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ON THE EXPONENTIAL DIOPHANTINE EQUATIONS $\mathbf{13^x + 17^y = z^2}$, $\mathbf{13^x + 37^y = z^2}$, $\mathbf{17^x + 37^y = z^2}$ and $\mathbf{17^x + 91^y = z^2}$

Anouar Gaha and Soufiane Mezroui

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Corresponding Author: Anouar Gaha

Abstract In this paper, we find that the exponential Diophantine equations $13^x + 17^y = z^2$, $13^x + 37^y = z^2$, $17^x + 37^y = z^2$ and $17^x + 91^y = z^2$ have no solutions in nonnegative integers x, y and z. These results are obtained by using Catalan's conjecture and some congruence theorems.

1 Introduction

For a very long time, the Diophantine equations have fascinated many mathematicians. The equation of the Fermat-Catalan conjecture and Beal's conjecture, $a^n + b^m = u^t$ with exponents subject to inequality restrictions, is an exponential Diophantine equation when it includes an additional variable or variables that occur as exponents. There isn't a general theory for these equations, although there have been discussions of particular examples such Catalan's conjecture, and many of the exponential Diophantine equations have been considered since then. In 1844, Catalan [3] posed a conjecture that the Diophantine equation $a^x - b^y = 1$ has a unique solution $(a, b, x, y) = (3, 2, 2, 3)$ in nonnegative integers a, b, x and y with $\min\{a, b, x, y\} > 1$. In 2004, Mihailescu [6] showed this conjecture of the Diophantine equation $p^x + 1 = z^2$ has a unique solution $(p, x, y) = (2, 3, 3)$, where p is prime and $\min\{p, x, y\} > 1$. In 2012, Chotchaisthit [4] proved that $(3, 0, 3)$ is the only nonnegative integer solution of the Diophantine equation $2^x + 11^y = z^2$. In the same year, Fergy and Rabago [5] investigated the Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$. They showed that the two equations have exactly two solutions in nonnegative integers (x, y, z) are $\{(1, 0, 2), (4, 1, 10)\}$ and $\{(1, 0, 2), (2, 1, 10)\}$, respectively. One year after, Sroysang [8] showed that the Diophantine equation $5^x + 7^y = z^2$ has no nonnegative integer solution. For instance, the same author [9] proved that the Diophantine equation $5^x + 23^y = z^2$ has no nonnegative integer solution. Two years after, he proved that the Diophantine equations $7^x + 19^y = z^2$ and $7^x + 91^y = z^2$ have no solutions, where x, y and z are nonnegative integers in [10]. More precisely, he showed that the Diophantine equation $5^x + 43^y = z^2$ has no nonnegative integer solution in [11]. In 2017, Nathan son [7] proved the Diophantine equation $x^n - y^n = z^{n+1}$ has infinitely many positive integral solutions for every positive integer n. Two years later, Borah and Dutta [1] showed that $(2, 1, 9)$ is a unique solution (x, y, z) for the Diophantine equation $7^x + 32^y = z^2$ in nonnegative integers x, y and z. For instance, Buosi, Lemos, Porto and Santiago [2] found all the nontrivial nonnegative integer solutions to the Diophantine equation $p^x - 2^y = z^2$ where $p = k^2 + 4$ is a prime number and $k \geq 1$, are given by $(1, 2, k)$ if $p \geq 13$ and $\{(1, 0, 2), (1, 2, 1), (3, 2, 11)\}$ if $p = 5$.

In this paper, we explore the following four specific Diophantine equations $13^x + 17^y = z^2$,

$13^x + 37^y = z^2$, $17^x + 37^y = z^2$ and $17^x + 91^y = z^2$. We show that these equations have no solutions for nonnegative integers x, y and z. To solve these equations, we will use the Catalan's conjecture and congruency theory.

2 Preliminaries

Proposition 2.1 (Catalan's conjecture, [3]). *The Diophantine equation $a^x - b^y = 1$ has a unique solution $(a, b, x, y) = (3, 2, 2, 3)$, where a, b, x and y are nonnegative integers with* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. *The Diophantine equation $13^x + 1 = z^2$ has no nonnegative integer solution, where x and z are nonnegative integers.*

Proof. Suppose x and z are nonnegative integers such that $13^x + 1 = z^2$. We now we consider three possibilities.

Case 1: If $x = 0$, then $z^2 = 2$, which is impossible.

Case 2: If $x = 1$, then $z^2 = 14$ is also impossible.

