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Existence of Solutions and Optimal Feedback Controls of SDEs driven by Spatial Parameters Local Martingale of McKean-Vlasov type

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Abstract In the first part of this paper, we are interested in the existence and uniqueness of solutions of distribution-dependent stochastic differential equations driven by spatially parameterized continuous local martingales. In this framework, the drift coefficient exhibits non-linear dependencies on both the state process and its distribution. To establish the existence of weak solutions, we first employ the Euler approximation and the martingale problem formulation. Next, we achieve the existence and pathwise uniqueness of its strong solution when non-Lipschitz conditions are satisfied. Finally, we demonstrate that the Euler approximation exhibits an optimal rate of strong convergence.

In the second part, we address a control problem in which the system is governed by a similar equation. Using the martingale problem formulation, we establish the existence of an optimal feedback control.

1 Introduction

To investigate nonlinear PDE in Vlasov's kinetic, Kac [13] in 1956, introduced the concept of propagation of chaos of mean-field particle systems, which inspired McKean [17] to explore nonlinear Fokker-Planck equations by employing SDEs with distribution-dependent drift coefficients. A comprehensive introduction can be found in [22]. These distribution-dependent stochastic differential equations (DDSDEs for short) are also called McKean-Vlasov SDEs or mean-field SDEs in the literature. DDSDEs have been extensively studied due to their broad range of applications, as demonstrated in [[2], [8], [1]].

Recently, many papers have been devoted to the study of the existence and uniqueness of solutions for DDSDEs. Let us mention some of these works. For McKean-Vlasov SDEs driven by distribution-dependent Brownian noises, the authors in [11] obtained the existence and uniqueness results under the Lipschitz condition. For the results under the non-Lipschitz condition for non-degenerate DDSDEs, references such as [[9], [18]] provided valuable insights. In [3], Ding et al. established the existence of weak solutions using the Euler approximation, under continuous coefficients and a linear growth condition; they also derived pathwise uniqueness under non-Lipschitz conditions. Additionally, in the works of [[23], [24]], the existence and uniqueness results are established only for the case that the diffusion coefficient is distribution-free. Several works have extended SDEs, backward SDEs, and DDSDEs to cases where the driving noise is not limited to Brownian motion but includes more general processes such as Lévy processes, stochastic integrals with respect to Poisson random measures, or, more generally, local martingales; see, for instance, [[5], [6], [8]], along with the references provided therein.

In this paper, we consider the following generalized DDSDE of the form,

$$\begin{cases} dX_t = & b(t, X_t, \mu_t)\, dt + M(dt, X_t), \\ X_0 = & \xi, \end{cases} \tag{1.1}$$

where $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ and $\{M(t,x)\}_{x\in\mathbb{R}^d}$ is a family of continuous d-dimensional local martingales with spatial parameter $x \in \mathbb{R}^d$ and local characteristic q, and defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ satisfying the usual conditions , $\mu_t := \mathbb{P} \circ X_t^{-1}$ denotes the law of the random variable X_t, and $X_0 = \xi$ is an $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable random variable. Its law is denoted by $\mu_0 := \mathbb{P} \circ \xi^{-1}$. Here $\mathcal{P}_2(\mathbb{R}^d)$ is the space of all probability measures on $\mathbb{R}^d$, endowed with the 2-Wasserstein metric (see Section 2 for details).

This form of continuous local martingales $M(t,x)$ and its corresponding stochastic calculus were initially developed by Kunita in [14] while investigating various stochastic problems related to SDEs and stochastic flows of SDEs driven by Brownian motion with values in the space of vector fields. In [14], Kunita also introduced SDEs driven by these martingales (a distribution-independent form of (1.1)), as an extension of the classical Itô's SDE, and established its existence and uniqueness of strong solutions under the fact that the pair (q, b) satisfies the Lipschitz condition and the linear growth condition via Picard iteration. This result was further extended by Liang in [15] to cover the non-Lipschitz case, using local time and composition techniques of auxiliary functions with appropriate processes. In discrete-time scenarios, Zhang Bo et al. in [25] developed regularity properties such as continuity, differentiability, and integrability for discrete-time martingales with spatial parameters. Zhang Bo et al. in [26] extensively studied nonlinear stochastic difference equations driven by these martingales and their solutions.

The primary focus of our work is to view equation (1.1) as a generalized DDSDE driven by infinite-dimensional Brownian motion (see Remark 2.4). In the first part, we investigate the existence and uniqueness of a strong solution of equation (1.1) under the assumption that the drift coefficient b and the local characteristic q are continuous, bounded, and satisfy non-Lipschitz conditions in the spatial variable, and b is Lipschitz continuous in the distribution variable with respect to the 2-Wasserstein metric. We establish the existence of strong solutions by deducing them from weak solutions, as demonstrated in Lemma 4.2. Furthermore, we prove the pathwise uniqueness of these solutions using Bihari's inequality.

In the second part, we address the existence of optimal controls for systems driven by the state equation

$$\begin{cases} dX_t^U = & b\left(t, X_t^U, \mu_t, U\left(t, X_t^U\right)\right) dt + M\left(dt, X_t^U, U\left(t, X_t^U\right)\right), \\ X_0^U = & \xi, \end{cases}$$

where this time b and the local martingale M are allowed to depend on the control $U(t,x) \in \mathbb{R}^k$ which is represented by a state feedback form. Here, a control is to be understood in a weak sense. Our purpose is to find a control, the so-called optimal control, which minimizes the cost functional

$$J(U) = \mathbb{E}\left[\int_0^T h\left(t, X_t^U, \mu_t, U\left(t, X_t^U\right)\right) dt + \ell\left(X_T^U, \mu_T\right)\right].$$

Adapting the ideas of Haussmann and Lepeltier [10] to our framework, we use the formulation by martingale problems for the above controlled DDSDE, to prove the existence result.

The organization of our paper is as follows. In Section 2, we introduce some facts, which come from Kunita [14], for the local martingale $M(t,x)$ so that equation (1.1) is well defined and state the main theorem (existence and uniqueness of strong solution), and we construct the Euler approximation for the solution of equation (1.1). In Section 3, we introduce the notion of weak solutions for this equation and the corresponding martingale problem. Then, under the condition that q and b are continuous and bounded, we use the Euler approximation and the formulation by the martingale problem to prove the weak existence. In Section 4, under the condition that q and b satisfy non-Lipschitz conditions, we obtain the existence and pathwise uniqueness of its strong solutions. In Section 5, we show that this approximation has an optimal strong convergence rate. In the last section, we are concerned with the existence of optimal feedback control.

2 DDSDE and strong solution

In this section, we examine the McKean-Vlasov Stochastic Differential Equation given by (1.1). We present a theorem that addresses the existence and uniqueness of its strong solution. Additionally, we construct the Euler approximation for the solution of equation (1.1). We shall first introduce some useful notations and assumptions and provide preliminaries on the integrals against local martingales with spatial parameters. For more details on the Itô-Kunita's integral, we refer to ([14], Chapter 03). We denote

- $\mathcal{S}^2([0,T];\mathbb{R}^d)$ to be the set of $\mathbb{R}^d$-valued predictable processes $\{X_t, t\in[0,T]\}$ and satisfy $\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t|^2\right] < +\infty$.
- $\mathcal{P}_2(\mathbb{R}^d)$ to be the space of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with finite second moment, i.e., $\int_{\mathbb{R}^d}|x|^2\mu(dx) < +\infty$.

For $\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$, the 2-Wasserstein distance $W_2(\mu,\nu)$ is defined by:

$$W_2(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)}\left\{\int_{\mathbb{R}^{2d}}|x-y|^2 d\pi(x,y)\right\}^{1/2}, \tag{2.1}$$

where $\Pi(\mu,\nu)$ denotes the set of probability measures on $\mathbb{R}^{2d}$ with μ and ν as respective first and second marginals. These probability measures are often called couplings between μ and ν. In the case $\mu := \mathbb{P}\circ X^{-1}$ and $\nu = \mathbb{P}\circ Y^{-1}$ are the laws of $\mathbb{R}^d$- valued random variables X and Y of order 2, then,

$$W_2(\mu,\nu) \leq \mathbb{E}\left[|X-Y|^2\right]^{1/2}. \tag{2.2}$$

Throughout this paper, we use the notation $\langle .,.\rangle$ to denote the quadratic covariation of two continuous local martingales. Also for two vectors $u,v\in\mathbb{R}^d$, $\langle u,v\rangle$ denotes the scalar product of u and v. $\|.\|$ and $|.|$ denote norms of matrix, respectively, of vectors.

Recall the DDSDE (1.1)

$$\begin{cases} dX_t = & b(t,X_t,\mu_t)\,dt + M(dt,X_t), \\ X_0 = & \xi. \end{cases}$$

Here $M(t,x) = \left(M^1(t,x),....,M^d(t,x)\right)$ is a continuous and $(\mathcal{F}_t)_{t\geq 0}$-adapted local martingale with parameter $x\in\mathbb{R}^d$. Suppose that there exists a predictable process $q^{ij}(s,x,y)$ with parameters (x,y) such that

$$\left\langle M^i(t,x), M^j(t,y)\right\rangle = \int_0^t q^{ij}(s,x,y)\,ds.$$

q is called the local characteristic of $M(t,x)$, where $q(t,x,y) = (q^{ij}(t,x,y))_{1\leq i,j\leq d}$ is a $d\times d$-matrix-valued such that $q^{ij}(t,x,y) = q^{ji}(t,y,x)$ a.s for all $x,y\in\mathbb{R}^d$, $t\in[0,T]$ and $1\leq i,j\leq d$. Moreover, $q(t,x,y)$ is a nonnegative-definite symmetric matrix $a.s$ for all $(t,x,y)\in[0,T]\times\mathbb{R}^{2d}$. If X_t is a $\mathbb{R}^d$-valued predictable process satisfying

$$\int_0^T q^{ij}(t,X_t,X_t)\,dt < \infty, \quad \text{a.s},$$

then the Itô-Kunita's stochastic integral $\int_0^t M(ds,X_s)$ of X_t based on the kernel $M(dt,x)$ is well defined. In particular, if the sample paths of X_t are continuous $a.s$, this integral can be approximated by the following Riemann sums:

$$\int_0^t M(ds,X_s) = \lim_{|\Delta|\to 0}\sum_{k=0}^{n-1}\left[M(t_{k+1}\wedge t, X_{t_k\wedge t}) - M(t_k\wedge t, X_{t_k\wedge t})\right],$$

uniformly in t on $[0,T]$, $\mathbb{P}$-$a.s$, where Δ is a partition of interval $[0,T]$ such that,

$$\begin{aligned} &\Delta = \{0 = t_0 < t_1 < ... < t_n = T\}, \\ &|\Delta| := \max_k \{|t_{k+1} - t_k|\} \to 0, \quad \text{as} \quad n \to \infty. \end{aligned} \tag{2.3}$$

The stochastic process $\int_0^t M(ds, X_s)$ is then a continuous local martingale.

To derive the existence and uniqueness of solutions of (1.1), we make the following assumptions.

(C.1) $b(t,x,\mu)$ and $q(t,x,y)$ are bounded, measurable and continuous in (t,x,μ) and (t,x,y), respectively.

(C.2) Let κ_1 and κ_2 be two positive, concave and increasing functions on $[0,\infty)$, and satisfying $\kappa_1(0) = \kappa_2(0) = 0$, $\int_{0^+} \frac{udu}{\kappa_1^2(u)+\kappa_2^2(u)} = \infty$. Assume that, for any $x, y \in \mathbb{R}^d$, $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $t \in [0,T]$, q and b satisfy

$$\begin{aligned} &|b(t,x,\mu) - b(t,y,\nu)| \leq M(\kappa_1(|x-y|) + W_2(\mu,\nu)), \\ &\|q(t,x,x) - 2q(t,x,y) + q(t,y,y)\| \leq L\kappa_2^2(|x-y|), \end{aligned}$$

where $L, M > 0$ are a constants.

