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# UNIQUENESS OF DIFFERENTIAL-DIFFERENCE MONOMIAL WITH FIXED POINT OF MEROMORPHIC FUNCTION

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**Abstract** *In this paper, we investigate* $f(z)$ *to be a transcendental meromorphic function of finite order* $\sigma(f)$ *and* $c_1, c_2, ..., c_k$ *be complex constants satisfying that at least one of them is non-zero. The authors establish fixed points about the differential-difference monomial* $(f^{(1)}(z+c_1))^{n_1}(f^{(2)}(z+c_2))^{n_2}...(f^{(k)}(z+c_k))^{n_k}$, *where* $\psi = \sum_{j=1}^{k} n_j$. *These results extend recent results obtained by Zhang and Chen [1].*

## 1 Introduction

*In this paper, we assume the reader is familiar with the fundamental results and the basic notations of the Nevanlinna theory of meromorphic functions (see [3] [2]) Let* $f(z)$ *be a non-constant meromorphic function defined in the complex plane* $\mathbb{C}$. *We will let* $\sigma(f)$ *and* $\mu(f)$ *denote the order of* $f(z)$ *respectively. We use* $S(r, f)$ *to denote any quantity of* $S(r, f) = O\{T(r, f)\}$ $(r \longrightarrow \infty)$, *possibly outside a set* $E$ *with finite logarithmic measure.*

*Let* $a \in \mathbb{C}$, *We use the notations* $\sigma(f)$ *to denote the order of* $f(z)$, $\lambda(f, a)$, *and* $\lambda(1/f)$, *respectively, to denote the exponent of convergence of zeros of* $f(z) - a$ *and poles of* $f(z)$. *Especially, if* $a = 0$, *we denote* $\lambda(f, 0) = \lambda(f)$. *A point* $z \in \mathbb{C}$ *is called as a fixed point of* $f(z)$ *if* $f(z) = z$. *There is a considerable number of results on the fixed points for meromorphic functions in the plane, we refer the reader to Chuang and Yang [4]. It follows Chen and Shon [5], we use the notation* $\tau(f)$ *to denote the exponent of convergence of fixed points of* $f$ *that is defined as*

$$\tau(f) = 1 - \lim_{r\to\infty} \sup \frac{\log N\left(r, \dfrac{1}{f-z}\right)}{logr},$$

*In 1993, Lahiri proved the following theorem*

***Theorem A.*** *[6] Let* $f$ *be a transcendental meromorphic function in the plane. Suppose that there exists* $a \in \mathbb{C}$ *with* $\delta(a, f) > 0$ *and* $\delta(\infty, f) = 1$. *Then* $f$ *has infinitely many fixed points.*

*Now-a-days investigating on the problems of sharing values by a meromorphic functions with its differences or shift become an extensive subfield of the modern uniqueness theory. We define the shift and difference operators of* $f(z)$ *as* $f(z+c)$ *and* $\Delta_c f(z) = f(z+c) - f(z)$, *where* $c$ *is a non-zero complex number.*

*When the order of $f$ is less than 1, Chen and Shon have proved the following.*

***Theorem B.*** *[5] Let $f$ be a transcendental meromorphic function of order of growth $\sigma(f) \leq 1$. Suppose that $f$ satisfies $\lambda(1/f) < \lambda(f) < 1$ or has infinitely many zeros (with $\lambda(f) = 0$) and finitely many poles. Then $\Delta f$ has infinitely many fixed points and satisfies the exponent of convergence of fixed points $\tau(\Delta f) = \sigma(f)$.*

*In 2014, Wu and Xu have proved the following.*

***Theorem C.*** *[7] Let $f$ be a transcendental meromorphic function of order of growth $\sigma(f) < 1$ and $a \in \mathbb{C}$. Suppose that $f$ satisfies $\lambda(1/f) < \sigma(f)$ and $\lambda(f, a) < \sigma(f)$. Then $\Delta f$ has infinitely many fixed points and satisfies the exponent of convergence of fixed points $\tau(\Delta f) = \sigma(f)$.*

*For the existence on the fixed points of differences, Cui and Yang have proved the following theorems.*

***Theorem D.*** *[8] Let $f$ be a function transcendental and meromorphic in the plane with the order $\sigma(f) = 1$. If $f$ has finitely many poles and infinitely many zeros with exponent of convergence of zeros $\lambda(f) \neq 1$, then $\Delta f$ has infinitely many zeros and fixed points.*

