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ON THE GEOMETRY OF THE NUMERICAL RANGE

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Abstract. It is well known that the numerical range $W(T)$ of a bounded linear operator T on a complex Hilbert space is included in the real line if and only if T is self-adjoint. Recently, it was proven that the numerical range $W(T)$ is a line segment if and only if there are scalars λ and μ such that $T^* = \lambda T + \mu I$, and it was given a process to determine the equation of the straight support of this numerical range in terms of λ and μ. In this paper, we complete these results and give the converse process, we determine the scalars λ and μ such that $T^* = \lambda T + \mu I$ from the equation of the straight support of the numerical range $W(T)$. Also, we generalize a condition for the selfadjointness of an operator and we give some properties concerned a small convex regions containing the numerical range.

1 Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space, and let $B(\mathcal{H})$ denote the Banach algebra of all bounded linear operators defined on the space $\mathcal{H}$. For an operator $A \in B(\mathcal{H})$, the numerical range of A is the subset of the complex numbers $\mathbb{C}$ given by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\},$$

and the numerical radius of A is the positive real given by

$$w(A) = \sup\{|z| : z \in W(A)\}.$$

The numerical range $W(A)$ is always convex, contains the eigenvalues of A and its closure contains the spectrum of A. In particular, if A is a normal matrix, then $W(A)$ is the convex hull of its spectrum. Certain analytic and algebraic properties of an operator can be concluded from the geometrical properties of its numerical range. As example, $A = \delta I$ if and only if $W(A)$ is the point $\{\delta\}$, A is self-adjoint if and only if $W(A)$ is real, and if $W(A)$ is a line segment, then A is normal [2]. Readers are referred to the the books ([2] , [3]) for more detail properties on the numerical range.

Chettouh and Bouzenada show in [1, th 2.2] that the numerical range $W(A)$ of an operator $A \in B(\mathcal{H})$ is a line segment if and only if there are scalars $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$ such that $A^* = \lambda A + \mu I$, and in [1, th 2.3] they determine the equation of the straight line containing $W(A)$ in terms of λ and μ as follows:

(i) If $|\lambda| \neq 1$, then $W(A)$ is the point $\{\delta\}$, where

$$\delta = \frac{\overline{\lambda}\mu + \overline{\mu}}{1 - |\lambda|^2}.$$

(ii) If $\lambda = 1$, then the equation of straight line containing $W(A)$ is

$$y = \frac{\mu}{2} i, \text{ with } Re\mu = 0.$$

(iii) If $\lambda = -1$, then the equation of straight line containing $W(A)$ is

$$x = \frac{\mu}{2}, \text{ with } \mu \in \mathbb{R}.$$

(iv) Otherwise, the equation of straight line containing $W(A)$ is

$$y = \left(\frac{-1 + Re\lambda}{Im\lambda}\right) x + \frac{Re\mu}{Im\lambda}.$$

In the present paper we give the converse theorem, we determine λ and μ from the equation of the straight line containing the numerical range. We also generalize a result of Toivo Nieminen [4] presented by George H. Orland in [5] for the seltadjointness of a bounded linear operator. In the second part of this paper we give a small convex region containing the numerical range and we give a necessary condition for the numerical range to be a disc.

In the following we will denote the spectrum, the spectral radius and the resolvent set of operators A by $\sigma(A)$, $r(A)$ and $\rho(A)$, respectively. Furthermore, we will denote the convex hull of a set $X \subset \mathbb{C}$ by conv(X). An operator A is said to be convexoid if $\overline{W(A)}$ coincides with the convex hull of their spectrum. Here, Similarly ans in reference [1], $\mathbb{S}(\mathcal{H})$ denotes the class of operators A in $B(\mathcal{H})$ which satisfy $A^* = \lambda A + \mu I$, for $\lambda, \mu \in \mathbb{C}$.

The following basic properties (see ([2], [3])) will be used.

(1) $W(A+S) \subset W(A) + W(B)$.

(2) $W(\alpha A + \beta I) = \alpha W(A) + \beta$, for all $\alpha, \beta \in \mathbb{C}$.

(3) $W(A \oplus B) = \text{conv}(W(A) \cup W(B))$.

(4) If $\overline{W(A)} = [m, M]$, then $\|A\| = \sup\{|m|, |M|\}$.