Case 3: If $x > 1$, then $z^2 = 13^x + 1 > 13^1 + 1 = 14$. This implies that $z > 3$. By Proposition 2.1, there is no nonnegative integer solution for $\min\{x, z\} > 1$. □

Lemma 2.3. *The Diophantine equation $1 + 17^y = z^2$ has no nonnegative integer solution where y and z are nonnegative integers.*

Proof. Let y and z are nonnegative integers. Now we consider three possibilities.

Case 1: If $y = 0$. Then $z^2 = 2$, which is impossible.

Case 2: If $y = 1$. Then $z^2 = 18$, also impossible.

Case 3: If $y > 1$. Then $z^2 = 1 + 17^y > 1 + 17^1 = 18$. This implies that $z > 4$. By Proposition 2.1, there is no nonnegative integer solution for $\min\{x, z\} > 1$. □

Lemma 2.4. *The Diophantine equation $1 + 37^y = z^2$ has no nonnegative integer solution where y and z are nonnegative integers.*

Proof. Suppose y and z are nonnegative integers such that $1 + 37^y = z^2$. Then we consider at the possibilities.

- If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geqslant 1$. Hence, $z^2 = 1 + 37^y \geqslant 1 + 37^1 = 38$. Then $z \geqslant 7$. Now we consider on the equation $z^2 - 37^y = 1$. By Proposition 2.1, we obtain $y = 1$. Then $z^2 = 38$. This is impossible. Therefore, the equation $1 + 37^y = z^2$ has no nonnegative integer solution.

□

Lemma 2.5. *The Diophantine equation $1 + 91^y = z^2$ has no nonnegative integer solution where y and z are nonnegative integers.*

Proof. Let y and z are nonnegative integers. Now we consider three possibilities.

Case 1: If $y = 0$. Then $z^2 = 2$. This is impossible.

Case 2: If $y = 1$. Then $z^2 = 92$, also impossible.

Case 3: If $y > 1$. Then $z^2 = 1 + 91^y > 1 + 91^1 = 92$. This implies that $z > 9$. By Proposition 2.1, there is no nonnegative integer solution for $\min\{x, z\} > 1$. □

3 Main Results

Theorem 3.1. *The Diophantine equation* $13^x + 17^y = z^2$ *has no nonnegative integer solution, where* x, y *and* z *are nonnegative integers.*

Proof. Suppose x, y and z are nonnegative integers. We have three possibilities.

Case 1: If $x = 0$, then by Lemma 2.3 there is no nonnegative integer solution.

Case 2: If $x \geqslant 1$ and $y = 0$, then by Lemma 2.2 there is no nonnegative integer solution.

Case 3: If $x \geqslant 1$ and $y \geqslant 1$, then 13^x and 17^y are both odd. Thus, z^2 is even. Therefore, z is even. So we let $z = 2n$ with $n \in \mathbb{N}^*$. Then $z^2 = 4n^2 \equiv 0 \pmod 4$. Since $13 \equiv 1 \pmod 4$ and $17 \equiv 1 \pmod 4$, $13^x \equiv 1 \pmod 4$ and $17^y \equiv 1 \pmod 4$. Therefore, $z^2 = 13^x + 17^y \equiv 2 \pmod 4$. Consequently, $4n^2 \equiv 2 \pmod 4$ which is impossible. □

Corollary 3.2. *The Diophantine equation* $13^x + 17^y = u^{2t}$ *has no nonnegative integer solution, where* x, y, t *and* u *are nonnegative integers.*

Proof. Consider the Diophantine equation $13^x + 17^y = u^{2t}$, where x, y, t and u are nonnegative integers. Let $z = u^t$. By Theorem 3.1, the Diophantine equation $13^x + 17^y = z^2$ has no nonnegative integer solution. □

Corollary 3.3. *The Diophantine equation* $13^x + 17^y = u^{2(t+1)}$ *has no nonnegative integer solution, where* x, y, t *and* u *are nonnegative integers.*

Proof. Let $z = u^{t+1}$. By Theorem 3.1, the Diophantine equation $13^x + 17^y = z^2$ has no nonnegative integer solution. □

Theorem 3.4. *The Diophantine equation* $13^x + 37^y = z^2$ *has no nonnegative integer solution, where* x, y *and* z *are nonnegative integers.*

Proof. Suppose x, y and z are nonnegative integers. We have three possibilities.

Case 1: If $x = 0$, then by Lemma 2.4 there is no nonnegative integer solution.

Case 2: If $x \geqslant 1$ and $y = 0$, then by Lemma 2.2 there is also no nonnegative integer solution.

