DYNAMICAL PROPERTIES OF CRITICAL EXPONENT FUNCTIONS

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Abstract In the last years the attention towards topological dynamical properties of highly discontinuous maps has increased significantly. In [1], a class of densely discontinuous interval maps, called "critical exponent maps", was introduced. These maps are defined through the word-combinatorics concept of critical exponent applied to the binary expansion of reals and show highly chaotically properties as well as some challenging problems. In this paper we identify an error in the proof of Theorem 7 in [1], a purely combinatorial result which in fact does not hold. We show that most of the results in [1], obtained there through Theorem 7, can be recovered. Moreover, we propose as a conjecture a weaker form of Theorem 7.

1 Introduction

In recent decades, emphasis has been placed on the concept of *critical exponent* of a word w over a given alphabet, that is the supremum of the reals α for which w contains an α-power. The critical exponents are significant in fields such that symbolic dynamics and transcendental number theory [2, 3]. In particular, this concept is closely linked to the problem of repetition threshold of an alphabet see [4, p. 126]), which was solved quite recently [5, 6, 7].

In [1], a family of interval maps based on the word-combinatorial concept of critical exponent was introduced. The generic n-*critical exponent function* $\kappa_n : [0, 1] \to [0, 1]$ is the map that associates to every $x \in [0, 1]$, the inverse of the critical exponent of the (infinite) base-n expansion of x (we assume $1/\infty = 0$). A word-combinatorial result, namely [1, Theorem 7], was presented and employed in that paper to prove some analytical and dynamical properties of such interval maps. However, [1, Theorem 7] does not hold. Denoting $\{0, 1\}^*$ and $\{0, 1\}^\omega$ as the sets of finite and infinite binary words, respectively, and using $E(w)$ to represent the critical exponent of a word (see Definition 2.2), Theorem 7 in [1] was formulated as follows:

> "*Let $w \in \{0, 1\}^*$. For every $\alpha > \max\{2, E(w)\}$,*
> *there exists $y \in \{0, 1\}^\omega$ such that $E(wy) = \alpha$*".

This claim is false. For instance, take $w \in \{0, 1\}^*$ of the form $w = (u1v)^k u$ with $u = (p0q)^{k'} p$ where $k, k' \in \mathbb{N}$ and $v, p, q \in \{0, 1\}^*$. We have:

$$E(w) \geq k + \frac{|u|}{|u| + |v| + 1}.$$

Meanwhile, $E(w0) \geq k' + \frac{|p|+1}{|p|+|q|+1}$ and $E(w1) \geq k + \frac{|u|+1}{|u|+|v|+1}$. In particular, we can take $v, p, q \in \{0, 1\}^*$ such that the previous inequalities become equalities. For instance, the simplest case is obtained by choosing $p = 0$, $q = v = \epsilon$, $k' = 1$ and $k = 2$. In this case, we have $w = 00100100$ and $E(w) = 8/3$. Moreover, both $w0$ and $w1$ are 3-powers, so that $E(w0) = E(w1) = 3$, from which follows that $E(wy) \geq 3 > 8/3$ for every $y \in \{0, 1\}^\omega$. We thank Saúl Rodríguez Martín for this observation.

We think that a result of similar flavor is true (see Conjecture 3.2). Even if in the present work we managed to walk around the use of [1, Theorem 7] to establish the main properties of

the critical exponent maps, we believe that the conjecture is interesting in itself.

The paper is organized as follows. In Section 2, we introduce the preliminary concepts and the notation that will be used throughout the paper. Section 3 presents our conjecture that would replace [1, Theorem 7], together with some remarks and weaker results useful in the subsequent analysis. Section 4 introduces the critical exponent functions and examines their properties, taking into account the error identified in [1, Theorem 7]. Finally, we highlight the claims made in [1] which are still not recovered.

2 Setting and notation

We denote the set of all natural numbers (i.e. positive integers) by $\mathbb{N}$ and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathcal{A} = \{a_0, a_1, \ldots, a_n\}$ be an alphabet. We indicate by $\mathcal{A}^*$ and $\mathcal{A}^\omega$ the set of the finite and infinite words over $\mathcal{A}$, respectively. We set $\mathcal{A}^\infty = \mathcal{A}^* \cup \mathcal{A}^\omega$. We let ϵ denote the empty word and we set $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\epsilon\}$. Given a word $w \in \mathcal{A}^\infty$, we indicate by w_i its i-th digit. Let $\ell \colon \mathcal{A}^* \to \mathbb{N}$ be the map that associates to the non-empty finite word $w = w_1 \ldots w_n$ the natural number n, while we set $\ell(\epsilon) = 0$. For $v, w \in \mathcal{A}^\infty$, we say that v is a *subword* of w if there exist $p \in \mathcal{A}^*$ and $s \in \mathcal{A}^\infty$ such that $w = pvs$. We say that $v \in \mathcal{A}^+$ is a *prefix* (*suffix*) of w if $w = vs$ ($w = pv$). For $w \in \mathcal{A}^\omega$, we indicate by $\mathcal{L}(w)$ the *language* of w, that is the set of all the finite subwords of w.