2 When the numerical range is a line segment

Theorem 2.1. *Let $A \in B(\mathcal{H})$ a non scalar operator such that $W(A)$ included in a complex line L. Then $A^* = \lambda A + \mu I$ such that:*

(1) *If the equation of L is $y = ax + b$ with $a \neq 0$, then*

$$\lambda = \frac{-(a+i)^2}{1+a^2}, \quad \textit{and} \quad \mu = -2b\frac{a+i}{1+a^2}.$$

(2) *If the equation of L is $y = b$, then*

$$\lambda = 1, \textit{ and } \mu = -2ib.$$

(3) *If the equation of L is $x = a$, then*

$$\lambda = -1, \textit{ and } \mu = 2a.$$

Proof. According to [1, th 2.2], there are $(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}$ such that $A^* = \lambda A + \mu I$, then $A = A + (\overline{\lambda}\mu + \overline{\mu}) I$, so that

$$-(\mu/\overline{\mu}) = \lambda, \quad \text{or} \quad \mu = 0. \tag{1}$$

Since A is non scalar operator and according to [1, th 2.3], we have

$$(Re\lambda)^2 + (Im\lambda)^2 = 1, \tag{2}$$

$$a = \frac{-1 + Re\lambda}{Im\lambda} \tag{3}$$

and

$$b = \frac{Re\mu}{Im\lambda}. \tag{4}$$

(1) From equation (3) we have $Re\lambda = 1 + a(Im\lambda)$. Substituting this into equation (2), we obtain

$$(1 + a^2)(Im\lambda)^2 + 2a(Im\lambda) = 0.$$

Then under [1, th 2.3] and that $(Im\lambda) \neq 0$, we get

$$Im\lambda = \frac{-2a}{1+a^2},$$

thus

$$Re\lambda = \frac{1-a^2}{1+a^2}$$

which implies that

$$\lambda = \frac{1-a^2-2ia}{1+a^2} = \frac{-(a+i)^2}{1+a^2}.$$

In the sequel from equation (4), we have $Re\mu = b\,(Im\lambda)$, then

$$Re\mu = \frac{-2ab}{1+a^2}.$$

From equation (1), we notice that

$$Im\mu = \frac{i\,(1+\lambda)}{(1-\lambda)} Re\mu.$$

In particular, we get

$$Im\mu = \frac{-2b}{1+a^2}$$

and hence

$$\mu = -2b\frac{a+i}{1+a^2}.$$

(2) If the equation of L is $y = b$, then according to $[1$, th $2.3]$, $\lambda = 1$ and $y = \mu i/2$ which yields

$$\lambda = 1, \quad \text{and} \quad \mu = -2ib.$$

(3) If the equation of L is $x = a$, then according to $[1$, th $2.3]$, $\lambda = -1$ and $x = \mu/2$, therefore

$$\lambda = -1, \quad \text{and} \quad \mu = 2a.$$

This complete the proof. □

Remark 2.2. If $W(A) = \{\delta\}$ such that $\delta \neq 0$, then

$$A^* = \frac{\overline{\delta}}{\delta} A.$$

Example 2.3. As a simple example, let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix},$$

then

$$A^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

We have $A^*A = AA^*$, then A is a normal matrice and $W(A)$ =conv$\sigma(A)$ =conv$\{1, i\}$, hence $W(A)$ is the line segment joining the points 1 and i. Therefore, $W(A)$ is included in the complex line whose equation $y = -x + 1$. From the first case of the theorem, we have

$$\lambda = \frac{-(-1+i)^2}{1+(-1)^2} = i \text{ and } \mu = -2\,(1)\,\frac{-1+i}{1+(-1)^2} = 1 - i.$$

Therefore,

$$\begin{aligned} A^* &= \lambda A + \mu I \\ &= i\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix} + (1-i)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix}. \end{aligned}$$

Example 2.4. For a general example, let $P \in B(\mathcal{H})$ an orthogonal projection and let's pose $A = (1+i)P + iI$. Then

$$W(A) = (1+i)W(P) + i,$$

hence $W(A)$ is the line segment joining the points i and $1+2i$, identically $W(A)$ is included in a complex line of the equation $y = x + 1$. Furthermore, we have $a = 1$ and $b = 1$ from the first case of the theorem, then $\lambda = -i$ and $\mu = -1 - i$. Therefore

$$A^* = -iA - (1+i)I = -i((1+i)P + iI) - (1+i)I = (1-i)P - iI,$$

which is equal to $((1+i)P + iI)^*$.