***Theorem E.*** *[8] Let $f$ be a function transcendental and meromorphic in the plane with the order $\sigma(f) = 1\ max\{\lambda(f), \lambda\frac{1}{f}\} = 1$. If $f$ has infinitely many zeros, then $\Delta f$ has infinitely many zeros and fixed points.*

*The conditions of Theorems D and E imply that $0, \infty$ are Borel exceptional values. If $\infty$ and $d \in \mathbb{C}$ are Borel exceptional values of $f$, Chen obtains the following theorem.*

***Theorem F.*** *[9] Let $f$ be a finite order meromorphic function such that $\lambda(\frac{1}{f}) < \sigma(f)$, and let $c \in \mathbb{C}\{0\}$ be a constant such that $f(z+c) \not\equiv f(z)$. If $f(z)$ has a Borel exceptional value $d \in \mathbb{C}$, then $\tau(\Delta_c f) = \sigma(f)$.*

*In 2016, Zhang and Chen we have obtained the following result.*

***Theorem G.*** *[1] Let $f$ be a finite order meromorphic function, and let $c \in \mathbb{C}\{0\}$ be a constant such that $f(z+c) \not\equiv f(z)$. If $f(z)$ has two Borel exceptional values, then $\tau(\Delta_c f) = \sigma(f)$.*

*Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$ and $a \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Nevanlinna's deficiency of $f$ with respect to $a$ is defined by*

$$\delta(a, f) = 1 - \lim_{r\to\infty} \sup \frac{N\left(r, \dfrac{1}{f-a}\right)}{T(r, f)},$$

*If $a = \infty$, then one should replace $N\left(r, \dfrac{1}{f-a}\right)$ in the above formula by $N(r, f)$. If $\delta(a, f) > 0$, then $a$ is called a Nevanlinna deficiency value of $f$.*

**Definition 1.1.** [10] The expression

$$M_j[f] = (f)^{n_{0j}}(f')^{n_{1j}} \cdots (f^{(k)})^{n_{kj}}$$

is called a *differential monomial* generated by $f$ of degree

$$d(M_j) = \sum_{i=0}^{k} n_{ij}$$

and weight

$$\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}.$$

The sum

$$P[f] = \sum_{j=1}^{t} b_j M_j[f]$$

is called a *differential polynomial* generated by $f$ of degree

$$d(P) = \max\{d(M_j) : 1 \leq j \leq t\}$$

and weight

$$\Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq t\},$$

where $T(r, b_j) = S(r, f)$ for $j = 1, 2, \ldots, t$.

The numbers

$$d(P) = \min\{d(M_j) : 1 \leq j \leq t\}$$

and $k$, the highest order of the derivative of $f$ in $P[f]$, are called respectively the *lower degree* and *order* of $P[f]$.

$P[f]$ is said to be *homogeneous* if

$$d(P) = d(P).$$

$P[f]$ is called a *linear differential polynomial* generated by $f$ if $d(P) = 1$. Otherwise, $P[f]$ is called a *non-linear differential polynomial*.

*For further investigation of the above theorem, we now define a differential-difference monomial $M(z)$ of an meromorphic function $f$ as follows:*

$$M(z) = (f^{(1)}(z+c_1))^{n_1}(f^{(2)}(z+c_2))^{n_2}...(f^{(k)}(z+c_k))^{n_k},$$

*where $k$ is a positive integer and $n_1, n_2, ..., n_k$ are non-negative integers, not all of them are zero. we call $k$ and $\psi = \sum_{j=1}^{k} n_j$ respectively the order and the degree of the monomial $M(z)$.*

*There has been significant research on various properties as well as the uniqueness of meromorphic functions, differential monomials, and differential polynomials, including differential-difference polynomials. For a detailed discussion of such studies, we refer to the articles [ [11], [12], [13], [14], [15], [16], [17], [23], [24], [25], [26]] and the references therein.*

*In this paper, we obtain Theorem G for differential-difference monomials defined above.*

**Theorem 1.2.** *Let $f(z)$ be a transcendental meromorphic function of finite order $\sigma(f)$ in the complex plan. let $c_1, c_2, ..., c_k$, $(k \in \mathbb{C})$ be complex constants satisfying that at least one of them is non-zero such that $M(z) \not\equiv 0$, If there is $a \in \mathbb{C}$ with $f$ is satisfies $\delta(\infty, f) = 1$ and $a$ is a Nevanlinna deficiency value of $f$, then $M(z)$ have infinitely many fixed points and $\tau(M(z)) = \sigma(f)$.*

