Palestine Journal of Mathematics

Vol 14(3)(2025) , 575–580 © Palestine Polytechnic University-PPU 2025

LIMIT CYCLES OF NONLINEAR TORSIONAL VIBRATION SYSTEM VIA AVERAGING THEORY

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Communicated by Martin Bohner

MSC 2010 Classifications: Primary 37G15; Secondary 37C80; Thirdly 37C30.

Keywords and phrases: Limit cycle, nonlinear torsional vibration system, averaging theory.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract We investigate the existence of limit cycles of the nonlinear torsional vibration equation of the form

$$\ddot{x} + \omega^2 x + \alpha x^2 + \beta x^3 + \gamma \dot{x} = G(t),$$

where $G(t)$ is a $\dfrac{2\pi}{\omega}$-periodic function with respect to the variable t, and $\alpha, \beta, \gamma, \omega$ are arbitrary parameters. In addition, we give an application.

1 Introduction and summary of the principal results

In rotating machinery equipments appear widely torsional vibration systems, for instance in rolling mill, turbine generator and steam turbine. Torsional vibration might be caused by unbalanced rotating parts, or due to torque fluctuations, or due to other mechanical reasons. The rotating axes or their complements may sustain harm or destruction if these vibrations are not controlled. For a mechanical drive system, torsional vibrations are crucial for performance and reliability. Consequently to study the stability or instability of torsional vibration mechanisms and their dynamics behaviors are important for the optimal design and vibration monitoring of a system.

Nonlinear torsional vibration systems have been studied intensively in these last years see for instance [2, 3, 4, 8]. But the strongly nonlinear torsional vibration systems became each time more important in engineering, and its dynamics behaviors have received less attention. In their study [10], the authors used Lagrange equations of motion to derive the dynamics equation for strongly nonlinear torsional vibration systems with external excitation quadratic and cubic nonlinear rigidity. They studied bifurcations and chaotic motions on these systems.

This study investigates the periodic solutions of the nonlinear torsional vibration systems considered in [10], more specifically, we are interested in examining the existence of limit cycles in the second–order non-autonomous differential equation of the following form.

$$\ddot{x} + \omega^2 x + \alpha x^2 + \beta x^3 + \gamma \dot{x} = G(t) \tag{1.1}$$

where $G(t)$ is a $\dfrac{2\pi}{\omega}$-periodic function in the variable t, and α, β, γ and $\omega \neq 0$ are arbitrary parameters. The derivative with respect to the time variable t is represented by the dot.

We define

$$\begin{aligned} S &= -\frac{1}{2\pi\omega^2} \int_0^{\frac{2\pi}{\omega}} G(t) \sin(\omega t)\, dt, \\ C &= \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} G(t) \cos(\omega t)\, dt. \end{aligned} \tag{1.2}$$

The following theorem gives our main result about the periodic solutions to the system (3.1).

Theorem 1.1. *Let the non-autonomous differential system* (3.1) *with* $\omega \neq 0$ *and* $C^2 + S^2 > 0$. *Let* $A = \gamma^3 C + 3\beta S^3 w^6$. *If the coefficients* α, β, γ *and the function* $G(t)$ *are of order* $O(\varepsilon)$ *being* ε *a small parameter, then, the following claims are true. System* (3.1) *has*

(a) *one limit cycle* $x(t)$ *if*

$$\beta A\big(27\beta^2\tilde{y}_0^4 + 54\beta^2\tilde{x}_0^2\tilde{y}_0^2\omega^2 + 27\beta^2\tilde{x}_0^4\omega^4 + 16\gamma^2\omega^6\big) \neq 0,$$

where we have that $\tilde{y}_0$ *is the unique real root of the cubic polynomial* $9\beta^2(C^2 + S^2\omega^6)y_0^3 - 12\beta\gamma^2 S\omega^6 y_0^2 + 4\gamma\omega^6\left(\gamma^3 + 9\beta CS\right)y_0 - 8\omega^6\left(\gamma^3 C - 3\beta S^3\omega^6\right)$, *and* $\tilde{x}_0 = \big(3\beta\gamma(C^2+S^2\omega^6)\tilde{y}_0^2 - 2\omega^6 S(\gamma^2 + 3\beta CS)\tilde{y}_0 + 8\gamma^2 CS\omega^6\big)/(2\omega^4 A)$;

(b) *one limit cycle* $x(t)$ *if* $\beta = 0$ *and* $\gamma \neq 0$;

(c) *one limit cycle* $x(t)$ *if* $\beta\gamma S \neq 0$ *and* $A = 0$;

(d) *one limit cycle* $x(t)$ *if* $\beta \neq 0$ *and* $A = S = 0$.