Remark 2.5. According to the previous theorem, if the operator A is convexoid and $\sigma(A)$ included in the line L of the equation $y = ax$ $(a \neq 0)$, then

$$A^* = \frac{-(a+i)^2}{1+a^2}A + -2b\frac{a+i}{1+a^2}I.$$

Theorem 2.6. [5] *If $\sigma(A)$ is real and $\|R_\alpha(A)\| \leq |\alpha|^{-1}$ for all nonzero purely imaginary α, then A is self-adjoint.*

We give a weaker sufficient condition on the operator A with $A^* = \lambda A + \mu I$, and $\lambda, \mu \in \mathbb{C}$. Similarly, we generalize the previous theorem for all complex line through the origin.

Theorem 2.7. *Let $A \in B(\mathcal{H})$. If $\sigma(A)$ included in the line L of the equation $y = ax$, $(a \neq 0)$ and $\left\|(A - \alpha I)^{-1}\right\| \leq |\alpha|^{-1}$ for all*

$$\alpha \in \left(1 - \frac{1}{a}i\right)\mathbb{R}^*,$$

then, $W(A)$ is a segment of the line L and

$$A^* = \frac{-(a+i)^2}{1+a^2}A.$$

Proof. Let $\alpha = (1 - i/a)\beta$ with $\beta \in \mathbb{R}^*$ and $\left\|(A - \alpha I)^{-1}\right\| \leq |\alpha|^{-1}$ for all real number different then zero β, we have

$$|\alpha| \|y\| \leq \|Ay - \alpha y\|, \ \forall y \in \mathcal{H}, \ \forall \beta \in \mathbb{R}^*,$$

and we have

$$|\alpha|^2 \|y\|^2 \leq \|Ay\|^2 + |\alpha|^2 \|y\|^2 - 2Re\langle Ay, \alpha y\rangle,$$

or

$$2Re(1 + i/a)\beta\langle Ay, y\rangle \leq \|Ay\|^2.$$

Hence

$$2\beta\left(Re\langle Ay, y\rangle - \frac{1}{a}Im\langle Ay, y\rangle\right) \leq \|Ay\|^2,$$

where β is real number different then zero, therefore one can get

$$Im\langle Ay,y\rangle = aRe\langle Ay,y\rangle\,,\ \forall y\in\mathcal{H}.$$

Thus, $W(A)$ is a line segment of the line L which is described by the equation $y = ax$. Moreover, according to the previous theorem we obtain

$$A^* = \frac{-(a+i)^2}{1+a^2}A.$$

□

Remark 2.8. If $\sigma(A)\subset i\mathbb{R}$, and $\left\|(A-\alpha I)^{-1}\right\| \leq |\alpha|^{-1}$ for all $\alpha\in\mathbb{R}^*$, we obtain $W(A)\subset i\mathbb{R}$ and A is anti-self-adjoint.

3 Boundedness of the numerical range

Theorem 3.1. *Let $A\in B(\mathcal{H})$. Then the intersection of the disk $D=\overline{D}(0,w(A))$ and the rectangle $R=[a,b]\times[c,d]$ is a small convex region containing $W(A)$ such that:*

$$a = \min_{\lambda\in\sigma(ReA)}(\lambda),\ b=\max_{\lambda\in\sigma(ReA)}(\lambda),\ c=\min_{\lambda\in\sigma(ImA)}(\lambda)\ \text{ and }\ d=\max_{\lambda\in\sigma(ImA)}(\lambda).$$

Moreover:
(i) The rectangle R is contained in the disk D if and only if

$$\sqrt{\|Re(A)\|^2+\|Im(A)\|^2} = w(A).$$

(ii) The disk D is contained in the rectangle R if and only if

$$w(A) = b = d = -a = -c.$$

Proof. The cartesian decomposition of an operator $A\in B(\mathcal{H})$ is

$$A = Re(A) + iIm(A),$$

where

$$Re(A) = \frac{A+A^*}{2} \quad\text{and}\quad Im(A) = \frac{A-A^*}{2i}.$$

$Re(A)$ and $Im(A)$ are self-adjoint operators, then $\overline{W(Re(A))}$ and $\overline{W(Im(A))}$ are the line segments $[a,b]$ and $[c,d]$ respectively. Hence, we have