**Theorem 1.3.** *Let $f(z)$ be a transcendental meromorphic function of finite order $\sigma(f)$ in the complex plan. let $c_1, c_2, ..., c_k$, $(k \in \mathbb{C})$ be complex constants satisfying that at least one of them is non-zero such that $M(z) \not\equiv 0$, If there is $a \in \mathbb{C}$ with $f$ is satisfies $\delta(\infty, f) = 1$ and $\delta(0, f) = 1$, then*

$$T(r, M(z)) \sim \psi T(r, f) \sim N\left(r, \frac{1}{M(z) - z}\right),$$

*as $r \longrightarrow \infty$, $r \notin E$, where $E$ is a possible exception set of $r$ with finite logarithmic measure.*

**Example 1.4.** Let $f(z) = e^z$ and $c_1 = \log 1$, $c_2 = \log 2$, ..., $c_k = \log k$ $(k \in \mathbb{C})$. Then $M(z) = f(z+c_1)f(z+c_2)...f(z+c_k) = e^{z+\log 1}.e^{z+\log 2}....e^{z+\ logk} = k!e^{kz}$. Obviously, we can get $\delta(0, f) = \delta(\infty, f) = 1$ and $M(z)$ have infinitely many fixed points and $\tau(M(z)) = \sigma(f)$.

$$T(r, M(z)) \sim kT(r, f) \sim N\left(r, \frac{1}{M(z) - z}\right),$$

as $r \longrightarrow \infty$, And above all, $M(z) = k!e^{kz} \neq 0$. Therefore, the condition $z \neq 0$ in (if assumed in the theorems above) is necessary.

1. *That the function $f(z) = e^z$ meets the required Nevanlinna deficiency conditions $\delta(0, f) = \delta(\infty, f) = 1$.*
2. *That the constructed $M(z) = k!e^{kz}$ is a concrete instance verifying the theorems' conclusions-especially the behavior of the characteristic function $T(r, M(z))$ and the existence of infinitely many fixed points.*
3. *The importance of the condition $M(z) \not\equiv 0$ (i.e., $z \neq 0$) in validating the asymptotic result.*

## 2 Some Lemmas

*In this section, we state some lemmas which will be needed in the sequel.*

**Lemma 2.1.** *[18] Let $f(z)$ be a transcendental meromorphic function of finite order, then*

$$m\left(r, \frac{f(z+c)}{f}\right) = S(r, f).$$

**Lemma 2.2.** *[19] Let $f(z)$ be a finite order meromorphic function, then, for each $k \in \mathbb{N}$, $\sigma(\Delta_c^k f) \leq \sigma(f)$.*

**Lemma 2.3.** *[20] Let $f$ be a transcendental meromorphic function of finite order. Then for any positive integer $n$, we have*

$$m\left(r, \frac{\Delta_c^n f(z)}{f(z)}\right) = S(r, f).$$

**Lemma 2.4.** *[21] Let $f$ be a transcendental meromorphic function of finite order. Then*

$$N(r, f(z+c)) = N(r, f) + S(r, f),$$

$$T(r, f(z+c)) = T(r, f) + S(r, f),$$

*where $S(r, f) = O(T(r, f))$ $(r \longrightarrow \infty)$, possibly outside a set $E$ of $r$ with finite logarithmic measure.*

**Lemma 2.5.** *[22] Suppose that $f(z)$ is a transcendental meromorphic function in the complex plane and $P(z) = a_0 z^n + a_1 z^{n-1} + ... + a_n$, where $a_0(\not\equiv 0)$, $a_1, ..., a_n$ are constants. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f),$$

**Lemma 2.6.** *[22] Let $F(r)$ and $G(r)$ be monotone increasing function such that $F(r) \leq G(r)$ outside of exceptional set $E$ that is of finite logarithmic measure. Then for any $\alpha > 0$, there exists $r_0 > 1$ such that $F(r) \leq G(\alpha r)$ for all $r > r_0$.*