Example 2.1. We give a few examples of the functions κ_1 and κ_2. Let $C > 0$ and let $0 \leq \eta \leq e^{-\frac{1}{e}}$. Define

(i) $\kappa_1(u) = \kappa_1(u) = Cu, \quad u \geq 0.$

(ii) $\kappa_1(u) = \kappa_1(u) = \begin{cases} u\left(\log u^{-1}\right)^{1/2}, & 0 \leq u \leq \eta, \\ \eta\left(\log \eta^{-1}\right)^{1/2} + \left(\log \eta^{-1}\right)^{1/2}\left(1 - \frac{1}{2\log \eta^{-1}}\right)(u-\eta), & \eta < u. \end{cases}$

Then κ_1 and κ_2 satisfy the conditions in **(C.2)**. We observe also that example (i) assures that our results covers the Lipschitz case.

Remark 2.2. Under condition **(C.1),** by Theorem 2.1.1 in ([14], pp. 43) and using Burkholder's inequality, we get that the process $\left(\int_0^t M(ds, X_s), t \in [0,T]\right)$ is a true continuous martingale. Moreover, it is a square integrable continuous (local) martingale.

Now, let's state one of our main results in the following theorem.

Theorem 2.3. *Let conditions (C.1) and (C.2) hold and $X_0 = \xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$. Then the McKean-Vlasov SDE (1.1) has a unique strong solution $X_t \in \mathcal{S}^2([0,T]; \mathbb{R}^d)$.*

Remark 2.4. Let $\left(B^l, ..., B^n\right)$ be n-dimensional standard Brownian motion and let $f_l(t,x,\mu), l = 1, 2, ...$ be $\mathbb{R}^d$- valued, continuous functions. Consider the following DDSDEs

$$X_t = x + \int_0^t b(s, X_s, \mu_s)\,ds + \sum_{l=1}^n \int_0^t f_l(s, X_s)\,dB_s^l. \tag{2.4}$$

Equation (2.4) is equivalent to equation (1.1) by setting $M(t,x) = \sum_{l=1}^n \int_0^t f_l(s,x)\,dB_s^l$. Its local characteristic is $q = \left(\sum_{l=1}^n f_l^i(t,x) f_l^j(t,y)\right)_{1\leq i,j\leq d}$. If b and the local characteristics q satisfy the conditions **(C.1)** and **(C.2)**, then we get existence and uniqueness of strong solutions of the DDSDE (2.4).

On the other hand, let $\left(B^l\right)_{l\in\mathbb{N}}$ be infinite independent copies of one dimensional standard Brownian motions such that the following equation

$$X_t = x + \int_0^t b(s, X_s, \mu_s)\,ds + \sum_{l=1}^\infty \int_0^t f_l(s, X_s)\,dB_s^l,$$

is equivalent to the McKean-Vlasov SDE (1.1) by setting $M(t,x) = \sum_{l=1}^\infty \int_0^t f_l(s,x)\,dB_s^l$. Its local characteristic is $q = \left(\sum_{l=1}^\infty f_l^i(t,x) f_l^j(t,y)\right)_{1\leq i,j\leq d}$. Thus, the equation (1.1) can be formally viewed as an DDSDE driven by infinite-dimensional Brownian motion.

We conclude this section by applying the Euler scheme to approximate the solution of (1.1). For $n > 0$, let Δ be the partition of interval $[0, T]$ and $|\Delta|$ be given by (2.3). For $0 \leq k \leq n-1$, we let $\Delta_t = t_{k+1} - t_k$, it is a fixed timestep. The Euler scheme applied to (1.1) has the form: $X_0^n = \xi$ and

$$X_{t_{k+1}}^n = X_{t_k}^n + b\left(t_k, X_{t_k}^n, \mu_{t_k}^n\right)(t_{k+1} - t_k) + M\left(t_{k+1}, X_{t_k}^n\right) - M\left(t_k, X_{t_k}^n\right),$$

for $0 \leq k \leq n-1$. This is an explicit scheme. It gives all the values of the approximation at the partition points $t_0, t_1, ..., t_n$.

Remark 2.5. Noting that $M(t_k, x)$ can be viewed as a discrete-time martingale with spatial parameters $x \in \mathbb{R}^d$. This type of martingales and their regularity properties were well studied by Zhang Bo et al. in [25]. When $b = 0$, the following nonlinear stochastic difference equations

$$\begin{cases} X_{t_{k+1}}^n = & X_{t_k}^n + M\left(t_{k+1}, X_{t_k}^n\right) - M\left(t_k, X_{t_k}^n\right), \\ X_0^n = & \xi, \end{cases}$$

was investigated by Zhang Bo et al. in [26]. Under that $M(t_k, x)$ is $\mathcal{F}_{t_k}-$adapted and continuous with respect to the spatial parameter x, they proved that this equation has a unique solution which is a stochastic sequence $\left(X_{t_k}^n\right)$ adapted to the filtration $\left(\mathcal{F}_{t_k}\right)$.

To compare the approximation solution with the original solution, we need to know the value of the approximation solution at all time instant. We shall use the following interpolation. Let X^n satisfies

$$X_t^n = X_{t_k}^n + b\left(t_k, X_{t_k}^n, \mu_{t_k}^n\right)(t - t_k) + M\left(t, X_{t_k}^n\right) - M\left(t_k, X_{t_k}^n\right),$$

for any $t \in [t_k, t_{k+1}), 0 \leq k \leq n-1$. In our analysis, it will be more natural to work with the equivalent definition

$$X_t^n = \xi + \int_0^t b\left(s, X_{\eta_n(s)}^n, \mu_{\eta_n(s)}^n\right) ds + \int_0^t M\left(ds, X_{\eta_n(s)}^n\right), t \in [0, T], \tag{2.5}$$

where $\eta_n(s) = t_k$ for any $s \in [t_k, t_{k+1})$, $0 \leq k \leq n-1$. The joint quadratic variation of $\int_0^t M^i\left(ds, X_{\eta_n(s)}^n\right)$ and $\int_0^t M^j\left(ds, X_{\eta_n(s)}^n\right)$ is given by

$$\begin{aligned} &\left\langle \int_0^t M^i\left(ds, X_{\eta_n(s)}^n\right), \int_0^t M^j\left(ds, X_{\eta_n(s)}^n\right)\right\rangle \\ &= \int_0^t q^{ij}\left(s, X_{\eta_n(s)}^n, X_{\eta_n(s)}^n\right) ds. \end{aligned}$$

Since that the drift coefficient b and the local characteristic q are bounded and $X_0 = \xi \in L^2\left(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d\right)$, it is easy to prove that there exists a positive constant C such that the solution X_t^n satisfies

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^n|^2\right] \leq C. \tag{2.6}$$

Indeed, by Burkholder's inequality and Cauchy-Schwartz's inequality, it holds that

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|X_t^n|^2\right] &\le 3\mathbb{E}\left[|\xi|^2+\sup_{0\le t\le T}\left|\int_0^t b\left(s,X^n_{\eta_n(s)},\mu^n_{\eta_n(s)}\right)ds\right|^2\right.\\&\quad\left.+\sup_{0\le t\le T}\left|\int_0^t M\left(ds,X^n_{\eta_n(s)}\right)\right|^2\right]\\&\le 3\left(\mathbb{E}\left[|\xi|^2\right]+T\mathbb{E}\left[\int_0^T\left|b\left(s,X^n_{\eta_n(s)},\mu^n_{\eta_n(s)}\right)\right|^2ds\right]\right.\\&\quad\left.+C_2\mathbb{E}\left[\sum_{i=1}^d\int_0^T q^{ii}\left(s,X^n_{\eta_n(s)},X^n_{\eta_n(s)}\right)ds\right]\right)\\&\le 3\mathbb{E}\left[|\xi|^2\right]+3TC_b+3C_2\mathbb{E}\left[\int_0^T\left\|q\left(s,X^n_{\eta_n(s)},X^n_{\eta_n(s)}\right)\right\|ds\right]\le C.\end{aligned}$$

3 Weak solution

3.1 Weak solution and martingale problem

In this subsection, we introduce the notion of weak solutions of equation (1.1) and the solution for the corresponding martingale problem. This weak solution may be defined as follows.

Definition 3.1. We say that equation (1.1) admits a weak solution with the initial distribution μ_0, if there exists a probability filtered space $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P}\right)$ satisfying the usual conditions, and a continuous $(\mathcal{F}_t)_{t\ge0}$-adapted process X such that

(i) $\mathbb{P}\circ\xi^{-1}=\mu_0$,

(ii) b and the local characteristic q of $M(t,x)$ satisfies

$$\int_0^T\left[\left|b^i(s,X_s,\mu_s)\right|+q^{ij}(s,X_s,X_s)\right]ds<\infty,\ \mathbb{P}\text{-a.s},\ 1\le i,j\le d,$$

(iii) the integral version of (1.1)

$$X_t=\xi+\int_0^t b(s,X_s,\mu_s)\,ds+\int_0^t M(ds,X_s)\,,\ 0\le t\le T,$$

holds almost surely. Such a weak solution is denoted by $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P},X\right)$.

As is well known, weak solutions for classical Itô's SDEs are equivalent to the existence of solutions for the corresponding martingale problem (see [12], [21]). Similarly, in the sequel of this subsection, we will introduce the concepts related to the solution for the (local) martingale problems associated with (1.1).

Let $C^k\left(\mathbb{R}^d\right)$ be the collection of all continuous functions which have continuous partial derivatives of every order up to k where k is a positive integer, and define the infinitesimal generator L associated with equation (1.1)

$$(Lf)(t,x,\mu)=\sum_{i=1}^d b^i(t,x,\mu)\,\partial_i f(x)+\frac{1}{2}\sum_{i,j=1}^d q^{ij}(t,x,x)\,\partial^2_{ij}f(x)\,,\quad f\in C^2\left(\mathbb{R}^d\right).$$

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, X\right)$ be a weak solution of (1.1). By applying Itô's formula to $f(X)$ (see Theorem 2.3.11 in [14], pp. 64), we get that $f(X)$ is a continuous semimartingale with decomposition

$$f(X_t) = f(X_0) + \int_0^t (Lf)(s, X_s, \mu_s)\, ds + \sum_{i=1}^{d} \int_0^t \frac{\partial f(X_s)}{\partial x_i} M^i(ds, X_s).$$

Clearly that the process $\left\{\sum_{i=1}^d \int_0^t \frac{\partial f(X_s)}{\partial x_i} M^i(ds, X_s), t \in [0,T]\right\}$ is a continuous local martingale, because according to Lemma 2.3.1 in ([14], pp. 56), one can get that the stochastic integral $\int_0^t U_s dM(s)$ is a limit of Riemann sums $\sum_{k=0}^{n-1} U_{t_k}(M(t_{k+1}) - M(t_k))$ as $|\Delta| \to 0$, where U standing for $\left(\frac{\partial f(X_s)}{\partial x_1}, ..., \frac{\partial f(X_s)}{\partial x_d}\right)$ and $dM(s) = M(ds, X_s)$, the partition Δ was defined in previous section.

Definition 3.2. X_t is called a solution to the local martingale problem corresponding to equation (1.1), if

$$N_t^f := f(X_t) - f(\xi) - \int_0^t (Lf)(s, X_s, \mu_s)\, ds, \quad f \in C^2(\mathbb{R}^d),$$

is a continuous local $(\mathcal{F}_t)_{t\geq 0}$-martingale under $\mathbb{P}$.

Furthermore, if the weak solution X satisfies the property in the above definition, then its law (denoted by $P = \mathbb{P}\circ X^{-1}$) on the (canonical) space $\left(C([0,T];\mathbb{R}^d), \mathcal{B}(C([0,T];\mathbb{R}^d))\right)$ is also a solution of a local martingale problem (see [12] for details).

For simplicity, we denote $\mathcal{W} := C([0,T];\mathbb{R}^d)$ and $\mathcal{W}_t := C([0,t];\mathbb{R}^d)$, $t \in [0,T]$, equipped with their Borel σ-fields which are denoted by $\mathbb{W} = \mathcal{B}(\mathcal{W})$ and $\tilde{\mathbb{W}}_t = \cap_{s>t}\mathcal{B}(\mathcal{W}_t)$, $t \in [0,T]$, respectively.