Case 3: If $x \geqslant 1$ and $y \geqslant 1$, then 13^x and 37^y are both odd. Thus, z^2 is even. Hence, $z^2 \equiv 1 \pmod 3$ or $z^2 \equiv 0 \pmod 4$. Since $13 \equiv 1 \pmod 3$ and $37 \equiv 1 \pmod 3$ or $13 \equiv 1 \pmod 4$ and $37 \equiv 1 \pmod 4$, $13^x \equiv 1 \pmod 3$ and $37^y \equiv 1 \pmod 3$ or $13^x \equiv 1 \pmod 4$ and $37^y \equiv 1 \pmod 4$. Consequently, $z^2 = 13^x + 37^y \equiv 2 \pmod 3$ or $z^2 = 13^x + 37^y \equiv 2 \pmod 4$ which is not possible. □

Corollary 3.5. *The Diophantine equation* $13^x + 37^y = u^{2t}$ *has no nonnegative integer solution, where* x, y, t *and* u *are nonnegative integers.*

Proof. Consider the Diophantine equation $13^x + 37^y = u^{2t}$, where x, y, t and u are nonnegative integers. Let $z = u^t$. By Theorem 3.4, the Diophantine equation reduces to $13^x + 37^y = z^2$ which has no nonnegative integer solution. □

Corollary 3.6. *The Diophantine equation* $13^x + 37^y = u^{2(t+1)}$ *has no nonnegative integer solution, where* x, y, t *and* u *are nonnegative integers.*

Proof. Let $z = u^{t+1}$. Then the Diophantine equation reduces to $13^x + 37^y = z^2$ which has no solution using, Theorem 3.4. □

Theorem 3.7. *The Diophantine equation* $17^x + 37^y = z^2$ *has no nonnegative integer solution, where* x, y *and* z *are nonnegative integers.*

Proof. Suppose x, y and z are nonnegative integers. We have three possibilities.

Case 1: If $x = 0$, then by Lemma 2.4 there is no nonnegative integer solution.

Case 2: If $x \geqslant 1$ and $y = 0$, then by Lemma 2.3 there is also no nonnegative integer solution.

Case 3: If $x \geqslant 1$ and $y \geqslant 1$, then 17^x and 37^y are both odd. Thus, z^2 is even. Then $z^2 \equiv 0 \pmod 4$. Since $17 \equiv 1 \pmod 4$ and $37 \equiv 1 \pmod 4$, $17^x \equiv 1 \pmod 4$ and $37^y \equiv 1 \pmod 4$. Consequently, $z^2 = 17^x + 37^y \equiv 2 \pmod 4$ which is not possible. □

Corollary 3.8. *The Diophantine equation* $17^x + 37^y = u^4$ *has no nonnegative integer solution, where* x, y *and* u *are nonnegative integers.*

Proof. Suppose x, y and u are nonnegative integers such that $17^x + 37^y = u^4$. Let $z = u^2$. Then the Diophantine equation reduces to $17^x + 37^y = z^2$. By Theorem 3.7, the equation $17^x + 37^y = z^2$ has no nonnegative integer solution. This implies that the equation $17^x + 37^y = u^4$ has no nonnegative integer solution. □

Corollary 3.9. *The Diophantine equation* $17^x + 37^y = u^{2(t+1)}$ *has no nonnegative integer solution, where* x, y, t *and* u *are nonnegative integers.*

Proof. Let $z = u^{t+1}$. Then the Diophantine equation reduces to $17^x + 37^y = z^2$ which has no nonnegative integer solution by Theorem 3.7. □

Theorem 3.10. *The Diophantine equation* $17^x + 91^y = z^2$ *has no nonnegative integer solution, where* x, y *and* z *are nonnegative integers.*

Proof. Suppose x, y and z are nonnegative integers. We consider the following remaining possibilities.

Case 1: If $x = 0$, then by Lemma 2.5 there is no nonnegative integer solution.

Case 2: If $y = 0$, then by Lemma 2.3 there is also no nonnegative integer solution.

Case 3: If $z = 0$, then $17^x + 91^y = 0$ which is not so since x and y are nonnegative integers.

Case 4: If $x = 1$, then $17 + 91^y = z^2$ and $17 + 91^y = z^2 \equiv 0 \pmod 6$. Thus, z^2 is even. Then z is even. Let $z = 2 \cdot 3n$, for $n \in \mathbb{N}^*$. Then $17 + 91^y = z^2 = 36n^2$. It follows that $17 + 91^y + 74 = 36n^2 + 74$ or $91 + 91^y = 36n^2 + 74$. As a result, $91(1 + 91^{y-1}) = 2(18n^2 + 37)$. So $1 + 91^{y-1} = 2$ and $18n^2 + 37 = 91$. Consequently, $y = 1$ and $n^2 = 3$ which is not possible.