**Lemma 2.7.** *Let $f$ be a transcendental meromorphic function of finite order $\sigma(f)$ in the complex plan. let $c_1, c_2, ..., c_k$, $(k \in \mathbb{C})$ be complex constants satisfying that at least one of them is non-zero such that $M(z) \not\equiv 0$ and $\delta(0, f) > 0$. Then $M(z)$ a transcendental and meromorphic function of finite order.*

*Proof.* Since $\delta(0, f) > 0$, from Lemma 2.2, we know that $\sigma(M(z)) \leq \sigma(f) < +\infty$. If $M(z)$ is a transcendental meromorphic function. Suppose that $M(z)$ is not a transcendental meromorphic function, then there is a rational $P(z)$ such that $P(z)M(z) \equiv 1$, i.e.,

$$\begin{aligned}\frac{1}{f^{\psi}} &\equiv P(z)\frac{M(z)}{f^{\psi}}\\ &\equiv P(z)\Big(\frac{f^{(1)}(z+c_1)}{f}\Big)^{n_1}\cdot\Big(\frac{f^{(2)}(z+c_2)}{f}\Big)^{n_2}\cdots\Big(\frac{f^{(k)}(z+c_k)}{f}\Big)^{n_k}\end{aligned}$$

Applying Lemma 2.1 and noting that $f(z)$ is transcendental, we can get

$$m\Big(r, \frac{1}{f^{\psi}}\Big) = S(r, f).$$

Therefore

$$\begin{aligned}m\Big(r, \frac{1}{f^{\psi}}\Big) + N\Big(r, \frac{1}{f^{\psi}}\Big) &\leq N\Big(r, \frac{1}{f^{\psi}}\Big) + S(r, f)\\ &\leq \psi N\Big(r, \frac{1}{f}\Big) + S(r, f).\end{aligned}$$

Apply Lemma 2.5 and the first fundamental theorem of Nevanlinna theory, we can get

$$\psi T(r, f) \leq \psi N\Big(r, \frac{1}{f}\Big) + S(r, f).$$

This contradicts with $\delta(0, f) > 0$. Thus $M(z)$ is a transcendental and meromorphic function of finite order. □

**Lemma 2.8.** *Let $f$ be a transcendental meromorphic function of finite order $\sigma(f)$ in the complex plan. let $c_1, c_2, ..., c_k$, $(k \in \mathbb{C})$ be complex constants satisfying that at least one of them is non-zero such that $M(z) \not\equiv 0$ and $\delta(0, f) > 0$, then*

$$\psi T(r, f) \leq \psi N\Big(r, \frac{1}{f}\Big) + 2\psi N(r, f) + N\Big(r, \frac{1}{M(z) - z}\Big) + S(r, f)$$

*Proof.* By Lemma 2.7, we know that $M(z)$ is a transcendental meromorphic function, then there is $\eta \in \mathbb{C}\{0\}$ such that $z\Delta_\eta M(z) - M(z) \not\equiv 0$.
Noticing

$$\frac{1}{f^{\psi}} = \frac{M(z)}{zf^{\psi}} - \frac{z\Delta_\eta M(z) - M(z)}{zf^{\psi}}\frac{M(z) - z}{z\Delta_\eta M(z) - M(z)}, \tag{2.1}$$

then

$$\begin{aligned}m\Big(r, \frac{1}{f^{\psi}}\Big) \leq& m\Big(r, \frac{M(z)}{zf^{\psi}}\Big) + m\Big(r, \frac{z\Delta_\eta M(z) - M(z)}{zf^{\psi}}\Big) + m\Big(r, \frac{M(z) - z}{z\Delta_\eta M(z) - M(z)}\Big) + O(1)\\ \leq& 2m\Big(r, \frac{M(z)}{f^{\psi}}\Big) + m\Big(r, \frac{\Delta_\eta M(a)}{f^{\psi}}\Big) + m\Big(r, \frac{M(z) - z}{z\Delta_\eta M(z) - M(z)}\Big) + O(\ log r).\end{aligned} \tag{2.2}$$