Let w_t be the coordinate process over $\mathcal{W}$, that is

$$w_t(\omega) = \omega_t, \omega \in \mathcal{W}.$$

Definition 3.3. A probability measure P on $(\mathcal{W}, \mathbb{W})$, under which

$$N_t^f := f(w_t) - f(\xi) - \int_0^t (Lf)(s, w_s, \mu_s)\, ds,\ f \in C^2(\mathbb{R}^d), \tag{3.1}$$

is a continuous local $\tilde{\mathbb{W}}_t-$martingale, is called a solution to the local martingale problem corresponding to equation (1.1) with the initial law μ_0, where $\mu_s = P \circ w_s^{-1}$denotes the law of w_s under P.

Remark 3.4. In both of the above definitions, if f is in $C_b^2(\mathbb{R}^d)$ (i.e., $f \in C^2(\mathbb{R}^d)$ and has bounded partial derivatives up to the second order), and if both b and q are bounded, then N_t^f is a continuous martingale, and the local martingale problem reduced to a martingale problem.

3.2 Weak existence

The main purpose of this subsection is to prove the existence of a solution of the martingale problem for (1.1) which is equivalent to the existence of its weak solution. Let X^n be a weak solution of the Euler approximation equation (2.5), then there exists a martingale solution $P^n := \mathbb{P} \circ (X^n)^{-1}$ on $(\mathcal{W}, \mathbb{W})$, such that

$$N_t^{f,n} := f(w_t^n) - f(\xi) - \int_0^t (L^n f)(s, w_s^n, \mu_s^n)\, ds,\ f \in C_b^2(\mathbb{R}^d),$$

is a continuous $\tilde{\mathbb{W}}_t-$martingale under P^n, where $\mu_{\eta_n(s)}^n := P^n \circ \left(w_{\eta_n(s)}^n\right)^{-1}$ and

$$(L^n f)(s, w_s^n, \mu_s^n)$$
$$= \sum_{i=1}^d b^i\left(s, w_{\eta_n(s)}^n, \mu_{\eta_n(s)}^n\right) \frac{\partial f(w_s^n)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^d q^{ij}\left(s, w_{\eta_n(s)}^n, w_{\eta_n(s)}^n\right) \frac{\partial^2 f(w_s^n)}{\partial x_i \partial x_j}.$$

We shall now show that the sequence of distributions (P^n) is tight on $\mathcal{W}$.

Lemma 3.5. *Under **(C.1)**, the sequence of distributions (P^n) is tight on the space $\mathcal{W}$ endowed with the topology of uniform convergence.*

Proof. Let $\mathcal{P}_2(\mathcal{W})$ be the space of probability measures on $(\mathcal{W}, \mathbb{W})$ equipped with the 2-Wasserstein metric (2.1). The proof is inspired from [[4], Theorem 3.4], in which to show that $(P^n)_{n\geq 1}$ is tight in $\mathcal{P}_2(\mathcal{W})$, we follow the approach used in [[21], Theorem 1.4.6], by showing that for each positive $f \in C_b^2(\mathbb{R}^d)$, there exists a constant Ξ_f^n such that: $f(w_t^n) + \Xi_f^n.t$ is a supermartingale under the distribution P^n on the canonical space $\mathcal{W}$. Let

$$\Xi_f^n = \sup\left\{\left|(L^n f)(s, w^n, \mu^n)\right|, (s, w^n, \mu^n) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\right\}.$$

By condition **(C.1)**, the coefficient b and q defining the operator $(L^n f)(s, w, \mu)$ are bounded, then Ξ_f^n is finite. Since for each n, $N_t^{f,n} := f(w_t^n) - f(\xi) - \int_0^t (L^n f)(s, w_s^n, \mu_s^n)\, ds$ is a P^n-martingale, then $f(w_t^n) + \Xi_f^n.t$ is a positive supermartingale. Then the sequence (P^n) is tight in $\mathcal{P}_2(\mathcal{W})$. □

Proposition 3.6. *Suppose that **(C.1)** holds and $X_0 = \xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$. Then there exists a solution to the martingale problem of (1.1).*

Proof. Recall that in the above lemma, we have seen that the sequence (P^n) of distributions of (X^n), solutions of (2.5), is tight on $\mathcal{W}$. Then there exists a subsequence still denoted by (P^n) and $P \in \mathcal{P}_2(\mathcal{W})$ such that (P^n) weakly converges to P. Now to prove that P on $(\mathcal{W}, \mathbb{W})$ is a solution of the martingale problem for (1.1), we just need to prove that N_t^f defined by (3.1), for $f \in C_b^2(\mathbb{R}^d)$, is a continuous $\tilde{\mathbb{W}}_t-$martingale under P. That is, for any continuous, bounded and $\tilde{\mathbb{W}}_s$-measurable functional $\mathcal{G}_s$,

$$\mathbb{E}^P\left[\left(N_t^f - N_s^f\right)\mathcal{G}_s\right] =$$
$$\int_{\mathcal{W}}\left[\left(f(w_t) - f(w_s) - \int_s^t (Lf)(r, w_r, \mu_r)\, dr\right)\mathcal{G}_s(w)\right] P(d\omega) = 0,\ 0 \leq s < t \leq T.$$

Since $N_t^{f,n}$ is a continuous $\tilde{\mathbb{W}}_t-$martingale under P^n, then

$$\mathbb{E}^{P^n}\left[\left(N_t^{f,n} - N_s^{f,n}\right)\mathcal{G}_s\right] = 0.$$

Note that P^n weakly converges to P. Let us prove that

$$\int_{\mathcal{W}}\left[\left(f(w_t) - f(w_s) - \int_s^t (Lf)(r, w_r, \mu_r)\, dr\right)\mathcal{G}_s(w)\right] P(d\omega)$$
$$= \lim_{n\to\infty}\int_{\mathcal{W}}\left[\left(f(w_t^n) - f(w_s^n) - \int_s^t (L^n f)(r, w_r^n, \mu_r^n)\, dr\right)\mathcal{G}_s(w^n)\right] P^n(d\omega^n). \tag{3.2}$$

Since (P^n) is tight in $\mathcal{P}_2(\mathcal{W})$, then by Skorokhod's representation theorem (see [14], Theorem 1.1.5, pp. 03), there exists a probability space $\left(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}\right)$ and $\mathcal{W}$-valued processes $\hat{X}$ and $\hat{X}^n$ on it, such that

- **(i)** The law of $\hat{X}^n$ (respectively, of $\hat{X}$) is P^n (respectively, is P), for each $n \geq 1$.
- **(ii)** $\hat{X}^n \to \hat{X}$ in $\mathcal{W}$, $\hat{\mathbb{P}}$-a.s, as $n \to \infty$.

Based on **(i)**, to prove (3.2) it is equivalent to prove that

$$\mathbb{E}^{\hat{\mathbb{P}}}\left[\left(f(\hat{X}_t) - f(\hat{X}_s) - \int_s^t (Lf)(r, \hat{X}_r, \mu_r)\, dr\right)\mathcal{G}_s(\hat{X})\right]$$
$$= \lim_{n\to\infty}\mathbb{E}^{\hat{\mathbb{P}}}\left[\left(f(\hat{X}_t^n) - f(\hat{X}_s^n) - \int_s^t (L^n f)(r, \hat{X}_r^n, \mu_r^n)\, dr\right)\mathcal{G}_s(\hat{X}^n)\right]. \tag{3.3}$$

On one hand, by **(ii)**, it holds that $\hat{X}_{\eta_n(r)}^n \overset{a.s}{\to} \hat{X}_r$ for $r \in [s,t]$ as $n \to \infty$. We also need to show that

$$\lim_{n\to\infty} W_2\left(\mu_{\eta_n(r)}^n, \mu_r\right) = 0, \tag{3.4}$$

indeed, by inequality (2.2), we have $W_2\left(\mu^n_{\eta_n(r)},\mu_r\right)\leq \mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}^n_{\eta_n(r)}-\hat{X}_r\right|^2\right]^{1/2}$, and we observe that for any $\theta>0$,

$$\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}^n_{\eta_n(r)}\right|^2\mathbf{1}_{\left\{\left|\hat{X}^n_{\eta_n(r)}\right|^2>\theta\right\}}\right]\leq \mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}^n_{\eta_n(r)}\right|^2\right].$$

By (2.6) we get

$$\sup_{n\geq 1}\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}^n_{\eta_n(r)}\right|^2\mathbf{1}_{\left\{\left|\hat{X}^n_{\eta_n(r)}\right|^2>\theta\right\}}\right]\leq \sup_{n\geq 1}\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}^n_{\eta_n(r)}\right|^2\right]=\sup_{n\geq 1}\mathbb{E}\left[\left|X^n_{\eta_n(r)}\right|^2\right]<\infty.$$

Then noting that

(i) $\left|\hat{X}^n_{\eta_n(r)}\right|^2\mathbf{1}_{\left\{\left|\hat{X}^n_{\eta_n(r)}\right|^2>\theta\right\}}\leq\left|\hat{X}^n_{\eta_n(r)}\right|^2$, for any $\theta>0$ and $n\geq 1$, i.e $\left|\hat{X}^n_{\eta_n(r)}\right|^2$ dominates $\left|\hat{X}^n_{\eta_n(r)}\right|^2\mathbf{1}_{\left\{\left|\hat{X}^n_{\eta_n(r)}\right|^2>\theta\right\}}$.

(ii) $\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}^n_{\eta_n(r)}\right|^2\right]<\infty$.

(iii) Since $\left|\hat{X}^n_{\eta_n(r)}\right|^2$ is integrable, it follows that $\lim_{\theta\to\infty}\mathbf{1}_{\left\{\left|\hat{X}^n_{\eta_n(r)}\right|^2>\theta\right\}}=0$.

By the dominated convergence theorem, we may deduce that

$$\lim_{\theta\to\infty}\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}^n_{\eta_n(r)}\right|^2\mathbf{1}_{\left\{\left|\hat{X}^n_{\eta_n(r)}\right|^2>\theta\right\}}\right]=\mathbb{E}^{\hat{\mathbb{P}}}\left[\lim_{\theta\to\infty}\left|\hat{X}^n_{\eta_n(r)}\right|^2\mathbf{1}_{\left\{\left|\hat{X}^n_{\eta_n(r)}\right|^2>\theta\right\}}\right]=0,$$

thus,

$$\lim_{\theta\to\infty}\sup_{n\geq 1}\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}^n_{\eta_n(r)}\right|^2\mathbf{1}_{\left\{\left|\hat{X}^n_{\eta_n(r)}\right|^2>\theta\right\}}\right]=\sup_{n\geq 1}\left[\lim_{\theta\to\infty}\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}^n_{\eta_n(r)}\right|^2\mathbf{1}_{\left\{\left|\hat{X}^n_{\eta_n(r)}\right|^2>\theta\right\}}\right]\right]=0,$$

and this proved that $\left\{\left|\hat{X}^n_{\eta_n(r)}\right|^2,n\geq 1\right\}$ is uniformly integrable. By Vitali convergence theorem, we know that $\hat{X}^n_{\eta_n(r)}\overset{a.s}{\to}\hat{X}_r$ and the uniform integrability of $\left\{\left|\hat{X}^n_{\eta_n(r)}\right|^2,n\geq 1\right\}$ imply that $\lim_{n\to\infty}\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}^n_{\eta_n(r)}-\hat{X}_r\right|^2\right]^{1/2}=0$ and furthermore $\lim_{n\to\infty}W_2\left(\mu^n_{\eta_n(r)},\mu_r\right)=0$.