$$W(A)\subset[a,b]+i[c,d].$$

Then, the rectangle R containing $W(A)$. Therefore $W(A)$ is contained in the intersection of the disk $D=\overline{D}(0,w(A))$ and the rectangle $R=[a,b]\times[c,d]$.
(i) Since

$$\sup_{\alpha\in R}|\alpha| = \sqrt{\|Re(A)\|^2+\|Im(A)\|^2},$$

and

$$\sqrt{\|Re(A)\|^2+\|Im(A)\|^2} \geqslant w(A),$$

because

$$W(A)\subset\overline{W(Re(A))}+i\overline{W(Im(A))} = \overline{Re(W(A))}+i\overline{Im(W(A))}.$$

Then

$$R\subset D \text{ if and only if } \sqrt{\|Re(A)\|^2+\|Im(A)\|^2} = w(A).$$

(ii) Since

$$w(A)\geqslant|\lambda|\,,\ \forall\lambda\in\{a,b,c,d\},$$

then

$$D\subset R \text{ if and only if } w(A) = b = d = -a = -c.$$

□

Remark 3.2. The three cases of this intersection : $D \subset R$, $R \subset D$, and $D \cap R$ different from D and different from R are possible according to the following examples:

Example 3.3. $a)$ Let A be the operator on $\mathbb{C}^2$ defined by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $W(A)$ is the closed disk of radius $1/2$, centred at the origin.

Therefore $w(A) = 1/2$, with the eigenvalues of $Re(A)$ are: $a = -1/2$, $b = 1/2$, and the eigenvalues of $Im(A)$ are: $c = -1/2$, $d = 1/2$.

Hence

$$W(A) = D \subset R = [-1/2, 1/2] \times [-1/2, 1/2] .$$

$b)$ Let $A = \text{diag}(0,\ 1,\ i,\ 1+i)$. Since A is normal, then $W(A)$ =conv$\sigma(A)$ =conv$\{0, 1, i, 1+i\}$, identically $W(A) = [0,1] \times [0,1]$. Therefore $w(A) = \sqrt{2}$.

The eigenvalues of $Re(A)$ are: $a = 0$, $b = 1$, and the eigenvalues of $Im(A)$ are: $c = 0$, $d = 1$.

Hence

$$W(A) = R \subset D = \overline{D}\left(0, \sqrt{2}\right).$$

$c)$ Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The matrix A can be written as $A_1 \oplus A_2$, where $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $A_2 = 1$ the identity operator on $\mathbb{C}$.

Then

$$\begin{aligned} W(A) &= \text{conv}\left(W(A_1) \cup W(A_2)\right) \\ &= \text{conv}\left(\overline{D}(0, 1/2) \cup \{1\}\right). \end{aligned}$$

Therefore

$$w(A) = 1 \text{ and } D = \overline{D}(0, w(A)) = \overline{D}(0, 1).$$

The eigenvalues of $Re(A)$ are: $a = -1/2$, $b = 1$, and the eigenvalues of $Im(A)$ are: $c = -1/2$, $d = 1/2$.

Then

$$R = [-1/2, 1] \times [-1/2, 1/2] .$$

Hence

$$D \cap R \neq D \text{ and } D \cap R \neq R.$$

Corollary 3.4. *Let $A \in B(\mathcal{H})$. If $W(A)$ is the disk centered at the origin (open or closed), then*

$$w(A) = \|Re(A)\| = \|Im(A)\|.$$

Proof. From previous theorem, if $W(A)$ is the disk $D(0, w(A))$ then: the disk $D(0, w(A))$ is contained in the rectangle R.

Hence

$$w(A) \leq |\alpha|, \ \forall \alpha \in \{-a, b, -c, d\}.$$

On the other hand

$$\|Re(A)\| = w(Re(A)) \leq \frac{1}{2}\left(w(A) + w(A^*)\right) = w(A),$$

and $\|Im(A)\| \leq w(A)$.

Therefore

$$w(A) = \|Re(A)\| = \|Im(A)\|.$$

□

Corollary 3.5. *Let $A \in B(\mathcal{H})$. If $W(A)$ is the disk $D(z, \rho)$ (open or closed) where $z \in \mathbb{C}, \rho \geq 0$, then*

$$w(A) = \|Re(A - zI)\| + |z| = \|Im(A - zI)\| + |z|.$$

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