Applying the first fundamental theorem, we get

$$m\Big(r, \frac{1}{f^{\psi}}\Big) = \psi T(r, f) - \psi N\Big(r, \frac{1}{f}\Big) + O(1). \tag{2.3}$$

$$\begin{aligned}m\Big(r, \frac{M(z) - z}{z\Delta_\eta M(z) - M(z)}\Big) =& m\Big(r, \frac{z\Delta_\eta M(z) - M(z)}{M(z) - z}\Big) + N\Big(r, \frac{z\Delta_\eta M(z) - M(z)}{M(z) - z}\Big)\\ &- N\Big(r, \frac{M(z) - z}{z\Delta_\eta M(z) - M(z)}\Big) + O(1)\\ \leq& m\Big(r, \frac{z\Delta_\eta M(z) - M(z)}{M(z) - z}\Big)\\ &+ N\Big(r, \frac{z\Delta_\eta M(z) - M(z)}{M(z) - z}\Big) + O(1).\end{aligned} \tag{2.4}$$

Combining (1)-(4) we have

$$\begin{aligned}\psi T(r,f) \leq & \psi N\left(r,\frac{1}{f}\right)+2m\left(r,\frac{M(z)}{f^{\psi}}\right)+m\left(r,\frac{\Delta_{\eta}M(z)}{f^{\psi}}\right)\\ &+m\left(r,\frac{z\Delta_{\eta}M(z)-M(z)}{M(z)-z}\right)\\ &+N\left(r,\frac{z\Delta_{\eta}M(z)-M(z)}{M(z)-z}\right)+O(\ logr)\\ \leq & \psi N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{M(z)-z}\right)+N(r,z\Delta_{\eta}M(z)-M(z))\\ &+2m\left(r,\frac{M(z)}{f^{\psi}}\right)+m\left(r,\frac{\Delta_{\eta}M(z)}{f^{\psi}}\right)+m\left(r,\frac{z\Delta_{\eta}M(z)-M(z)}{M(z)-z}\right)\\ &+O(\ logr).\end{aligned} \tag{2.5}$$

Since

$$\begin{aligned}\Delta_{\eta}M(z)=&\Delta_{\eta}((f^{(1)}(z+c_1))^{n_1}(f^{(2)}(z+c_2))^{n_2}...(f^{(k)}(z+c_k))^{n_k})\\ =&(f^{(1)}(z+c_1+\eta))^{n_1}(f^{(2)}(z+c_2+\eta))^{n_2}...(f^{(k)}(z+c_k+\eta))^{n_k}\\ &-(f^{(1)}(z+c_1))^{n_1}(f^{(2)}(z+c_2))^{n_2}...(f^{(k)}(z+c_k))^{n_k}\\ \Delta_{\eta}(M(z)-z)=&\Delta_{\eta}((f^{(1)}(z+c_1))^{n_1}(f^{(2)}(z+c_2))^{n_2}...(f^{(k)}(z+c_k))^{n_k}-z)\\ =&(f^{(1)}(z+c_1+\eta))^{n_1}(f^{(2)}(z+c_2+\eta))^{n_2}...(f^{(k)}(z+c_k+\eta))^{n_k}\\ &-(f^{(1)}(z+c_1))^{n_1}(f^{(2)}(z+c_2))^{n_2}...(f^{(k)}(z+c_k))^{n_k}-(z+\eta),\end{aligned} \tag{2.6}$$

then, we can get

$$\begin{aligned}z\Delta_{\eta}M(z)-M(z)=&z((f^{(1)}(z+c_1+\eta))^{n_1}(f^{(2)}(z+c_2+\eta))^{n_2}...\\ &(f^{(k)}(z+c_k+\eta))^{n_k})-(z+1)((f^{(1)}(z+c_1))^{n_1}(f^{(2)}\\ &(z+c_2))^{n_2}...(f^{(k)}(z+c_k))^{n_k})\\ z\Delta_{\eta}(M(z)-z)-(M(z)-z)=&z((f^{(1)}(z+c_1+\eta))^{n_1}(f^{(2)}(z+c_2+\eta))^{n_2}...\\ &(f^{(k)}(z+c_k+\eta))^{n_k})-(z+1)((f^{(1)}(z+c_1))^{n_1}\\ &(f^{(2)}(z+c_2))^{n_2}...(f^{(k)}(z+c_k))^{n_k})-z((z+\eta)-1).\end{aligned} \tag{2.7}$$