On the other hand, we have $f\in C^2_b\left(\mathbb{R}^d\right)$, then, using Hölder's inequality and condition **(C.1)**, we get

$$\begin{aligned}&\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\int_s^t\sum_{i,j=1}^d q^{ij}\left(r,\hat{X}^n_{\eta_n(r)},\hat{X}^n_{\eta_n(r)}\right)\frac{\partial^2 f\left(\hat{X}^n_r\right)}{\partial x_i\partial x_j}dr\right|\right]\\&\leq d\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_s^t\left(\sum_{i,j=1}^d\left|q^{ij}\left(r,\hat{X}^n_{\eta_n(r)},\hat{X}^n_{\eta_n(r)}\right)\frac{\partial^2 f\left(\hat{X}^n_r\right)}{\partial x_i\partial x_j}\right|^2\right)^{\frac12}dr\right]\\&<\infty.\end{aligned}$$

Thus, by the continuity of q and the dominated convergence theorem we obtain

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}^{\hat{\mathbb{P}}}\left[\left(\int_s^t q^{ij}\left(r,\hat{X}^n_{\eta_n(r)},\hat{X}^n_{\eta_n(r)}\right)\frac{\partial^2 f\left(\hat{X}^n_r\right)}{\partial x_i\partial x_j}dr\right)\mathcal{G}_s\left(\hat{X}^n\right)\right]\\&=\mathbb{E}^{\hat{\mathbb{P}}}\left[\left(\int_s^t q^{ij}\left(r,\hat{X}_r,\hat{X}_r\right)\frac{\partial^2 f\left(\hat{X}_r\right)}{\partial x_i\partial x_j}dr\right)\mathcal{G}_s\left(\hat{X}\right)\right].\end{aligned}\tag{3.5}$$

In the same way we get also

$$\lim_{n\to\infty} \mathbb{E}^{\hat{\mathbb{P}}} \left[\left(\int_s^t b^i \left(r, \hat{X}^n_{\eta_n(r)}, \mu^n_{\eta_n(r)}\right) \frac{\partial f\left(\hat{X}^n_r\right)}{\partial x_i} dr \right) \mathcal{G}_s\left(\hat{X}^n\right) \right] \tag{3.6}$$
$$= \mathbb{E}^{\hat{\mathbb{P}}} \left[\left(\int_s^t b^i \left(r, \hat{X}_r, \mu_r\right) \frac{\partial f\left(\hat{X}_r\right)}{\partial x_i} dr \right) \mathcal{G}_s\left(\hat{X}\right) \right].$$

By **(ii)** it is clear that

$$\mathbb{E}^{\hat{\mathbb{P}}} \left[\left(f\left(\hat{X}_t\right) - f\left(\hat{X}_s\right) \right) \mathcal{G}_s\left(\hat{X}\right) \right]$$
$$= \lim_{n\to\infty} \mathbb{E}^{\hat{\mathbb{P}}} \left[\left(f\left(\hat{X}^n_t\right) - f\left(\hat{X}^n_s\right) \right) \mathcal{G}_s\left(\hat{X}^n\right) \right]. \tag{3.7}$$

Finally, combining (3.5), (3.6) and (3.7) yields (3.3) which in turn gives (3.2) by **(i)**. The proof is thus complete. □

4 Existence and uniqueness of strong solution

This section is to prove Theorem 2.3. In the next lemma, we present a result on the existence of strong solutions deduced from weak solutions.

Remark 4.1. As is well known, the Yamada–Watanabe principle is a fundamental tool for establishing the strong well-posedness of SDEs via the combination of weak existence and pathwise uniqueness. However, these classical techniques do not apply directly to DDSDE (1.1), since the drift coefficient depends explicitly on the law of the solution. Instead, in our work, we adopt an indirect approach: we first construct a weak solution to the DDSDE (1.1) (as shown in the previous section), then fix the flow μ_t and consider the associated SDE with a fixed measure. If this auxiliary SDE satisfies pathwise uniqueness, then together with the previously established weak existence, we can apply the Yamada–Watanabe principle to this associated SDE. By uniqueness in law, this leads to the existence of a strong solution for the original DDSDE (1.1). In the following lemma, inspired by Lemma 3.4 in [11], we present a modified version of the Yamada–Watanabe principle adapted to this context, which allows the deduction of strong existence results for DDSDE (1.1).

Lemma 4.2. *Let $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \left(\tilde{\mathcal{F}}_t\right)_{t\geq 0}, \tilde{\mathbb{P}}, \tilde{X}\right)$ be a weak solution to (1.1) with $\mu_t = \tilde{\mathbb{P}} \circ \tilde{X}_t^{-1}$. If the SDE*

$$X^\mu_t = b\left(t, X^\mu_t, \mu_t\right) dt + M\left(dt, X^\mu_t\right), \ \ X^\mu_0 = \xi, \quad t \in [0, T], \tag{4.1}$$

has a unique strong solution X^μ_t up to life time with value $\mu_0 = \mathbb{P} \circ \xi^{-1}$. Then DDSDE (1.1) has a strong solution.

Proof. Since $\mu_t = \tilde{\mathbb{P}} \circ \tilde{X}_t^{-1}$, $\tilde{X}_t$ is also a weak solution to (4.1). By Yamada-Watanabe principle, the strong uniqueness of (4.1) implies the weak uniqueness, so that X^μ_t is nonexplosive with $\mu_t = \mathbb{P} \circ (X^\mu_t)^{-1}$, $t \geq 0$. Therefore, X^μ_t is a strong solution to (1.1). □

The first part of the proof of Theorem 2.3 (existence of strong solutions) is constructed by the following steps.

(i) Under conditions **(C.1)**-**(C.2)** and by Theorem 1.2 in [15], we get that the SDE (4.1) has a unique nonexplosive strong solution.

(ii) In Proposition 3.6, under condition **(C.1)**, we established the existence of a martingale solution to equation (1.1), which, in turn, guarantees the weak existence of a solution.

Thus, by the results (i)-(ii) and Lemma 4.2, we have shown the existence of strong solutions of equation (1.1).
To accomplish the proof of Theorem 2.3, the pathwise uniqueness for (1.1) is obtained in Proposition 4.3, and since b and q satisfy **(C.1)**, and $X^n_0 = \xi \in L^2\left(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d\right)$, as in (2.6) it is easy to prove that this solution X_t satisfies

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^2\right] \leq \infty. \tag{4.2}$$

Proposition 4.3. *Under conditions **(C.1)** and **(C.2)**, the pathwise uniqueness holds for DDSDE (1.1).*

Proof. Suppose that X and $\tilde{X}$ are two strong solutions to (1.1) with $X_0 = \tilde{X}_0$. Set $Z_t = X_t - \tilde{X}_t$, then

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T} |Z_t|^2\right] &\leq 2\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \left(b\left(s,X_s,\mu_s\right) - b\left(s,\tilde{X}_s,\tilde{\mu}_s\right)\right) ds\right|^2\right] \\
&+ 2\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t M\left(ds,X_s\right) - \int_0^t M\left(ds,\tilde{X}_s\right)\right|^2\right] \\
&= \mathbb{E}\left[\sup_{0\leq t\leq T} I_1\left(t\right)\right] + \mathbb{E}\left[\sup_{0\leq t\leq T} I_2\left(t\right)\right]. \qquad (4.3)
\end{aligned}$$

For $I_2(t)$, by applying Doob's inequality, Burkholder's inequality and condition **(C.2)** gives that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T} I_2\left(t\right)\right] \\
&\leq C_2\mathbb{E}\left[\sum_{i=1}^d \left\langle \int_0^T M^i\left(ds,X_s\right) - \int_0^T M^i\left(ds,\tilde{X}_s\right)\right\rangle\right] \\
&\leq C_2\mathbb{E}\left[\sum_{i=1}^d \int_0^T \left[q^{ii}\left(s,X_s,X_s\right) - 2q^{ii}\left(s,X_s,\tilde{X}_s\right) + q^{ii}\left(s,\tilde{X}_s,\tilde{X}_s\right)\right] ds\right] \\
&\leq C_L\mathbb{E}\left[\int_0^T \kappa_2^2\left(|Z_s|\right) ds\right]. \qquad (4.4)
\end{aligned}$$

For $I_1(t)$, by Cauchy-Schwartz's inequality, condition (**C.2**) and the property of the 2-wasserstien metric (2.2), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T} I_1\left(t\right)\right] \\
&\leq 2T\mathbb{E}\left[\int_0^T \left|b\left(s,X_s,\mu_s\right) - b\left(s,\tilde{X}_s,\tilde{\mu}_s\right)\right|^2 ds\right] \\
&\leq C_M\mathbb{E}\int_0^T \kappa_1^2\left(|Z_s|\right) + W_2^2\left(\mu_s,\tilde{\mu}_s\right) ds \\
&\leq C_M\mathbb{E}\int_0^T \left(\kappa_1^2\left(|Z_s|\right) + \mathbb{E}\left[|Z_s|^2\right]\right) ds. \qquad (4.5)
\end{aligned}$$

Combining (4.3)-(4.5), we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Z_t|^2\right] \leq C\mathbb{E}\int_0^T \left(\kappa_1^2\left(\sup_{0\leq u\leq s} |Z_u|\right) + \kappa_2^2\left(\sup_{0\leq u\leq s} |Z_u|\right) + \mathbb{E}\left[\sup_{0\leq u\leq s} |Z_u|^2\right]\right) ds.$$

Let $g(z) = \kappa_1^2\left(z^{\frac{1}{2}}\right) + \kappa_2^2\left(z^{\frac{1}{2}}\right)$. Since $\frac{\kappa_1(z)}{z}$, $\frac{\kappa_2(z)}{z}$, $(\kappa_1')_+(z)$ and $(\kappa_2')_+(z)$ are nonnegative and non-increasing functions, the right derivative of g, that is, $g_+'(z) = \left(\frac{\kappa_1\left(z^{\frac{1}{2}}\right)}{z^{\frac{1}{2}}}\right)(\kappa_1')_+(z) + \left(\frac{\kappa_2\left(z^{\frac{1}{2}}\right)}{z^{\frac{1}{2}}}\right)(\kappa_2')_+(z)$ is a nonnegative, non-increasing function, therefore g is a nonnegative, concave and increasing. Thus, by Jensen's inequality, we have

$$\mathbb{E}\left[\sup_{0\le t\le T}|Z_t|^2\right] \le C\mathbb{E}\int_0^T \left(g\left(\mathbb{E}\left[\sup_{0\le u\le s}|Z_u|^2\right]\right) + \mathbb{E}\left[\sup_{0\le u\le s}|Z_u|^2\right]\right) ds.$$

Since $\int_0^\delta \frac{dz}{g(z)} = 2\int_0^{\delta^{\frac{1}{2}}} \frac{ydy}{\kappa_1^2(y)+\kappa_2^2(y)} = +\infty$, then by Lemma 144 (see [20], pp113), we get that $g(z)+z$ is a nonnegative, concave and increasing, and satisfies $\int_{0^+} \frac{dz}{g(z)+z} = +\infty$. Consequently, by Bihari's inequality (see [16], Lemma 3.6) we get $\mathbb{E}\left[\sup_{0\le t\le T}|Z_t|^2\right] = 0$ for all $t \in [0,T]$. Therefore $X_t = \tilde{X}_t$ $a.s$ for every $t \ge 0$ because T is arbitrary. Thus $\mathbb{P}\left(X_t = \tilde{X}_t \text{ for all } t \ge 0\right) = 1$ due to the fact that X_t and $\tilde{X}_t$ are continuous stochastic processes. The proof of Proposition 4.3 is completed. □

5 Rate of convergence for the Euler scheme

In this section, we show that the Euler approximation (2.5) has an optimal strong convergence rate.

Proposition 5.1. *Assume that q and b satisfy **(C.1)** and*

(C.3) *Let $\sigma_1(x)$ and $\sigma_2(x)$ be a positive continuous functions, bounded on $[1,\infty[$, and satisfying $\lim_{x\downarrow 0} \frac{\sigma_i(x)}{\log(x^{-1})} = \gamma_i < \infty$, $i = 1,2$. Assume that, for any $x, y \in \mathbb{R}^d$, $\mu, \nu \in \mathcal{P}_2\left(\mathbb{R}^d\right)$ and $t \in [0,T]$, q and b satisfy*

$$\begin{aligned} &|b(t,x,\mu) - b(t,y,\nu)| \le k\left(|x-y|\,\sigma_1(|x-y|) + W_2(\mu,\nu)\right), \\ &\|q(t,x,x) - 2q(t,x,y) + q(t,y,y)\| \le k|x-y|^2\sigma_2(|x-y|), \end{aligned}$$

where $k > 0$ is a constant, and $\mathbb{E}\left[|\xi|^2\right] < \infty$. Then, for the Euler approximation (2.5) the estimate

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t^n - X_t|\right] \le C\Delta_t^{1/2},$$

holds, where the constant C does not depend on Δ_t.