Moreover when the limit cycle exists it is $O(\varepsilon)$ *near of the periodic solution*

$$x(t) = \tilde{x}_0\cos(\omega t) + \frac{\tilde{y}_0}{\omega}\sin(\omega t),$$

where in case (a) *the values of* $\tilde{y}_0$ *and* $\tilde{x}_0$ *are the ones defined in statement* (a); *in case* (b) *we have that* $(\tilde{x}_0, \tilde{y}_0) = (2S\omega^2/\gamma, 2C/\gamma)$; *in case* (c) *we have that*

$$(\tilde{x}_0, \tilde{y}_0) = \left(\frac{2y_0 S\omega^2}{\gamma}\left(1 - \frac{18\beta^2 S^4\omega^6}{\gamma^6 + 9\beta^2 S^4\omega^6}\right), -\frac{12\beta\gamma^2 S^3\omega^6}{\gamma^6 + 9\beta^2 S^4\omega^6}\right);$$

and in case (d) *we have that* $(\tilde{x}_0, \tilde{y}_0) = \big(2(C/(3\beta))^{1/3}, 0\big)$.

The proof of Theorem 1.1 is in Section 3; it is based in the averaging theory of first order. Section 2 provides a summary of the results needed to prove Theorem 1.1. For further examples of the averaging theory in the study of periodic solutions, refer to, for instance [7] and [9].

Note that case (a) of Theorem 1.1 is the more generic because in it we do not need that any equality holds.

We give an application of Theorem 1.1.

Corollary 1.2. *Examine the non-autonomous second-order differential equation given by* (1.1) *under the assumptions of Theorem 1.1 with* $\omega = 1$, $\beta = \gamma = \varepsilon$, $G(t) = \varepsilon\sin t$ *and* $\alpha = \varepsilon\alpha_1$. *Then for* $\varepsilon \neq 0$ *sufficiently small this differential equation has one stable limit cycle* $x(t)$ *near the periodic solution*

$$\begin{aligned} x(t) = & \left(\frac{2(\sqrt{921}^{\frac{1}{3}} - 27)}{3\cdot 3^{\frac{2}{3}}} - \frac{8}{3(3(\sqrt{921}-27))^{\frac{1}{3}}}\right)\cos t + \\ & \frac{1}{9}\left((550 - 18\sqrt{921})^{\frac{1}{3}} + (550 + 18\sqrt{921} - 8)^{\frac{1}{3}}\right)\sin t, \end{aligned} \tag{1.3}$$

of $\ddot{x} + x = 0$ *when* $\varepsilon \to 0$.

Corollary 1.2 is demonstrated in Section 3.

2 Fundamental results on averaging theory

The fundamental averaging theory result that will be used to prove Theorem 1.1 is presented in this section.

The problem of investigating T-periodic solutions of differential systems is formulated as

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}) + F_1(t, \mathbf{x})\varepsilon + F_2(t, \mathbf{x}, \varepsilon)\varepsilon^2, \tag{2.1}$$

with ε is small enough, the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are of class $\mathcal{C}^2$ and T-periodic with respect to the variable t, Ω consists of an open subset of $\mathbb{R}^n$. For the unperturbed system.

$$\dot{\mathbf{x}} = F_0(t, \mathbf{x}), \tag{2.2}$$

There is a submanifold with periodic solutions. The averaging theory gives a solution to this problem.
The solution of system (2.2) with the initial condition $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$ is shown by $\mathbf{x}(t, \mathbf{z})$. Along a periodic solution $\mathbf{x}(t, \mathbf{z})$, the linearized system of the unperturbed system (2.2) is

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}. \tag{2.3}$$

Let $M_{\mathbf{z}}(t)$ be the fundamental matrix of the linear differential system (2.3). Assume that there exists an open set V with $V \subset M$, such that $\mathbf{x}(t, \mathbf{z})$ is T-periodic for every $\mathbf{z} \in Cl(V)$. Then, for sufficiently small ε, the following conclusion gives sufficient conditions for the periodic solutions of V to persist.