Therefore,

$$\begin{aligned}\frac{z\Delta_{\eta}M(z)-M(z)}{M(z)-z}=&\frac{z\Delta_{\eta}(M(z)-z)-(M(z)-z)}{M(z)-z}\\ =&\frac{z\Delta_{\eta}(M(z)-z)}{M(z)-z}-1\end{aligned} \tag{2.8}$$

$$\begin{aligned}N(r,z\Delta_{\eta}M(z)-M(z))\leq & N(r,(f^{(1)}(z+c_1+\eta))^{n_1}(f^{(2)}(z+c_2+\eta))^{n_2}...(f^{(k)}(z+c_k+\eta))^{n_k})\\ &+N(r,(f^{(1)}(z+c_1))^{n_1}(f^{(2)}(z+c_2))^{n_2}...(f^{(k)}(z+c_k))^{n_k})\end{aligned} \tag{2.9}$$

Thus from Lemma 2.4 and (9), we deduce

$$N(r,z\Delta_{\eta}M(z)-M(z))\leq 2\psi N(r,f(z))+O(\ logr) \tag{2.10}$$

By Lemmas 2.2 and 2.7 we know that $M(z)-z$ is a transcendental meromorphic function of finite order. It follows from Lemma 2.3 and (8), that

$$
\begin{aligned}
m\left(r, \frac{M(z)}{f^{\psi}}\right) &= S(r,f) \\
m\left(r, \frac{\Delta_{\eta} M(z)}{f^{\psi}}\right) &= S(r,f) \\
m\left(r, \frac{z\Delta_{\eta} M(z) - M(z)}{M(z)-z}\right) &= S(r,f).
\end{aligned} \tag{2.11}
$$

From (5) and (10)-(11), we have

$$
\psi T(r,f) \leq \psi N\left(r, \frac{1}{f}\right) + 2\psi N(r,f) + N\left(r, \frac{1}{M(z)-z}\right) + S(r,f). \tag{2.12}
$$

□

## 3 Proof of Theorem

### Theorem 1.9.

*Proof.* Denoting $g = f - a$ by (12) we derive,

$$
\begin{aligned}
\psi T(r,f) \leq& \psi T(r,g) + O(1) \\
\leq& \psi N\left(r, \frac{1}{g}\right) + 2\psi N(r,g) + N\left(r, \frac{1}{M(z)-z}\right) + S(r,g) \\
=& \psi N\left(r, \frac{1}{f-a}\right) + 2\psi N(r,f) + N\left(r, \frac{1}{M(z)-z}\right) + S(r,f).
\end{aligned} \tag{3.1}
$$

Since $\delta(0,f) > 0$ and $\delta(\infty,f) = 1$, there is a positive number $\theta < 1$ such that

$$
N\left(r, \frac{1}{f-a}\right) < \theta T(r,f) \tag{3.2}
$$

$$
N(r,f) = O(1)T(r,f) \tag{3.3}
$$

If $M(z)$ has only a finite number of fixed points, then from (13), (14) and (15) we would have

$$
\psi(1 - O(1) - \theta)T(r,f) \leq N\left(r, \frac{1}{M(z)-z}\right), r \notin E, r \longrightarrow \infty, \tag{3.4}
$$

where $E$ is a possible exceptional set with finite logarithmic measure. Noticing $f$ is transcendental, applying Lemma 2.6 and (16), we can get that $M(z)$ assumes has infinitely many fixed points and $\tau(M(z)) = \sigma(f)$. □

### Theorem 1.10.

*Proof.* Since $\delta(0,f) = 1$ and $\delta(\infty,f) = 1$,

$$
N\left(r, \frac{1}{f}\right) = S(r,f) \tag{3.5}
$$

$$
N(r,f) = S(r,f) \tag{3.6}
$$

Since

$$
T(r,f) \leq \sum_{j=1}^{k} \psi_i T(r, f^{(j)}(z+c_j)) + O(1). \tag{3.7}
$$

Using Lemma 2.4, we can derive from (19) that

$$
T(r, M(z)) \leq \psi T(r,f) + S(r,f) \tag{3.8}
$$

From (17), (18), (19), (20), we have

$$\begin{aligned} \psi T(r,f) \leq & N\left(r, \frac{1}{M(z)-z}\right) + S(r,f) \\ \leq & T(r, M(z)) + S(r,f) \\ \leq & \psi T(r,f) + S(r,f) \end{aligned} \tag{3.9}$$

Since $f$ is transcendental, (21) means that $M(z)$ assumes has infinitely many fixed points and

$$T(r, M(z)) \sim \psi T(r,f) \sim N\left(r, \frac{1}{M(z)-z}\right),$$

as $r \notin E, r \longrightarrow \infty$, where $E$ is a possible exception set of $r$ with finite logarithmic measure. □

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