Proof. Set $Y_t^n := X_t^n - X_t$ and then Y_t^n satisfies the following equation

$$Y_t^n = \int_0^t \left(b\left(s, X_{\eta_n(s)}^n, \mu_{\eta_n(s)}^n\right) - b(s, X_s, \mu_s)\right) ds + \left(\int_0^t M\left(ds, X_{\eta_n(s)}^n\right) - \int_0^t M(ds, X_s)\right).$$

By Itô's formula we obtain that

$$\begin{aligned}
|Y_t^n|^2 &= 2\int_0^t \left\langle Y_s^n, b\left(s, X_{\eta_n(s)}^n, \mu_{\eta_n(s)}^n\right) - b\left(s, X_s, \mu_s\right)\right\rangle ds \\
&+ 2\int_0^t \left\langle Y_s^n, M\left(ds, X_{\eta_n(s)}^n\right) - M\left(ds, X_s\right)\right\rangle \\
&+ \sum_{i=1}^d \left\langle \int_0^t M^i\left(ds, X_{\eta_n(s)}^n\right) - \int_0^t M^i\left(ds, X_s\right)\right\rangle \\
&= A_1(t) + A_2(t) + A_3(t).
\end{aligned}$$

For $T > 0$, $A_1(t)$ and $A_3(t)$, by condition **(C.3),** Young's inequality, Fubini theorem, linearity of expectation and inequality (2.2), we obtain

$$\begin{aligned}
&\mathbb{E}\left[A_1(t)\right] \\
&\leq 2\mathbb{E}\left[\int_0^t |Y_s^n| \left|b\left(s, X_{\eta_n(s)}^n, \mu_{\eta_n(s)}^n\right) - b\left(s, X_s, \mu_s\right)\right| ds\right] \\
&\leq 2\mathbb{E}\left[\int_0^t \left(|Y_s^n| \left|b\left(s, X_{\eta_n(s)}^n, \mu_{\eta_n(s)}^n\right) - b\left(s, X_s^n, \mu_s^n\right)\right| + |Y_s^n| \left|b\left(s, X_s^n, \mu_s^n\right) - b\left(s, X_s, \mu_s\right)\right|\right) ds\right] \\
&\leq \mathbb{E}\left[\int_0^t \left(|Y_s^n|^2 + \left|b\left(s, X_{\eta_n(s)}^n, \mu_{\eta_n(s)}^n\right) - b\left(s, X_s^n, \mu_s^n\right)\right|^2\right) ds\right] \\
&+ 2k\mathbb{E}\left[\int_0^t \left(|Y_s^n|^2 \sigma_1\left(Y_s^n\right) + |Y_s^n| W_2\left(\mu_s^n, \mu_s\right)\right) ds\right] \\
&\leq \mathbb{E}\left[\int_0^t |Y_s^n|^2 ds\right] + 2k^2\mathbb{E}\left[\int_0^t \left(\left|X_{\eta_n(s)}^n - X_s^n\right|^2 \sigma_1^2\left(\left|X_{\eta_n(s)}^n - X_s^n\right|\right) + W_2^2\left(\mu_{\eta_n(s)}^n, \mu_s^n\right)\right) ds\right] \\
&+ 2k\mathbb{E}\left[\int_0^t \left(|Y_s^n|^2 \sigma_1\left(|Y_s^n|\right) + \frac{|Y_s^n|^2}{2} + \frac{W_2^2\left(\mu_s^n, \mu_s\right)}{2}\right) ds\right] \\
&\leq \mathbb{E}\left[\int_0^t |Y_s^n|^2 ds\right] + 2k^2\mathbb{E}\left[\int_0^t \left|X_{\eta_n(s)}^n - X_s^n\right|^2 \sigma_1^2\left(\left|X_{\eta_n(s)}^n - X_s^n\right|\right) ds\right] \\
&+ 2k^2\int_0^t \mathbb{E}\left[\left|X_{\eta_n(s)}^n - X_s^n\right|^2\right] ds + 2k\mathbb{E}\left[\int_0^t |Y_s^n|^2 \sigma_1\left(|Y_s^n|\right) ds\right] + 2k\int_0^t \mathbb{E}\left[|Y_s^n|^2\right] ds \\
&\leq C_k\mathbb{E}\left[\int_0^t |Y_s^n|^2 ds\right] + C_1\mathbb{E}\left[\int_0^t \left|X_{\eta_n(s)}^n - X_s^n\right|^2 \sigma_1^2\left(\left|X_{\eta_n(s)}^n - X_s^n\right|\right) ds\right] \\
&+ C_1\int_0^t \mathbb{E}\left[\left|X_{\eta_n(s)}^n - X_s^n\right|^2\right] ds + C_1\mathbb{E}\left[\int_0^t |Y_s^n|^2 \sigma_1\left(|Y_s^n|\right) ds\right],
\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[A_3(t)\right] &= \mathbb{E}\left[\sum_{i=1}^{d}\left\langle \int_0^t M^i\left(ds, X^n_{\eta_n(s)}\right) - \int_0^t M^i\left(ds, X^n_s\right) + \int_0^t M^i\left(ds, X^n_s\right) - \int_0^t M^i\left(ds, X_s\right)\right\rangle\right] \\ &\leq 2\mathbb{E}\left[\sum_{i=1}^{d}\left\langle \int_0^t M^i\left(ds, X^n_{\eta_n(s)}\right) - \int_0^t M^i\left(ds, X^n_s\right)\right\rangle\right] \\ &\quad + 2\mathbb{E}\left[\sum_{i=1}^{d}\left\langle \int_0^t M^i\left(ds, X^n_s\right) - \int_0^t M^i\left(ds, X_s\right)\right\rangle\right] \\ &\leq 2\mathbb{E}\left[\int_0^t \left\| q\left(s, X^n_{\eta_n(s)}, X^n_{\eta_n(s)}\right) - 2q\left(s, X^n_{\eta_n(s)}, X^n_s\right) + q\left(s, X^n_s, X^n_s\right)\right\| ds\right] \\ &\quad + 2\mathbb{E}\left[\int_0^t \left\| q\left(s, X^n_s, X^n_s\right) - 2q\left(s, X^n_s, X_s\right) + q\left(s, X_s, X_s\right)\right\| ds\right] \\ &\leq C_2\mathbb{E}\left[\int_0^t \left|X^n_{\eta_n(s)} - X^{(n)}_s\right|^2 \sigma_2\left(\left|X^n_{\eta_n(s)} - X^n_s\right|\right) ds\right] + C_2\mathbb{E}\left[\int_0^t \left|Y^n_s\right|^2 \sigma_2\left(\left|Y^n_s\right|\right) ds\right].\end{aligned}$$

Since the functions σ_i, $i = 1, 2$ satisfy the condition in (**C.3**), then, by the same technique using in the proof of Lemma 2.1 in **[19]**, there exist an $0 < \eta < \frac{1}{e}$ such that

$$\begin{aligned} c_2 x \sigma_i(x) &\leq \kappa_\eta(x) \\ c_2 x^2 \sigma_i(x) &\leq \kappa_\eta\left(x^2\right), \ i = 1, 2, \quad x > 0, \end{aligned}$$

where

$$\kappa_\eta(x) = \begin{cases} 0, & x = 0, \\ x \log x^{-1}, & 0 < x \leq \eta, \\ \left(\log \eta^{-1} - 1\right) x + \eta, & \eta < x, \end{cases}$$

is a positive, concave, strictly increasing function on $[0, \infty)$, and satisfying $\kappa_\eta(0) = 0$, and $\int_{0^+} \frac{du}{\kappa_\eta(u) + u} = +\infty$. Then

$$\begin{aligned} &\mathbb{E}\left[A_1(t)\right] + \mathbb{E}\left[A_3(t)\right] \\ &\leq C_k \mathbb{E}\left[\int_0^t \left|Y^n_s\right|^2 ds\right] + C_1 \mathbb{E}\left[\int_0^t \kappa^2_\eta\left(\left|X^n_{\eta_n(s)} - X^n_s\right|\right) ds\right] + C_1 \int_0^t \mathbb{E}\left[\left|X^n_{\eta_n(s)} - X^n_s\right|^2\right] ds \\ &C_2 \mathbb{E}\left[\int_0^t \kappa_\eta\left(\left|X^n_{\eta_n(s)} - X^n_s\right|^2\right) ds\right] + \left(C_1 + C_2\right) \mathbb{E}\left[\int_0^t \kappa_\eta\left(\left|Y^n_s\right|^2\right) ds\right], \end{aligned}$$

and furthermore, by Jensen's inequality,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T} A_1(t)\right]+\mathbb{E}\left[\sup_{0\le t\le T} A_3(t)\right]\\
&\le C_k\int_0^T \mathbb{E}\left[\sup_{0\le u\le s}|Y_u^n|^2\right]ds + C_1\int_0^T \kappa_\eta^2\left(\mathbb{E}\left[\sup_{0\le u\le s}\left|X_{\eta_n(u)}^n - X_u^n\right|^2\right]^{1/2}\right)ds\\
&+C_1\int_0^T \mathbb{E}\left[\sup_{0\le u\le s}\left|X_{\eta_n(u)}^n - X_u^n\right|^2\right]ds\\
&+C_2\int_0^T \kappa_\eta\left(\mathbb{E}\left[\sup_{0\le u\le s}\left|X_{\eta_n(u)}^n - X_u^n\right|^2\right]\right)ds+(C_1+C_2)\int_0^T\kappa_\eta\left(\mathbb{E}\left[\sup_{0\le u\le s}|Y_u^n|^2\right]\right)ds. \quad (5.1)
\end{aligned}$$

For $A_2(t)$, using condition **(C.3)**, Burkholder's inequality, Young's inequality and (2.2) gives that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T} A_2(t)\right]\\
&=2\mathbb{E}\left[\sup_{0\le t\le T}\int_0^t\left\langle Y_s^n, M\left(ds, X_{\eta_n(s)}^n\right)-M\left(ds,X_s^n\right)\right\rangle+\sup_{0\le t\le T}\int_0^t\left\langle Y_s^n, M\left(ds,X_s^n\right)-M\left(ds,X_s\right)\right\rangle\right]\\
&\le 2C\mathbb{E}\left[\sum_{i=1}^d\int_0^T\left|Y_s^{n,i}\right|^2\left(q^{ii}\left(s,X_{\eta_n(s)}^n,X_{\eta_n(s)}^n\right)-2q^{ii}\left(s,X_{\eta_n(s)}^n,X_s^n\right)+q^{ii}\left(s,X_s^n,X_s^n\right)\right)ds\right]^{1/2}\\
&+2C\mathbb{E}\left[\sum_{i=1}^d\int_0^T\left|Y_s^{n,i}\right|^2\left(q^{ii}\left(s,X_s^n,X_s^n\right)-2q^{ii}\left(s,X_s^n,X_s,\right)+q^{ii}\left(s,X_s,X_s\right)\right)ds\right]^{1/2}\\
&\le 2C\mathbb{E}\left[\sup_{0\le s\le T}|Y_s^n|^2\int_0^T\left\|q\left(s,X_{\eta_n(s)}^n,X_{\eta_n(s)}^n\right)-2q\left(s,X_{\eta_n(s)}^n,X_s^n\right)+q\left(s,X_s^n,X_s^n\right)\right\|ds\right]^{1/2}\\
&+2C\mathbb{E}\left[\sup_{0\le s\le T}|Y_s^n|^2\int_0^T\left\|q\left(s,X_s^n,X_s^n\right)-2q\left(s,X_s^n,X_s\right)+q\left(s,X_s,X_s\right)\right\|ds\right]^{1/2}\\
&\le\frac{1}{2}\mathbb{E}\left[\sup_{0\le s\le T}|Y_s^n|^2\right]+C_2\mathbb{E}\left[\int_0^T\left|X_{\eta_n(s)}^n-X_s^n\right|^2\sigma_2\left(\left|X_{\eta_n(s)}^n-X_s^n\right|\right)ds\right]\\
&+C_2\mathbb{E}\left[\int_0^T|Y_s^n|^2\sigma_2\left(|Y_s^n|\right)ds\right]\\
&\le\frac{1}{2}\mathbb{E}\left[\sup_{0\le s\le T}|Y_s^n|^2\right]+C_2\int_0^T\kappa_\eta\left(\mathbb{E}\left[\sup_{0\le u\le s}\left|X_{\eta_n(u)}^n-X_u^n\right|^2\right]\right)ds\\
&+C_2\int_0^T\kappa_\eta\left(\mathbb{E}\left[\sup_{0\le u\le s}|Y_u^n|^2\right]\right)ds. \quad (5.2)
\end{aligned}$$

