-2\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n|\varphi(T_k(u_n)-T_k(u))|\frac{d(|u_n|)}{b(|u_n|)}S_h(u_n)e^{B(|u_n|)}\,dx\\
&\quad+\int_{\{|u_n|\le k\}}|T_k(u_n)|^{p_0-2}T_k(u_n)\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}\omega_0\,dx\\
&\quad+\int_{\{k<|u_n|\le 2h\}}|u_n|^{p_0-1}|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)e^{B(|u_n|)}\omega_0\,dx\\
&\le \varepsilon_3(n,h)+\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} d(|u_n|)|D^iu_n|^{p_i}\left|\varphi\left(T_k(u_n)-T_k(u)\right)\right|S_h(u_n)e^{B(|u_n|)}\omega_i\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} d(|u_n|)|D^iu_n|^{p_i}\left|\varphi\left(T_k(u_n)-T_k(u)\right)\right|S_h(u_n)e^{B(|u_n|)}\omega_i\,dx.
\end{aligned}\tag{5.36}$$

Thus, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\left(D^iT_k(u_n)-D^iT_k(u)\right)\varphi'\left(T_k(u_n)-T_k(u)\right)S_h(u_n)e^{B(|u_n|)}\,dx\\
&\quad-\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_k(u)\varphi'\left(T_k(u_n)-T_k(u)\right)S_h(u_n)e^{B(|u_n|)}\,dx\\
&\quad-3\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n|\varphi(T_k(u_n)-T_k(u))|\frac{d(|u_n|)}{b(|u_n|)}S_h(u_n)e^{B(|u_n|)}\,dx\\
&\quad+\int_{\{|u_n|\le k\}}|T_k(u_n)|^{p_0-2}T_k(u_n)\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}\omega_0\,dx\\
&\quad\le\varepsilon_3(n,h).
\end{aligned}\tag{5.37}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n)-D^iT_k(u))\\
&\qquad\times\left(\varphi'(T_k(u_n)-T_k(u))-3\varphi(T_k(u_n)-T_k(u))\frac{d(|u_n|)}{b(|u_n|)}\right)e^{B(|u_n|)}\,dx\\
&\quad+\int_{\{|u_n|\le k\}}(|T_k(u_n)|^{p_0-2}T_k(u_n)-|T_k(u)|^{p_0-2}T_k(u))\varphi(T_k(u_n)-T_k(u))\omega_0\,dx\\
&\le e^{B(\infty)}\left(\varphi'(2k)+3\varphi(2k)\left\|\frac{d(|\cdot|)}{b(|\cdot|)}\right\|_{L^\infty(I\!R)}\right)\sum_{i=1}^{N}\int_\Omega|a_i(x,T_k(u_n),\nabla T_k(u))|\,|D^iT_k(u_n)-D^iT_k(u)|\,dx\\
&\quad+3e^{B(\infty)}\left\|\frac{d(|\cdot|)}{b(|\cdot|)}\right\|_{L^\infty(I\!R)}\sum_{i=1}^{N}\int_\Omega|a_i(x,T_k(u_n),\nabla T_k(u_n))|\,|D^iT_k(u)|\,|\varphi(T_k(u_n)-T_k(u))|\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_k(u)\varphi'\left(T_k(u_n)-T_k(u)\right)S_h(u_n)e^{B(|u_n|)}\,dx\\
&\quad+\int_{\{|u_n|\le k\}}|T_k(u)|^{p_0-2}T_k(u)\varphi(T_k(u_n)-T_k(u))\omega_0\,dx+\varepsilon_4(n,h).
\end{aligned}\tag{5.38}$$