For every $w \in \mathcal{A}^*$ and $n \in \mathbb{N}$, we indicate by w^n the concatenation of n-copies of w, that is

$$w^n = \underbrace{ww \ldots w}_{n\text{-times}}.$$

We call *deletion operator* the function $\delta \colon \mathcal{A}^\infty \to \mathcal{A}^\infty$ that removes the first digit of a word. Hence, for a word $w = w_0 w_1 w_2 \ldots \in \mathcal{A}^\infty$, we have that

$$\delta(w) = \delta(w_0 w_1 w_2 \ldots) = w_1 w_2 \ldots$$

and we set $\delta(\epsilon) = \epsilon$. Consequently, we can define for every $n \in \mathbb{N}$, the operator δ^n that removes the first n letters of a word. Thus, if $w = w_0 w_1 w_2 \ldots \in \mathcal{A}^\infty$, we have that

$$\delta^n(w) = \delta^n(w_0 w_1 \ldots w_n w_{n+1} \ldots) = w_n w_{n+1} \ldots \ .$$

For every $x \in [0, 1]$, we denote by $w_x \in \{0, 1\}^\omega$ the binary expansion of x, that is

$$x = (0.w_x)_2 = \sum_{i=0}^{\infty} \frac{(w_x)_i}{2^{i+1}}.$$

Remark 2.1. If $x \in [0, 1]$ is a dyadic rational, we consider its binary expansion w_x, whose digits are ultimately 1, for instance $w_{\frac{1}{2}} = 01111\ldots$.

For any finite binary word $w \in \{0, 1\}^+$, we denote by I_w the open cylinder characterized by w, hence $I_w := \left((0.w)_2, (0.w)_2 + 2^{-\ell(w)}\right)$.

For positive integers p and q, we say that a word $w \in \mathcal{A}^+$ is a *p/q-power* if it is of the form $w = x^n y$, where $x \in \mathcal{A}^+$, $y \in \mathcal{A}^*$ is a prefix of x, $\ell(w) = p$ and $\ell(x) = q$ (cf. [8]). Let $\alpha \in \mathbb{Q}^+$. A word $w \in \mathcal{A}^\infty \setminus \{\epsilon\}$ *avoids α-powers* if none of its subwords is an r-power for any rational $r \geq \alpha$. Otherwise, we say that w *contains an α-power*.

Definition 2.2. The *critical exponent* of a word is the function $E \colon \mathcal{A}^\infty \to \mathbb{R}$, given by

$$E(w) = \begin{cases} \sup\{r \in \mathbb{Q} : w \text{ contains an } r\text{-power}\}, & \text{if } w \neq \epsilon, \\ 0, & \text{if } w = \epsilon. \end{cases}$$

Let us define the well known Thue-Morse sequence, which is important for fields such as number theory, differential geometry, combinatorics of words, and Morse theory ([9]).

Definition 2.3. The *Thue-Morse sequence* is the element $\tau = \tau_0 \tau_1 \tau_2 \ldots \tau_n \ldots \in \{0, 1\}^\omega$ given by

$$
\begin{cases}
\tau_0 = 0, \\
\tau_{2n} = \tau_n, \\
\tau_{2n+1} = 1 - \tau_n.
\end{cases}
$$

The Thue-Morse sequence starts as follows:

$$\tau = 0\ 1\ 10\ 1001\ 10010110\ 1001011001101001\ \ldots\ .$$

It has been proved that $E(\tau) = 2$ (see the classical paper [10]).

Remark 2.4. By recursively using bitwise negation, an alternative construction can be employed to define the Thue-Morse sequence. The first digit of the sequence is $\tau_0 = 0$. Then, for any $n \in \mathbb{N}_0$, if the first 2^n digits are given, then the next 2^n digits of the sequence are the bitwise negations of the formers. In other words, if we have $\tau_0, \tau_1, \ldots, \tau_{2^n-1}$, then $\tau_{k+2^n} = 1 - \tau_k$ for every $k = 0, \ldots, 2^n - 1$.

3 A conjecture and some combinatorial results

In this section, we present our conjecture replacing the [1, Theorem 7] along with some remarks and weaker results. The latter results, expressed in Proposition 3.3 and 3.5, will be useful for the analysis of dynamical and analytical properties of critical exponent functions done in the next section.
In order to present the conjecture, we give the following notation.