Combining (5.1) and (5.2), we get

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n|^2\right] &\le C\int_0^T \mathbb{E}\left[\sup_{0\le u\le s}|Y_u^n|^2\right]ds + C\int_0^T \kappa_\eta\left(\mathbb{E}\left[\sup_{0\le u\le s}|Y_u^n|^2\right]\right)ds\\ &+ C\int_0^T \kappa_\eta^2\left(\mathbb{E}\left[\sup_{0\le u\le s}\left|X^n_{\eta_n(u)}-X_u^n\right|^2\right]^{1/2}\right)ds\\ &+ C\int_0^T \kappa_\eta\left(\mathbb{E}\left[\sup_{0\le u\le s}\left|X^n_{\eta_n(u)}-X_u^n\right|^2\right]\right)ds + C\int_0^T \mathbb{E}\left[\sup_{0\le u\le s}\left|X^n_{\eta_n(u)}-X_u^n\right|^2\right]ds,\end{aligned}$$

where C is a constant depending on C_1 and C_2. Next, for $u\in[t_k,t_{k+1})$, $0\le k\le n-1$,

$$X_u^n = X_{t_k}^n + \int_{t_k}^u b\left(r, X^n_{\eta_n(r)}, \mu^n_{\eta_n(r)}\right)dr + \int_{t_k}^u M\left(dr, X^n_{\eta_n(r)}\right).$$

Then, by Burkholder's inequality, Cauchy-Schwartz's inequality and **(C.1)** admits us to get

$$\begin{aligned}&\mathbb{E}\left[\sup_{t_k\le u<t_{k+1}}\left|X^n_{\eta_n(u)}-X_u^n\right|^2\right]\\ &\le 2\Delta_t\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left|b\left(r,X^n_{\eta_n(r)},\mu^n_{\eta_n(r)}\right)\right|^2dr\right] + C_2\mathbb{E}\left[\sum_{i=1}^d\int_{t_k}^{t_{k+1}} q^{ii}\left(r,X^n_{\eta_n(r)},X^n_{\eta_n(r)}\right)dr\right]\\ &\le C_b\Delta_t^2 + C_2\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left\|q\left(r,X^n_{\eta_n(r)},X^n_{\eta_n(r)}\right)\right\|dr\right]\\ &\le C\Delta_t,\end{aligned}$$

where the constant C is independent of Δ_t. Thus, we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n|^2\right] &\le C\int_0^T \mathbb{E}\left[\sup_{0\le u\le s}|Y_u^n|^2\right]ds + C\int_0^T\kappa_\eta\left(\mathbb{E}\left[\sup_{0\le u\le s}|Y_u^n|^2\right]\right)ds\\ &+ CT\kappa_\eta^2\left(C\Delta_t^{1/2}\right) + CT\kappa_\eta\left(C\Delta_t\right) + CT\Delta_t\\ &\le C\int_0^T\left(\mathbb{E}\left[\sup_{0\le u\le s}|Y_u^n|^2\right] + \kappa_\eta\left(\mathbb{E}\left[\sup_{0\le u\le s}|Y_u^n|^2\right]\right)\right)ds\\ &+ CT\kappa_\eta^2\left(C\Delta_t^{1/2}\right) + CT\kappa_\eta\left(C\Delta_t\right) + CT\Delta_t.\end{aligned}$$

By Lemma 2.1 in [27] it holds that

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n|^2\right] \le A^{\exp(-CT)},$$

where $A = CT\kappa_\eta^2\left(C\Delta_t^{1/2}\right) + CT\kappa_\eta\left(C\Delta_t\right) + CT\Delta_t$. Thus, we get

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t^n - X_t|^2\right] \le C\Delta_t.$$

The proof is thus complete. □

6 Optimal feedback control problem

6.1 Problem statement

In this subsection, we present the formulation of our mean-field optimal feedback control problem. Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \ \mathbb{P}\right)$ be a filtered probability space satisfying the usual conditions. Let $\mathbb{U}$ be some subset of $\mathbb{R}^k$. We deal with the optimal control problem

$$\begin{cases} dX_t = & b\left(t, X_t^U, \mu_t, U\left(t, X_t^U\right)\right) dt + M\left(dt, X_t^U, U\left(t, X_t^U\right)\right), \\ X_0 = & \xi, \end{cases} \tag{6.1}$$

with the cost functional

$$J(U) = \mathbb{E}\left[\int_0^T h\left(t, X_t^U, \mu_t, U\left(t, X_t^U\right)\right) dt + \ell\left(X_T^U, \mu_T\right)\right], \tag{6.2}$$

where $U(t,x) \in \mathbb{U}$ is a feedback control, $\mu_t := \mathbb{P}_{X_t^U}$, the functions $(b,h) : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2\left(\mathbb{R}^d\right) \times \mathbb{U} \to \left(\mathbb{R}^d, \mathbb{R}_+\right)$ and $\ell : \mathbb{R}^d \times \mathcal{P}_2\left(\mathbb{R}^d\right) \to \mathbb{R}_+$, and $M(t,x,u) = \left(M^1(t,x,u), ..., M^d(t, x, u)\right)$ is a continuous $(\mathcal{F}_t)_{t\geq 0}$-adapted local martingale with parameter $(x,u) \in \mathbb{R}^d \times \mathbb{U}$. For $1 \leq i, j \leq d$, the joint quadratic of $M^i(t,x,u)$ and $M^j(t,y,v)$ is given by

$$\left\langle M^i(t,x,u), M^j(t,y,v)\right\rangle = \int_0^t q^{ij}(s,x,u,y,v)\, ds.$$

We now state our main assumptions for the mappings in this part.

(C.4) The functions b and q are measurable and continuous in (t,x,μ,u) and (t,x,u,y,v), respectively, and satisfy for $x, u \in \mathbb{R}^d \times \mathbb{U}$, $\mu \in \mathcal{P}_2\left(\mathbb{R}^d\right)$ and $t \in [0,T]$,

$$|b(t,x,u,\mu)| \leq M_1\left(1 + |x| + |u| + W_2(\mu, \delta_0)\right),$$
$$\|q(t,x,u,x,u)\| \leq g(t)\left(1 + |x|^2 + |u|^2\right),$$

where $g(t)$ is a non-negative non-random function with $\int_0^t g(s)\, ds < \infty$, $M_1 > 0$ is a constant and δ_0 is the Dirac measure at 0.

(C.5) The functions h and ℓ are nonnegatives, measurables and continuous in (t,x,μ,u) and (x,μ), respectively, and satisfy for $x, u \in \mathbb{R}^d \times \mathbb{U}$, $\mu \in \mathcal{P}_2\left(\mathbb{R}^d\right)$ and $t \in [0,T]$,

$$|h(t,x,\mu,u)| \leq M_2\left(1 + |x| + W_2(\mu,\delta_0) + |u|\right),$$
$$|\ell(x,\mu)| \leq M_2\left(1 + |x| + W_2(\mu,\delta_0)\right)$$

where $M_2 > 0$ is a constant.

Since our aim is to prove the existence of optimal feedback controls, we give here its definition as follows.

Definition 6.1.

(1) Admissible controls are defined as functions $U(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{U}$ satisfying the following condition.

- **(i)** For each control $U(t,x)$, equation (6.1) with initial condition $X_0 = \xi$ has a weak solution.

(2) To obtain sufficient conditions for the existence of an optimal control, we assume

- **(ii)** There exists an $M_3 > 0$ such that, for any $t \in [0,T]$ and $x \in \mathbb{R}^d$,

$$|U(t,x)| \leq M_3(1 + |x|).$$

(iii) For each $R > 0$, there exists a constant $M_R > 0$ (depending on R) such that, for $t, s \in [0, T]$ and $x_1, x_2 \in \mathbb{R}^d$ with $|x_1| < R$ and $|x_2| < R$,

$$|U(t, x_1) - U(s, x_2)| \leq M_R (|t - s| + |x_1 - x_2|).$$

(3) An admissible control $U^*(t, x)$ is said to be optimal and the corresponding solution $X_t^{U^*}$ of equation (6.1) is referred to as an optimal solution if the pair $\left(U^*\left(t, X_t^{U^*}\right), X_t^{U^*}\right)$ satisfies

$$J(U^*) = \mathbb{E}\left[\int_0^T h\left(t, X_t^{U^*}, \mu_t^*, U^*\left(t, X_t^{U^*}\right)\right) dt + \ell\left(X_T^{U^*}, \mu_T^*\right)\right] = \inf_{U \in \mathcal{U}} J(U).$$

We denote by $\mathcal{U}$ the set of controls satisfying conditions **(i)**-**(iii)**, and we call it admissible for the control problem [(6.1)-(6.2)].

Since the weak solutions of equation (6.1) is equivalent to the existence of solutions for the corresponding martingale problem, then we introduce the following definition which is equivalent to Definition 3.2

Definition 6.2. Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, X^U\right)$ be a weak solution of equation (6.1). We say that X^U is a solution of the martingale problem if:

$$f\left(X_t^U\right) - f(\xi) - \int_0^t (Lf)\left(s, X_s^U, \mu_s, U\left(s, X_s^U\right)\right) ds, \ f \in C_b^2\left(\mathbb{R}^d\right),$$

is a continuous $\mathbb{P}$-martingale, where

$$\begin{aligned}(Lf)\left(s, X_s^U, \mu_s, U\left(s, X_s^U\right)\right) &= \sum_{i=1}^d b^i\left(s, X_s^U, \mu_s, U\left(s, X_s^U\right)\right) \frac{\partial f\left(X_s^U\right)}{\partial x_i} \\ &+ \frac{1}{2} \sum_{i,j=1}^d q^{ij}\left(s, X_s^U, U\left(s, X_s^U\right), X_s^U, U\left(s, X_s^U\right)\right) \frac{\partial^2 f\left(X_s^U\right)}{\partial x_i \partial x_j}. \qquad (6.3)\end{aligned}$$

6.2 Existence of optimal feedback control

In the following theorem, we state our last main result of this paper.

Theorem 6.3. *Let assumptions **(C.4)** and **(C.5)** hold and* $\mathbb{E}\left[|\xi|^2\right] < \infty$. *Then the feedback control problem [(6.1)-(6.2)] has a solution in the class of admissible controls $\mathcal{U}$, i.e., there exists an admissible control $U^*(t, x)$ minimizing the cost function (6.2).*

Proof. Firstly, we have $J(U) \geq 0$, then $J(U)$ has nonnegative lower bounds, which also implies that there exists a sequence of admissible controls $U^n(t, x) \in \mathcal{U}$ such that

$$\lim_{n \to \infty} J(U^n) = \inf_{U \in \mathcal{U}} J(U).$$

For this sequence $(U^n(t, x))$ and by Definition 6.1:

(1) The linear growth condition **(ii)** implies that the sequence $(U^n(t, x))$ is uniformly locally bounded, i.e.,

$$\sup_{n \geq 0} |U^n(t, x)| \leq M_3 (1 + |x|) \leq M_3 (1 + R),$$

holds for any $t \in [0, T]$ and $x \in \mathbb{R}^d$ with $|x| < R$.

(2) The local Lipschitz condition **(iii)** yields that $U^n(t, x)$ is uniformly equicontinuous, i.e,

$$\begin{aligned}&\sup_{\substack{n \geq 0 \\ |t-s| < \delta_1, |x_1 - x_2| < \delta_2}} |U^n(t, x_1) - U^n(s, x_2)| \\ &\leq M_R (|t - s| + |x_1 - x_2|) \\ &\leq M_R (\delta_1 + \delta_2) \to 0, \text{ as } \delta_1, \delta_2 \to 0,\end{aligned}$$

for $t, s \in [0, T]$ and $x_1, x_2 \in \mathbb{R}^d$ with $|x_1| < R$ and $|x_2| < R$.