For the first term on the right-hand side of (5.38). In view of (5.18), we have $a_i(x,T_k(u_n),\nabla T_k(u))$ converges to $a_i(x,T_k(u),\nabla T_k(u))$ strongly in $L^{p_i'}(\Omega,\omega_i^*)$, and since $D^iT_k(u_n)\rightharpoonup D^iT_k(u)$ weakly in $L^{p_i}(\Omega,\omega_i)$, it follows that

$$\varepsilon_5(n)=\sum_{i=1}^{N}\int_\Omega|a_i(x,T_k(u_n),\nabla T_k(u))|\,|D^iT_k(u_n)-D^iT_k(u)|\,dx\to 0\quad\text{as}\quad n\to\infty.\tag{5.39}$$

Concerning the second term on the right-hand side of (5.38), we have $(a_i(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $L^{p_i'}(\Omega,\omega_i^*)$, then there exists a measurable function $\zeta_k\in L^{p_i'}(\Omega,\omega_i^*)$ such that $a_i(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup\zeta_k$ weakly in $L^{p_i'}(\Omega,\omega_i^*)$, and since $|D^iT_k(u)|\,|\varphi(T_k(u_n)-T_k(u))|\to 0$ strongly in $L^{p_i}(\Omega,\omega_i)$, we deduce that

$$\varepsilon_6(n)=\sum_{i=1}^{N}\int_\Omega|a_i(x,T_k(u_n),\nabla T_k(u_n))|\,|D^iT_k(u)|\,|\varphi(T_k(u_n)-T_k(u))|\,dx\to 0\text{ as }n\to\infty.\tag{5.40}$$

For the third term on the right-hand side of (5.38), we have $a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))$ is bounded in $L^{p_i'}(\Omega,\omega_i^*)$, then there exists a measurable function $\varsigma_{2h}\in L^{p_i'}(\Omega,\omega_i^*)$ such that $a_i(x,T_{2h}(u_n),$

$\nabla T_{2h}(u_n)) \rightharpoonup \varsigma_{2h}$ weakly in $L^{p_i'}(\Omega, \omega_i^*)$. Thus, we obtain

$$\begin{aligned}\varepsilon_7(n) &= \left|\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n)) D^i T_k(u)\varphi'\left(T_k(u_n) - T_k(u)\right) S_h(u_n) e^{B(|u_n|)}\, dx\right| \\ &\le e^{B(\infty)}\varphi'(2k)\sum_{i=1}^{N}\int_{\{k<|u_n|<2h\}} |a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n))|\,|D^i T_k(u)|\, dx \\ &\longrightarrow e^{B(\infty)}\varphi'(2k)\sum_{i=1}^{N}\int_{\{k<|u|<2h\}} |\varsigma_{2h}|\,|D^i T_k(u)|\, dx = 0 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{5.41}$$

Concerning the last term on the right-hand side of (5.38). Since $|T_k(u)|^{p_0-2}T_k(u) \in L^1(\Omega)$ and $T_k(u_n) \rightharpoonup T_k(u)$ weak$-*$ in $L^\infty(\Omega)$, we get

$$\varepsilon_8(n) = \int_{\{|u_n|\le k\}} |T_k(u)|^{p_0-2}T_k(u)(T_k(u_n) - T_k(u))\omega_0\, dx\, dx \to 0 \text{ as } n\to\infty. \tag{5.42}$$

We have $\varphi'(s) - \gamma|\varphi(s)| \ge \dfrac{1}{2}$ for any $s \in I\!R$. Thus, by combining (5.38) and (5.39) – (5.42), we conclude that

$$\begin{aligned}0 &\le \frac{1}{2}\sum_{i=1}^{N}\int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))(D^i T_k(u_n) - D^i T_k(u))\, dx \\ &\le \sum_{i=1}^{N}\int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))(D^i T_k(u_n) - D^i T_k(u)) \\ &\qquad \times\Big(\varphi'(T_k(u_n) - T_k(u)) - 3\varphi(T_k(u_n) - T_k(u))\frac{d(|u_n|)}{b(|u_n|)}\Big)e^{B(|u_n|)}\, dx \\ &\le \varepsilon_9(n,h) \longrightarrow 0 \quad \text{as} \quad n, h\to\infty.\end{aligned} \tag{5.43}$$