Case 5: If $z = 1$, then $17^x + 91^y = 1$ which is not possible for nonnegative integers x and y.

Case 6: $x, y, z > 1$. Suppose that $17^x + 91^y = z^2$ is true for nonnegative integers x, y and z. We have $17^x \equiv 1 \pmod 4$ and $91^y \equiv 1 \pmod 4$ for an even integer y and $91^y \equiv 3 \pmod 4$ for an odd integer y. Since $z^2 \equiv 0 \pmod 4$, y must be odd and z must be even. We have two possibilities for x.

- If $x = 2m$, for $m \in \mathbb{N}^*$. Then $17^x + 91^y = z^2$ reduces to $17^{2m} + 91^y = z^2$. So $91^y = (z - 17^m)(z + 17^m)$. Take $z - 17^m = 91^t$ and $z + 17^m = 91^{y-t}$, where $y > 2t$ with t a nonnegative integer. It follows that $91^{y-t} - 91^t = 91^t(91^{y-2t} - 1) = 2 \cdot 17^m$ which implies that $91^t = 1$ and $91^{y-2t} - 1 = 2 \cdot 17^m$. Thus, $t = 0$ and $91^y - 1 = 2 \cdot 17^m$. Since $m \geqslant 1$, then $91^y = 1 + 2 \cdot 17^m \geqslant 1 + 2 \cdot 17^1 = 35$ which is impossible.
- If $x = 2m + 1$, for $m \in \mathbb{N}$. Then $17^x + 91^y = z^2$ reduces to $17^{2m+1} + 91^y = z^2$. So $91^y - 8 \cdot 17^{2m} = z^2 - 25 \cdot 17^{2m}$ Or $(z - 5 \cdot 17^m)(z + 5 \cdot 17^m) = 91^y - 8 \cdot 17^{2m}$. If $z - 5 \cdot 17^m = 1$ and $z + 5 \cdot 17^m = 91^y - 8 \cdot 17^{2m}$, then $10 \cdot 17^m + 8 \cdot 17^{2m} = 91^y - 1$. So $17^m(10 + 8 \cdot 17^m) = 91^y - 1$. Obviously, we see that there are impossible values for m and y to hold the equality. On the other hand, if $z - 5 \cdot 17^m = 91^y - 8 \cdot 17^{2m}$ and $z + 5 \cdot 17^m = 1$, then $17^m(10 - 8 \cdot 17^m) = 1 - 91^y$ which has no nonnegative integer solution.

□

Corollary 3.11. *The Diophantine equation* $17^x + 91^y = u^{2t}$ *has no nonnegative integer solution, where* x, y, t *and* u *are nonnegative integers.*

Proof. Suppose x, y, t and u are nonnegative integers such that $17^x + 91^y = u^{2t}$. Let $z = u^t$. Then the Diophantine equation reduces to $17^x + 91^y = z^2$. By Theorem 3.10, the equation $17^x + 91^y = z^2$ has no nonnegative integer solution. This implies that the equation $17^x + 91^y = u^{2t}$ has no nonnegative integer solution. □

Corollary 3.12. *The Diophantine equation* $17^x + 91^y = u^{2(t+1)}$ *has no nonnegative integer solution, where* x, y, t *and* u *are nonnegative integers.*

Proof. Let $z = u^{t+1}$. Then the Diophantine equation reduces to $17^x + 91^y = z^2$ which has no nonnegative integer solution by Theorem 3.10. □

4 Conclusions

We demonstrated that the exponential Diophantine equations $13^x + 17^y = z^2$, $13^x + 37^y = z^2$, $17^x + 37^y = z^2$ and $17^x + 91^y = z^2$ have no solutions in the nonnegative integers x, y and z.

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Author information

Anouar Gaha, Mathematics and Intelligent Systems (MASI), National School of Applied Sciences of Tangier (ENSAT), Abdelmalek Essaadi University, Tangier, Morocco.
E-mail: anouar.gaha@etu.uae.ac.ma

Soufiane Mezroui, Mathematics and Intelligent Systems Team (MASI), Higher School of Technology of Tetouan (ESTTe), Abdelmalek Essaadi University, Tetouan, Morocco.
E-mail: smezroui@uae.ac.ma

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