Since relations **(1)** and **(2)** satisfy the conditions of Arzelà-Ascoli theorem, then the sequence $(U^n(t,x))$ has a subsequence (still denoted by $(U^n(t,x))$) which converges to $U^*(t,x)$ pointwise for any $t \in [0,T]$ and $x \in \mathbb{R}^d$. Clearly that $U^*(t,x)$ satisfies conditions **(i)** and **(ii)** in Definition 6.1.

Now, let us show that there exists a weak solution of equation (6.1) with the control $U^*(t,x)$.

By condition **(i)** in Definition 6.1 and since that $U^n(t,x)$ are admissible controls, we know that there exists a weak solution $X_t^{U^n}$ of the form

$$X_t^{U^n} = \xi + \int_0^t b\left(s, X_s^{U^n}, \mu_s^n, U^n\left(s, X_s^{U^n}\right)\right) ds + \int_0^t M\left(ds, X_s^{U^n}, U^n\left(s, X_s^{U^n}\right)\right),$$

where $\mu_s^n := \mathbb{P}_{X_s^{U^n}}$. We need to show that these solutions $X_t^{U^n}$ are:

(A) uniformly bounded, i.e., $\lim_{R\to\infty}\sup_{n\geq 0}\mathbb{P}\left(\left|X_t^{U^n}\right| > R\right) = 0$,

(B) and uniformly continuous, i.e., $\lim_{h\to 0}\sup_{\substack{n\geq 0\\ |t-s|<h}}\mathbb{P}\left(\left|X_t^{U^n} - X_s^{U^n}\right| > \varepsilon\right) = 0$, for $t,s \in [0,T]$, $s<t$ and for any $\varepsilon > 0$.

For relation **(A)**, by Bulkhorder's inequality, Hölder's inequality, theorem of Fubini and the uniform linear growth condition **(C.4),** it holds that

$$\begin{aligned}
\mathbb{E}\left[\left|X_t^{U^n}\right|^2\right] &\leq 3\Bigg(\mathbb{E}\left[|\xi|^2\right] + \mathbb{E}\left[\left|\int_0^t b\left(s, X_s^{U^n}, \mu_s^n, U^n\left(s, X_s^{U^n}\right)\right) ds\right|^2\right] \\
&\quad + \mathbb{E}\left[\left|\int_0^t M\left(ds, X_s^{U^n}, U^n\left(s, X_s^{U^n}\right)\right)\right|^2\right]\Bigg) \\
&\leq 3\Bigg(\mathbb{E}\left[|\xi|^2\right] + T\mathbb{E}\left[\int_0^t 3M_1^2\left(1 + \left|X_s^{U^n}\right|^2 + \left|U^n\left(s, X_s^{U^n}\right)\right|^2 + W_2^2\left(\mu_s^n, \delta_0\right)\right) ds\right] \\
&\quad + C_2\mathbb{E}\left[\int_0^t \left\|q\left(s, X_s^{U^n}, U^n\left(s, X_s^{U^n}\right), X_s^{U^n}, U^n\left(s, X_s^{U^n}\right)\right)\right\| ds\right]\Bigg) \\
&\leq 3\Bigg(\mathbb{E}\left[|\xi|^2\right] + T\mathbb{E}\left[\int_0^t 3M_1^2\left(1 + \left|X_s^{U^n}\right|^2 + \left|U^n\left(s, X_s^{U^n}\right)\right|^2 + W_2^2\left(\mu_s^n, \delta_0\right)\right) ds\right] \\
&\quad + C_2\mathbb{E}\left[\int_0^t g(s)\left(1 + \left|X_s^{U^n}\right|^2 + \left|U^n\left(s, X_s^{U^n}\right)\right|^2\right) ds\right]\Bigg).
\end{aligned}$$

By the property of the 2-Wasserstien metric (2.2) and the linear growth condition **(ii)** of U^n, we get

$$\begin{aligned}
\mathbb{E}\left[\left|X_t^{U^n}\right|^2\right] &\leq 3\Bigg(\mathbb{E}\left[|\xi|^2\right] + T\mathbb{E}\left[\int_0^t 3M_1^2\left(1 + 2\left|X_s^{U^n}\right|^2 + 2M_3^2\left(1 + \left|X_s^{U^n}\right|^2\right)\right) ds\right] \\
&\quad + C_2\mathbb{E}\left[\sup_{0\leq t\leq T} g(t)\int_0^t \left(1 + \left|X_s^{U^n}\right|^2 + 2M_3^2\left(1 + \left|X_s^{U^n}\right|^2\right)\right) ds\right]\Bigg) \\
&\leq 3\left(\mathbb{E}\left[|\xi|^2\right] + \left(6\left(TM_1\right)^2 + C_2T\right)\left(1 + M_3^2\right)\right) \\
&\quad + 3\left(6TM_1^2\left(1 + M_3^2\right) + C_2\left(1 + 2M_3^2\right)\right)\int_0^t \mathbb{E}\left[\left|X_s^{U^n}\right|^2\right] ds \\
&\leq L + N\mathbb{E}\left[\int_0^t \left|X_s^{U^n}\right|^2 ds\right], 0 \leq t \leq T,
\end{aligned}$$

where $L = 3\left(\mathbb{E}\left[|\xi|^2\right] + \left(6\left(TM_1\right)^2 + C_2T\right)\left(1 + M_3^2\right)\right)$ and $N = 3\left(6TM_1^2\left(1 + M_3^2\right) + C_2\left(1\right.\right.$

$+2M_3^2)\big)$ are two consrtants since $\mathbb{E}\left[|\xi|^2\right] < \infty$ and $\int_0^T g(t)\,dt < \infty$. This, together with Gronwall's inequality (see [7]), implies

$$\mathbb{E}\left[\left|X_t^{U^n}\right|^2\right] \leq Le^{TN}. \tag{6.4}$$

Therefore, by Chebychev's inequality,

$$\sup_{\substack{n\geq 0\\ 0\leq t\leq T}} \mathbb{P}\left(\left|X_t^{U^n}\right| > R\right) \leq \sup_{\substack{n\geq 0\\ 0\leq t\leq T}} \frac{\mathbb{E}\left[\left|X_t^{U^n}\right|^2\right]}{R^2} \leq \frac{Le^{TN}}{R^2} \to 0 \text{ as } R \to \infty.$$

For relation **(B)**, using the similar deduction to above, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|X_t^{U^n} - X_s^{U^n}\right|^2\right] \\
&\leq 2h\mathbb{E}\left[\int_s^t \left|b\left(r, X_r^{U^n}, \mu_r^n, U^n\left(r, X_r^{U^n}\right)\right)\right|^2 dr\right] \\
&\quad + 2C_2\mathbb{E}\left[\int_s^t \left\|q\left(r, X_r^{U^n}, U^n\left(r, X_r^{U^n}\right), X_r^{U^n}, U^n\left(r, X_r^{U^n}\right)\right)\right\| dr\right] \\
&\leq 2h\mathbb{E}\left[\int_s^t 3M_1^2\left(1 + \left|X_r^{U^n}\right|^2 + \left|U^n\left(r, X_r^{U^n}\right)\right|^2 + W_2^2\left(\mu_r^n, \delta_0\right)\right) dr\right] \\
&\quad + 2C_2\mathbb{E}\left[\sup_{0\leq r\leq T} g(r) \int_s^t \left(1 + \left|X_r^{U^n}\right|^2 + \left|U^n\left(s, X_r^{U^n}\right)\right|^2\right) dr\right] \\
&\leq 2h\mathbb{E}\left[\int_s^t 3M_1^2\left(1 + 2\left|X_r^{U^n}\right|^2 + 2M_3^2\left(1 + \left|X_r^{U^n}\right|^2\right)\right) dr\right] \\
&\quad + 2C_2\mathbb{E}\left[\sup_{0\leq r\leq T} g(r) \int_s^t \left(1 + \left|X_r^{U^n}\right|^2 + 2M_3^2\left(1 + \left|X_r^{U^n}\right|^2\right)\right) dr\right] \\
&\leq 2h\left(\left(6hM_1^2\left(1 + M_3^2\right) + C_2\left(1 + 2M_3^2\right)\right)\left(1 + Le^{TN}\right)\right).
\end{aligned}$$

Therefore, by Chebychev's inequality,

$$\begin{aligned}
&\sup_{\substack{n\geq 0\\ |t-s|<h}} \mathbb{P}\left(\left|X_t^{U^n} - X_s^{U^n}\right| > \varepsilon\right) \\
&\leq \sup_{\substack{n\geq 0\\ |t-s|<h}} \frac{2h\left(\left(6hM_2^2\left(1 + M_3^2\right) + C_2\left(1 + 2M_3^2\right)\right)\left(1 + Le^{TN}\right)\right)}{\varepsilon^2} \to 0 \text{ as } h \to 0.
\end{aligned}$$

Since relations **(A)** and **(B)** are satisfied then by the standard tightness criteria on the space $\mathcal{P}_2(\mathcal{W})$ (see [12], Theorem 4.2, pp. 17) we can easily conclude that $\left(X^{U^n}\right)$ is tight. Thus, we can extract from $\left(X^{U^n}\right)$ a weakly convergent subsequence which still denoted by $\left(X^{U^n}\right)$. Therefore, by Skorohod's representation theorem, there exists a probability space $\left(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}\right)$ and processes $\hat{X}^{U^*}$ and $\hat{X}^{U^n}$ on it such that

(a) The laws of $\hat{X}^{U^n}$ and X^{U^n} coincide for each $n \geq 0$.

(b) $\hat{X}^{U^n} \to \hat{X}^{U^*}$, $\hat{\mathbb{P}}$-a.s, as $n \to \infty$.

Similarly to the proof of Proposition 3.6, to show that $\hat{X}^{U^*}$ is a solution of equation (6.1) corresponding to the control U^*, we just need to show that

$$\begin{aligned}
&\mathbb{E}^{\hat{\mathbb{P}}}\left[\left(f\left(\hat{X}_t^{U^*}\right)-f\left(\hat{X}_s^{U^*}\right)-\int_s^t (Lf)\left(r,\hat{X}_r^{U^*},\mu_r^*,U^*\left(r,\hat{X}_r^{U^*}\right)\right)dr\right)\mathcal{G}_s\right]\\
&=\lim_{n\to\infty}\mathbb{E}^{\hat{\mathbb{P}}}\left[\left(f\left(\hat{X}_t^{U^n}\right)-f\left(\hat{X}_s^{U^n}\right)-\int_s^t (L^nf)\left(r,\hat{X}_r^{U^n},\mu_r^n,U^n\left(r,\hat{X}_r^{U^n}\right)\right)dr\right)\mathcal{G}_s\right]\\
&=\lim_{n\to\infty}\mathbb{E}\left[\left(f\left(X_t^{U^n}\right)-f\left(X_s^{U^n}\right)-\int_s^t (L^nf)\left(r,X_r^{U^n},\mu_r^n,U^n\left(r,X_r^{U^n}\right)\right)dr\right)\mathcal{G}_s\right] \quad (6.5)\\
&=0, \quad (6.6)
\end{aligned}$$

for any continuous, bounded and $\tilde{\mathbb{W}}_s$-measurable functional $\mathcal{G}_s$, where (Lf) and (L^nf) are defined according to (6.3) for $\hat{X}^{U^*}$ and $\hat{X}^{U^n}$, respectively, with $\mu_s^* := \hat{\mathbb{P}}_{\hat{X}_s^{U^*}}$. In one side, by (b), we have $\hat{X}_r^{U^n} \overset{a.s}{\to} \hat{X}_r^{U^*}$ for $r\in[s,t]$ as $n\to\infty$, and by the same way used in the proof of (3.4), we get $\lim_{n\to\infty} W_2\left(\mu_r^n,\mu_r^*\right)=0$. In other side, we have $f\in C_b^2\left(\mathbb{R}^d\right)$, i.e, there exists a constant R_1 such that the inequalities $|f(x)|\le R_1$, $\left|\frac{\partial f}{\partial x_i}\right|\le R_1$ and $\left|\frac{\partial^2 f}{\partial x_i x_j}\right|\le R_1$ hold for $x\in\mathbb{R}^d$ and $1\le i,j\le n$. Then,