Theorem 2.1 (Perturbations of an isochronous set). *Assume that there exists an open and bounded set V with $Cl(V) \subset \Omega$ such that, for each $\mathbf{z} \in Cl(V)$, the solution $\mathbf{x}(t, \mathbf{z})$ is T-periodic, considering a function $\mathcal{F} : Cl(V) \to \mathbb{R}^n$ as defined by*

$$\mathcal{F}(\mathbf{z}) = \frac{1}{T} \int_0^T M_{\mathbf{z}}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}))\, dt. \tag{2.4}$$

If there exists $a \in V$ such that $\mathcal{F}(a) = 0$ and $\det((d\mathcal{F}/d\mathbf{z})(a)) \neq 0$, then there exists a T-periodic solution $\varphi(t, \varepsilon)$ to system (2.1) such that $\varphi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

Theorem 2.1 is based on the work of Malkin [5] and Roseau [6]; see [1] for a proof.

3 Proofs

Proof of Theorem 1.1. To represent the second-order differential equation (1.1) in normal form (2.1) of the averaging theory we use the assumption that α, β, γ and the function $G(t)$ are of order $O(\varepsilon)$ being ε a small parameter, i.e. $\alpha = \varepsilon a$, $\beta = \varepsilon b$, $\gamma = \varepsilon c$, $G(t) = \varepsilon g(t)$, then equation (1.1), thus corresponds to (3.1)

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega^2 x + \varepsilon(-ax^2 - bx^3 - cy + g(t)). \end{aligned} \tag{3.1}$$

Let us investigate the non-autonomous differential system (3.1). For $\varepsilon = 0$, the unique singular point of system (3.1) is $(x, y) = (0, 0)$. At this singular point, the linearized system of (3.1) has $\pm i\omega$ eigenvalues. The unperturbed system's solution $(x(t), y(t))$, such that $(x(0), y(0)) = (x_0, y_0)$ is

$$\begin{aligned} x(t) &= x_0 \cos(\omega t) + \frac{y_0}{\omega} \sin(\omega t), \\ y(t) &= -x_0 w \sin(\omega t) + y_0 \cos(\omega t). \end{aligned} \tag{3.2}$$

We remark that these solutions have the period $\frac{2\pi}{\omega}$. According to (2.1), we deduce

$$\begin{aligned} \mathbf{x} &= (x, y), \\ \mathbf{z} &= (x_0, y_0), \\ F_0(t, \mathbf{x}) &= \left(y, -\omega^2 x\right), \\ F_1(t, \mathbf{x}) &= \left(0, -ax^2 - bx^3 - cy + g(t)\right), \\ F_2(t, \mathbf{x}) &= (0, 0). \end{aligned}$$

Since the fundamental matrix solution $M_{\mathbf{z}}(t)$ is independent of $\mathbf{z}$, we will write it as $M(t)$. A computation shows that

$$M(t) = \begin{pmatrix} \cos(\omega t) & \dfrac{\sin(\omega t)}{\omega} \\ -\omega \sin(\omega t) & \cos(\omega t) \end{pmatrix}.$$

According to Theorem 2.1 we must look for the zeros (x_0, y_0) of the function $\mathcal{F}(x_0, y_0)$ given in (2.4). More precisely, we have $\mathcal{F}(x_0, y_0) = (\mathcal{F}_1(x_0, y_0), \mathcal{F}_2(x_0, y_0))$ such that

$$\begin{aligned}
\mathcal{F}_1(x_0, y_0) &= \frac{3by_0^3}{8\omega^6} + \frac{3bx_0^2y_0}{8\omega^4} - \frac{cx_0}{d\omega^2} + S_0, \\
\mathcal{F}_2(x_0, y_0) &= -\frac{3bx_0^3}{8} - \frac{cy_0}{2} - \frac{3bx_0y_0^2}{8\omega^2} + C_0.
\end{aligned} \tag{3.3}$$

where $S_0 = S/\varepsilon$ and $C_0 = C/\varepsilon$ and C and S are defined in (1.2). For solving this system we use the Groebner basis method. Thus we obtain an other polynomial system of ten equations in the variables x_0 and y_0, which has the same solutions than system $\mathcal{F}(x_0, y_0) = (0, 0)$. We only use from these ten equations the following two:

$$\begin{aligned}
e_1 = &\; 9b^2\left(C_0^2 + S_0^2\omega^6\right)y_0^3 - 12bc^2S_0\omega^6y_0^2 + \\
&\; 4c\omega^6\left(c^3 + 9bC_0S_0\right)y_0 - 8\omega^6\left(c^3C_0 - 3bS_0^3\omega^6\right) = 0, \\
e_2 = &\; 2\omega^4(3bS_0^3\omega^6 + c^3C_0)x_0 - 3bc(C_0^2 + S_0^2\omega^6)y_0^2 + \\
&\; 2\omega^6S_0(c^3 + 3bC_0S_0)y_0 - 8c^2C_0S_0\omega^6 = 0.
\end{aligned}$$