By letting n goes to infinity, we deduce that

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)(D^i T_k(u_n) - D^i T_k(u))\, dx \\ &\quad + \int_\Omega \left(|T_k(u_n)|^{p_0-2}T_k(u_n)\omega_0 - |T_k(u)|^{p_0-2}T_k(u)\omega_0\right)\left(T_k(u_n) - T_k(u)\right)\, dx \longrightarrow 0.\end{aligned} \tag{5.44}$$

Thanks to Lemma 3.1, we deduce that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in} \quad W_0^{1,\vec{p}}(\Omega, \vec{\omega}) \quad \text{and} \quad D^i u_n \longrightarrow D^i u \quad \text{a.e. in} \quad \Omega. \tag{5.45}$$

Moreover, since $a_i(x, T_k(u_n), \nabla u_n)D^i u_n$ tends to $a_i(x, u, \nabla u)D^i u$ almost everywhere in Ω as n goes to infinity. In view of (5.27) and Fatou's lemma, we conclude that

$$\begin{aligned}&\lim_{h\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u|\le h\}} a_i(x, u, \nabla u)D^i u\, dx \\ &\qquad \le \lim_{h\to\infty}\liminf_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}} a_i(x, T_k(u_n), \nabla u_n)D^i u_n\, dx \\ &\qquad \le \lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}} a_i(x, T_k(u_n), \nabla u_n)D^i u_n\, dx = 0.\end{aligned} \tag{5.46}$$

Step 5: Equi-integrability of the sequence $(g_n(x, u_n, \nabla u_n))_n$.

In this part, we will show that

$$g_n(x, u_n, \nabla u_n) \longrightarrow g(x, u, \nabla u) \quad \text{strongly in} \quad L^1(\Omega). \tag{5.47}$$

In view of (5.45), we have

$$g_n(x, u_n, \nabla u_n) \longrightarrow g(x, u, \nabla u) \quad \text{a.e. in} \quad \Omega. \tag{5.48}$$

For any measurable subset E of Ω, we have

$$\int_E |g_n(x, u_n, \nabla u_n)| \; dx \leq \int_E |g_n(x, T_h(u_n), \nabla T_h(u_n))| \; dx + \int_{\{|u_n|>h\}} |g_n(x, u_n, \nabla u_n)| \; dx. \tag{5.49}$$

From (5.30) and the fact that $g_0(x) \in L^1(\Omega)$, we deduce that

$$\int_{\{|u_n|>h\}} |g_n(x, u_n, \nabla u_n)| \, dx \leq \int_{\{|u_n|>h\}} |g_0(x)| \, dx + \sum_{i=1}^{N} \int_{\{|u_n|>h\}} d(|u_n|)|D^i u_n|^{p_i} \omega_i dx \longrightarrow 0$$

as $h \to \infty$. Thus, for all $\varepsilon > 0$, there exists $h_0(\varepsilon) > 0$ such that

$$\int_{\{|u_n|>h\}} |g_n(x, u_n, \nabla u_n)| \; dx \leq \frac{\varepsilon}{2} \qquad \text{for any} \quad h \geq h_0(\varepsilon). \tag{5.50}$$

On the other hand, in view of (5.45), there exists $\beta(\varepsilon) > 0$ small enough such that: for all $E \subset \Omega$ with $meas(E) \leq \beta(\varepsilon)$, we have

$$\int_E |g_n(x, T_h(u_n), \nabla T_h(u_n))| \; dx \leq \int_E |g_0(x)| \; dx + \sum_{i=1}^{N} \int_E d\left(|T_h(u_n)|\right) |D^i T_h(u_n)|^{p_i} \omega_i \, dx \leq \frac{\varepsilon}{2}. \tag{5.51}$$