$$\begin{aligned}
&\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|f\left(\hat{X}_t^{U^n}\right)-f\left(\hat{X}_s^{U^n}\right)-\int_s^t (L^nf)\left(r,\hat{X}_r^{U^n},\mu_r^n,U^n\left(r,\hat{X}_r^{U^n}\right)\right)dr\right|\right]\\
&\le \mathbb{E}^{\hat{\mathbb{P}}}\left[\left|f\left(\hat{X}_t^{U^n}\right)\right|\right]+\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|f\left(\hat{X}_s^{U^n}\right)\right|\right]+\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\int_s^t (L^nf)\left(r,\hat{X}_r^{U^n},\mu_r^n,U^n\left(r,\hat{X}_r^{U^n}\right)\right)dr\right|\right]\\
&\le 2\left(R_1+\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\int_s^t\sum_{i=1}^d b^i\left(r,\hat{X}_r^{U^n},\mu_r^n,U^n\left(r,\hat{X}_r^{U^n}\right)\right)\frac{\partial f\left(\hat{X}_r^{U^n}\right)}{\partial x_i}dr\right|\right]\right.\\
&\left.+\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\int_s^t\sum_{i,j=1}^d q^{ij}\left(r,\hat{X}_r^{U^n},U^n\left(r,\hat{X}_r^{U^n}\right),\hat{X}_r^{U^n},U^n\left(r,\hat{X}_r^{U^n}\right)\right)\frac{\partial^2 f\left(\hat{X}_r^{U^n}\right)}{\partial x_i\partial x_j}dr\right|\right]\right). \quad (6.7)
\end{aligned}$$

Using Hölder's inequality, the linear growth conditions **(ii)** and **(C.4)** for q and inequality (6.4) gives us

$$\begin{aligned}
&\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\int_s^t\sum_{i,j=1}^d q^{ij}\left(r,\hat{X}_r^{U^n},U^n\left(r,\hat{X}_r^{U^n}\right),\hat{X}_r^{U^n},U^n\left(r,\hat{X}_r^{U^n}\right)\right)\frac{\partial^2 f\left(\hat{X}_r^{U^n}\right)}{\partial x_i\partial x_j}dr\right|\right]\\
&\le \mathbb{E}^{\hat{\mathbb{P}}}\left[\sum_{i,j=1}^d\left|\int_s^t q^{ij}\left(r,\hat{X}_r^{U^n},U^n\left(r,\hat{X}_r^{U^n}\right),\hat{X}_r^{U^n},U^n\left(r,\hat{X}_r^{U^n}\right)\right)\frac{\partial^2 f\left(\hat{X}_r^{U^n}\right)}{\partial x_i\partial x_j}dr\right|\right]\\
&\le R_1 d\,\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_s^t\left\|q\left(r,\hat{X}_r^{U^n},U^n\left(r,\hat{X}_r^{U^n}\right),\hat{X}_r^{U^n},U^n\left(r,\hat{X}_r^{U^n}\right)\right)\right\|dr\right]\\
&\le R_1 d\,\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_s^t g(r)\left(1+\left|\hat{X}_r^{U^n}\right|^2+\left|U^n\left(r,\hat{X}_r^{U^n}\right)\right|^2\right)dr\right]\\
&\le R_1 d\,\mathbb{E}^{\hat{\mathbb{P}}}\left[\sup_{0\le r\le T} g(r)\int_s^t\left(1+\left|\hat{X}_r^{U^n}\right|^2+2M_3^2\left(1+\left|\hat{X}_r^{U^n}\right|^2\right)\right)dr\right]\\
&\le R_1 d\,(t-s)\left(1+2M_3^2\right)+R_1 d\left(1+2M_3^2\right)\int_s^t\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}_r^{U^n}\right|^2\right]dr\\
&\le R_1 d\,(t-s)\left(1+2M_3^2\right)\left(1+Le^{TN}\right). \quad (6.8)
\end{aligned}$$

By Cauchy-Schwartz's inequality, the linear growth conditions **(ii)** and **(C.4)** for b, the property of the 2-Wasserstien metric (2.2) and inequality (6.4), we obtain

$$\begin{aligned}
&\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\int_s^t \sum_{i=1}^d b^i\left(r,\hat{X}_r^{U^n},\mu_r^n,U^n\left(r,\hat{X}_r^{U^n}\right)\right)\frac{\partial f\left(\hat{X}_r^{U^n}\right)}{\partial x_i}dr\right|^2\right]\\
&\leq \mathbb{E}^{\hat{\mathbb{P}}}\left[d\sum_{i=1}^d\left|\int_s^t b^i\left(r,\hat{X}_r^{U^n},\mu_r^n,U^n\left(r,\hat{X}_r^{U^n}\right)\right)\frac{\partial f\left(\hat{X}_r^{U^n}\right)}{\partial x_i}dr\right|^2\right]\\
&\leq 3\left(M_1R_1\right)^2 d\left(t-s\right)\sum_{i=1}^d \mathbb{E}^{\hat{\mathbb{P}}}\left[\int_s^t\left(1+\left|\hat{X}_{i,r}^{U^n}\right|^2+\left|U_i^n\left(r,\hat{X}_{i,r}^{U^n}\right)\right|^2+W_2^2\left(\mu_{i,r}^n,\delta_0\right)\right)dr\right]\\
&\leq 3\left(M_1R_1\right)^2 d\left(t-s\right)\sum_{i=1}^d \mathbb{E}^{\hat{\mathbb{P}}}\left[\int_s^t\left(1+2\left|\hat{X}_{i,r}^{U^n}\right|^2+2M_3^2\left(1+\left|\hat{X}_{i,r}^{U^n}\right|^2\right)\right)dr\right]\\
&\leq 3\left(M_1R_1\right)^2 d^2\left(t-s\right)^2\left(1+2M_3^2\right)+6\left(M_1R_1\right)^2 d\left(t-s\right)\left(1+M_3^2\right)\sum_{i=1}^d \mathbb{E}^{\hat{\mathbb{P}}}\left[\int_s^t\left|\hat{X}_{i,r}^{U^n}\right|^2 dr\right]\\
&\leq 6\left(M_1R_1\right)^2 d^2\left(t-s\right)^2\left(1+M_3^2\right)\left(1+Le^{TN}\right).
\end{aligned}\tag{6.9}$$

Combining inequalies (6.7)-(6.9), we get

$$\begin{aligned}
&\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|f\left(\hat{X}_t^{U^n}\right)-f\left(\hat{X}_s^{U^n}\right)-\int_s^t\left(L^n f\right)\left(r,\hat{X}_r^{U^n},\mu_r^n,U^n\left(r,\hat{X}_r^{U^n}\right)\right)dr\right|\right]\\
&\leq 2\left(6\left(M_1R_1\right)^2 d^2\left(t-s\right)^2\left(1+M_3^2\right)\left(1+Le^{TN}\right)\right)^{1/2}\\
&+2R_1\left(1+d\left(t-s\right)\left(1+2M_3^2\right)\left(1+Le^{TN}\right)\right)\\
&<\infty.
\end{aligned}$$

Thus, by the dominated convergence theorem we get (6.6). Then $\hat{X}_t^{U^*}$ is the weak solution of (6.1) associated with the control $U^*\left(t,x\right)$.
To finish the proof of Theorem 6.3, it remains to show that $U^*\left(t,x\right)$ is an optimal control.
By Cauchy-Schwartz's inequality, conditions **(C.5),** and inequality (6.4), we obtain

$$\begin{aligned}
&\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\int_0^T h\left(t,\hat{X}_t^{U^n},\mu_t^n,U^n\left(t,\hat{X}_t^{U^n}\right)\right)dt+\ell\left(\hat{X}_T^{U^n},\mu_T^n\right)\right|^2\right]\\
&\leq 2\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\int_0^T h\left(t,\hat{X}_t^{U^n},\mu_t^n,U^n\left(t,\hat{X}_t^{U^n}\right)\right)dt\right|^2\right]+2\mathbb{E}\left[\left|\ell\left(\hat{X}_T^{U^n},\mu_T^n\right)\right|^2\right]\\
&\leq 8M_2^2T\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_0^T\left(1+\left|\hat{X}_t^{U^n}\right|^2+\left|U^n\left(t,\hat{X}_t^{U^n}\right)\right|^2+W_2^2\left(\mu_t^n,\delta_0\right)\right)dt\right]\\
&+6M_2^2\mathbb{E}^{\hat{\mathbb{P}}}\left[1+\left|\hat{X}_T^{U^n}\right|^2+W_2^2\left(\mu_T^n,\delta_0\right)\right]\\
&\leq 8M_2^2T\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_0^T\left(1+2\left|\hat{X}_t^{U^n}\right|^2+\frac{\left(2M_3\right)^2}{2}\left(1+\left|\hat{X}_t^{U^n}\right|^2\right)\right)dt\right]\\
&+6M_2^2\left(1+2\mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}_T^{U^n}\right|^2\right]\right)
\end{aligned}$$

$$\leq 8\left(M_2 T\right)^2\left(1+\frac{\left(2 M_3\right)^2}{2}\right)+8 M_2^2 T\left(2+\frac{\left(2 M_3\right)^2}{2}\right) \int_0^T \mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}_t^{U^n}\right|^2\right] dt$$
$$+6 M_2^2\left(1+2 \mathbb{E}^{\hat{\mathbb{P}}}\left[\left|\hat{X}_T^{U^n}\right|^2\right]\right)$$
$$\leq 8\left(M_2 T\right)^2\left(1+2 L e^{T N}+\frac{\left(2 M_3\right)^{2 k}}{2}\left(1+L e^{T N}\right)\right)+6 M_2^2\left(1+2 L e^{T N}\right)$$
$$<\infty .$$

Therefore, according to the above properties (a)-(b) and by the dominated convergence theorem, it follows that

$$\begin{aligned}
\inf_{U \in \mathcal{U}} J(U) &= \lim_{n \to \infty} J\left(U^n\right) \\
&= \lim_{n \to \infty} \mathbb{E}\left[\int_0^T h\left(t, X_t^{U^n}, \mu_t^n, U^n\left(t, X_t^{U^n}\right)\right) dt+\ell\left(X_T^{U^n}, \mu_T^n\right)\right] \\
&= \lim_{n \to \infty} \mathbb{E}^{\hat{\mathbb{P}}}\left[\int_0^T h\left(t, \hat{X}_t^{U^n}, \mu_t^n, U^n\left(t, \hat{X}_t^{U^n}\right)\right) dt+\ell\left(\hat{X}_T^{U^n}, \mu_T^n\right)\right] \\
&= \mathbb{E}^{\hat{\mathbb{P}}}\left[\int_0^T \lim_{n \to \infty} h\left(t, \hat{X}_t^{U^n}, \mu_t^n, U^n\left(t, \hat{X}_t^{U^n}\right)\right) dt+\lim_{n \to \infty} \ell\left(\hat{X}_T^{U^n}, \mu_T^n\right)\right] \\
&= \mathbb{E}^{\hat{\mathbb{P}}}\left[\int_0^T h\left(t, \hat{X}_t^{U^*}, \mu_t^*, U^*\left(t, \hat{X}_t^{U^*}\right)\right) dt+\ell\left(\hat{X}_T^{U^*}, \mu_T^*\right)\right] \\
&= J\left(U^*\right).
\end{aligned}$$

Consequently, the function $U^*(t, x)$ is an optimal control.

The proof of Theorem 6.3 is thus completed. □

Remark 6.4. It should be noted that the control produced by Theorem 6.3 is called Markovian because it is given in the form $U^*(t, x)$, i.e., it depends only on the current time and the current state, but it does not necessarily render the state process X a Markov process. Although the dynamics appear to be Markovian, the process X is a solution of a potentially ill-posed martingale problem, and it is well known (see [21], Chapter 12) that uniqueness in law is required to guarantee the solution is Markovian.

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