The equation $e_1 = 0$ and $e_2 = 0$ can be written as

$$\begin{aligned}
E_1 = &\; 9\beta^2\left(C^2 + S^2\omega^6\right)y_0^3 - 12\beta\gamma^2S\omega^6y_0^2 + \\
&\; 4\gamma\omega^6\left(\gamma^3 + 9\beta CS\right)y_0 - 8\omega^6\left(\gamma^3C - 3\beta S^3\omega^6\right) = 0, \\
E_2 = &\; 2\omega^4(3\beta S^3\omega^6 + \gamma^3C)x_0 - 3\beta\gamma(C^2 + S^2\omega^6)y_0^2 + \\
&\; 2\omega^6S(\gamma^2 + 3\beta CS)y_0 - 8\gamma^2CS\omega^6 = 0,
\end{aligned}$$

after multiplied by ε^4. Moreover the system $(\mathcal{F}_1(x_0, y_0), \mathcal{F}_2(x_0, y_0)) = (0, 0)$ defined by (3.3) is equivalent to the system

$$\begin{aligned}
\varepsilon\mathcal{F}_1(x_0, y_0) &= \frac{3\beta y_0^3}{8\omega^6} + \frac{3\beta x_0^2y_0}{8\omega^4} - \frac{\gamma x_0}{d\omega^2} + S = 0, \\
\varepsilon\mathcal{F}_2(x_0, y_0) &= -\frac{3\beta x_0^3}{8} - \frac{\gamma y_0}{2} - \frac{3\beta x_0y_0^2}{8\omega^2} + C = 0.
\end{aligned} \tag{3.4}$$

Proof of statement (a) *of Theorem 1.1.* Equation $E_1 = 0$ is a polynomial of degree three with respect to the variable y_0. It is known that if the polynomial $a_3x^3 + a_2x^2 + a_1x + a_0$ has $\Delta = 18a_3a_2a_1a_0 - 4a_2^3a_0 + a_2^2a_1^2 - 4a_3a_1^3 - 27a_3^2a_0^2 < 0$, then it has only one real root and two complex (non-real) roots. In our case the Δ for the cubic equation $E_1 = 0$ is

$$-576\beta^2\omega^{12}(\gamma^3C + 3\beta S^3\omega^6)^2(243\beta^2C^4 + 4\gamma^6\omega^6 + 486C^2S^2\omega^6 + 243\beta^2S^4\omega^{12}).$$

Since $\beta \neq 0$, $\omega \neq 0$, $A = \gamma^3C + 3\beta S^3\omega^6 \neq 0$ and $C^2 + S^2 > 0$, the factor $243\beta^2C^4 + 4\gamma^6\omega^6 + 486C^2S^2\omega^6 + 243\beta^2S^4\omega^{12} > 0$, and consequently $\Delta < 0$. So the cubic polynomial equation $E_1 = 0$ has a unique real root $\widetilde{y}_0$. Substituting this root in the equation $E_2 = 0$ we obtain a unique value $\widetilde{x}_0$ for x_0. Now it is easy to check that this root $(\widetilde{x}_0, \widetilde{y}_0)$ verifies system (3.4), and if its Jacobian $(27\beta^2\widetilde{y}_0^4 + 54\beta^2\widetilde{x}_0^2\widetilde{y}_0^2\omega^2 + 27\beta^2\widetilde{x}_0^4\omega^4 + 16\gamma^2\omega^6)/(64\omega^8) \neq 0$. So, Theorem 2.1 provides a unique limit cycle for the differential equation (1.1) for the values $\tilde{x}_0$ and $\tilde{y}_0$ given in the statement of the theorem. The proof for statement (a) of Theorem 1.1 is complete. □

Proof of statement (b) *of Theorem 1.1.* If $\beta = 0$ and $\gamma \neq 0$, then for these values the Groebner basis of system (3.4) with respect to the variables x_0 and y_0 becomes

$$\begin{aligned}
\gamma y_0 - 2C &= 0, \\
Cx_0 - S\omega^2y_0 &= 0, \\
\gamma x_0 - 2S\omega^2 &= 0.
\end{aligned}$$