According to (5.49), (5.50) and (5.51), we obtain

$$\int_E |g_n(x, u_n, \nabla u_n)| \; dx \leq \varepsilon \qquad \text{for any} \;\; E \subset \Omega \quad \text{such that} \quad meas(E) \leq \beta(\varepsilon). \tag{5.52}$$

As a result, the sequence $\left(g_n(x, u_n, \nabla u_n)\right)_n$ is uniformly equi-integrable, and thanks to (5.48) and Vitali's theorem, we conclude that

$$g_n(x, u_n, \nabla u_n) \; \longrightarrow g(x, u, \nabla u) \qquad \text{strongly in} \quad L^1(\Omega). \tag{5.53}$$

Step 6: Passage to the limit.

Let $\varphi \in K_\psi \cap L^\infty(\Omega)$ and $M = k + \|\varphi\|_\infty$.
By using $v = u_n - \eta T_k(u_n - \varphi)$ as a test function for the approximate problem (5.3), we obtain

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi) \, dx + \int_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) \, dx \\ + \int_\Omega |u_n|^{p_0 - 2} u_n T_k(u_n - \varphi) \omega_0 \, dx \leq \int_\Omega f_n T_k(u_n - \varphi) \, dx. \end{aligned} \tag{5.54}$$

Since $\{|u_n - \varphi| \leq k\} \subseteq \{|u_n| \leq M\}$, then we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi) \, dx \\ &= \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n)) \left(D^i T_M(u_n) - D^i \varphi\right) \chi_{\{|u_n - \varphi| \leq k\}} \, dx \\ &= \sum_{i=1}^{N} \int_\Omega \left(a_i(x, T_M(u_n), \nabla T_M(u_n)) - a_i(x, T_M(u_n), \nabla \varphi)\right) \left(D^i T_M(u_n) - D^i \varphi\right) \chi_{\{|u_n - \varphi| \leq k\}} \, dx \\ &\quad + \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla \varphi) \left(D^i T_M(u_n) - D^i \varphi\right) \chi_{\{|u_n - \varphi| \leq k\}} \, dx. \end{aligned} \tag{5.55}$$

According to Fatou's Lemma, we have

$$\begin{aligned}
&\liminf_{n\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx\\
&\geq \sum_{i=1}^{N}\int_\Omega \left(a_i(x,T_M(u),\nabla T_M(u))-a_i(x,T_M(u),\nabla\varphi)\right)\left(D^iT_M(u)-D^i\varphi\right)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&\qquad+\sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u),\nabla\varphi)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&=\sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u),\nabla T_M(u))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&=\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla u)D^iT_k(u-\varphi)\,dx.
\end{aligned} \tag{5.56}$$

Having in mind that $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weak$-*$ in $L^\infty(\Omega)$ as n goes to infinity, and in view of (5.31) and (5.53), we obtain

$$\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega g(x,u,\nabla u)T_k(u-\varphi)\,dx, \tag{5.57}$$

and

$$\int_\Omega |u_n|^{p_0-2}u_nT_k(u_n-\varphi)\omega_0\,dx\quad\longrightarrow\int_\Omega |u|^{p_0-2}uT_k(u-\varphi)\omega_0\,dx. \tag{5.58}$$

Moreover, since $f_n(x)\to f(x)$ strongly in $L^1(\Omega)$, we get

$$\int_\Omega f_nT_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega fT_k(u-\varphi)\,dx. \tag{5.59}$$

By combining (5.54) and (5.56) – (5.59), we conclude that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla u)D^iT_k(u-\varphi)\,dx+\int_\Omega g(x,u,\nabla u)T_k(u-\varphi)\,dx\\
&\quad+\int_\Omega |u|^{p_0-2}uT_k(u-\varphi)\omega_0\,dx\leq\int_\Omega f_nT_k(u-\varphi)\,dx,
\end{aligned} \tag{5.60}$$

for any $\varphi\in K_\psi\cap L^\infty(\Omega)$, which complete the proof of theorem 5.2.

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