It has the unique solution $(\tilde{x}_0, \tilde{y}_0) = (2S\omega^2/\gamma, 2C/\gamma)$, and the Jacobian of system (3.4) evaluated at this solution is $\gamma^2/(4\omega^2) \neq 0$. Then, according to Theorem 2.1, statement (b) of Theorem 1.1 follows. □

Proof of statement (c) *of Theorem 1.1.* We assume that $\beta\gamma S \neq 0$ and $A = 0$. This implies that $C = -3\beta S^3\omega^6/\gamma^3$. In this case, by using the Groebner basis method, we obtain the equivalent polynomial system

$$\begin{aligned}
&(-2\gamma^2 + 3\beta S y_0)(\gamma^6 y_0 + 12\beta\gamma^2 S^3\omega^6 + 9\beta^2 S^4\omega^6 y_0) = 0,\\
&(-2\gamma^2 + 3\beta S y_0)(\gamma x_0^3 - 2\gamma^2 S\omega^2 - 3\beta S^2\omega^2 y_0) = 0,\\
&\gamma^4 y_0^2 + \gamma^4\omega^2 x_0^2 - 2\gamma^3 S\omega^4 x_0 + 6\beta S^3\omega^4 y_0 = 0,
\end{aligned}$$

to polynomial system (3.4). By solving this system we obtain a unique real solution, namely

$$(\tilde{x}_0, \tilde{y}_0) = \left(\frac{2y_0 S\omega^2}{\gamma}\left(1 - \frac{18\beta^2 S^4\omega^6}{\gamma^6 + 9\beta^2 S^4\omega^6}\right), -\frac{12\beta\gamma^2 S^3\omega^6}{\gamma^6 + 9\beta^2 S^4\omega^6}\right).$$

Since the Jacobian of system (3.4) evaluated at this solution is $(C^6 + 27\beta^2 S^4\omega^6)/\,(4\gamma^4\omega^2) \neq 0$, Theorem 2.1 is used to prove assertion (c) of Theorem 1.1. □

Proof of statement (d) *of Theorem 1.1.* If $S = 0$, from $\gamma^3 C + 3\beta S^3\omega^6 = 0$ we deduce that $\gamma = 0$. Solving in this case the system (3.4) we get a unique real solution $(\tilde{x}_0, \tilde{y}_0) = \big(2(C/(3\beta))^{1/3}, 0\big)$, and the Jacobian of system (3.4) evaluated at this solution is $3(3\beta C^2)^{2/3}/(4\omega^4) \neq 0$. So, by Theorem 2.1 statement (d) of Theorem 1.1 follows. □

This concludes the proof of Theorem 1.1. □

Proof of Corollary 1.2. We consider the non-autonomous second-order differential equation (1.1) with $\omega = 1$, $\beta = \gamma = \varepsilon$, $G(t) = \varepsilon \sin t$ and $\alpha = \varepsilon\alpha_1$ arbitrary. Then the constants C and S defined in (1.2) now take the values $C = 0$ and $S = -\varepsilon/2$. Therefore, using the notation of Theorem 1.1 we have that $\beta A = -3\varepsilon^5/8 \neq 0$. So the values of $(\tilde{x}_0, \tilde{y}_0)$ corresponding to the case (a) of Theorem 1.1 are now

$$\begin{aligned}
\tilde{x}_0 &= \frac{2(\sqrt{921} - 27)^{1/3}}{3 \cdot 3^{2/3}} - \frac{8}{3(3(\sqrt{921} - 27))^{1/3}},\\
\tilde{y}_0 &= \frac{1}{9}\left((550 - 18\sqrt{921})^{1/3} + (550 + 18\sqrt{921})^{1/3} - 8\right),
\end{aligned}$$

where $\tilde{x}_0$ is the unique real real root of the cubic polynomial $16 + 16x_0 + 9x_0^3$, and $\tilde{y}_0$ is the unique real real root of the cubic polynomial $-12 + 16y_0 + 24y_0^2 + 9y_0^3$. The eigenvalues of the Jacobian matrix of system (3.4) evaluated at the solution $(\tilde{x}_0, \tilde{y}_0)$ are complex, having real part equal to $-1/2$, consequently the limit cycle (3.1) associated to this solution is stable. This demonstrates Corollary 1.2. □

Acknowledgements

The first author is partially supported by a MINECO grant MTM2013-40998-P, an AGAUR grant number 2014SGR-568, and the grants FP7-PEOPLE-2012-IRSES 318999 and 316338.

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Received: 2024-09-24
Accepted: 2024-11-05