Definition 3.1. For any $w \in \{0, 1\}^*$, we denote by $\mathcal{P}(w) \subset \{0, 1\}^\omega$ the set of binary words of infinite length with prefix w, hence

$$\mathcal{P}(w) := \{wy : y \in \{0, 1\}^\omega\}, \tag{3.1}$$

and we denote by $\mathcal{E}_w$ the infimum among all the critical exponent of the words in $\mathcal{P}(w)$, namely

$$\mathcal{E}_w := \inf_{z \in \mathcal{P}(w)} E(z). \tag{3.2}$$

Notice that, in general, we have $E(wy) \neq E(y)$ for $w \in \{0, 1\}^*$ and $y \in \{0, 1\}^\infty$, even if $E(w) \leq E(y)$. In particular, we can have $E(wy) > \max\{E(w), E(y)\}$. For instance, by taking $w = 00$ and $y = \tau$, we have that

$$E(00\tau) = 3 > \max\{E(00), E(\tau)\} = 2.$$

Let us now state the conjecture.

Conjecture 3.2. For any $w \in \{0, 1\}^*$ and $\alpha \geq \mathcal{E}_w$, there exists $y \in \{0, 1\}^\omega$ such that $E(wy) = \alpha$. In other words,
$$E(\mathcal{P}(w)) = \{E(z) : z \in \mathcal{P}(w)\} = [\mathcal{E}_w, +\infty].$$

Even proving that for any $w \in \{0, 1\}^*$ the infimum $\mathcal{E}_w$ is attained does not appear straightforward. However, this is not the primary focus of the result. In particular, we find the most intriguing aspect to be the proof that $E(\mathcal{P}(w))$ cannot be a disconnected set; hence, it must be either $E(\mathcal{P}(w)) = (\mathcal{E}_w, +\infty]$ or $E(\mathcal{P}(w)) = [\mathcal{E}_w, +\infty]$.
Notice that the previous conjecture holds true if $w = \epsilon$, as proved by J. D. Currie and N. Rampersad in [11]. Moreover, by using the construction of [11], it holds also whenever w is a finite subword of the Thue-Morse sequence. More formally, we have the following result, which in itself is an (easy) generalization of the result proved by Currie and Rampersad.

Proposition 3.3. *Let* $w \in \{0, 1\}^*$ *be a finite subword of the Thue-Morse sequence. Then, for any* $\alpha \geq 2$ *there exists* $y \in \{0, 1\}^\omega$ *such that* $E(wy) = \alpha$.

Remark 3.4. To provide a clearer and more readable proof of Proposition 3.3, let us briefly review the key points we need from the construction presented in [11] for a word $z \in \{0,1\}^\omega$ with critical exponent α, for any $\alpha \geq 2$. The interest reader can find the details in the original paper.

If $\alpha = 2$, then just consider the Thue-Morse sequence, namely $z = \tau$.

Otherwise, let $(\beta_i)_{i \in \mathbb{N}} \subset \mathbb{R}$ a sequence of real numbers converging to α such that for any $i \in \mathbb{N}$ we have $\beta_i < \alpha$ and $\beta_i = r_i - t_i/2^{s_i}$ for some positive integers r_i, t_i, s_i with $s_i \geq 3$ and $t_i < 2^{s_i}$. Moreover, denoting by μ the *Thue-Morse morphism*, i.e., $\mu(0) = 01$ and $\mu(1) = 10$, assume that $\delta^{t_i}\mu^{s_i}(0)$ begins with 00. By [11, Lemma 5], the word $\delta^{t_i}\mu^{s_i}(0^{r_i})$ has critical exponent in $[\beta_i, \alpha)$, for any $i \in \mathbb{N}$. For any word w (here some technical details are omitted), let us denote by $\phi_i(w)$ the word given by $\delta^{t_i}\mu^{s_i}(0^{r_i}w)$. Then, by construction and by [11, Lemma 5], the sequence of words $(w_n)_{n \in \mathbb{N}}$ defined as

$$w_n = \phi_1(\phi_2(\ldots(\phi_n(\epsilon)))),$$

is such that $\ell(w_n)$ is diverging to infinity and w_n is a prefix of w_{n+1}, so that the word $z = \lim_{n \to \infty} w_n$ is well defined. Moreover, $E(z) \geq E(w_n)$ for every n and, at the same time, $E(p) < \alpha$ for every prefix p of z, implying that $E(z) = \alpha$.

By the previous construction, we have that z attains its critical exponent only "at the infinity", meaning that $E(\delta^n(z)) = \alpha$ for any $n \in \mathbb{N}$. Moreover, z always begins with 00 and it doesn't contain any 000, since it has been obtained through an iterative application of the Thue-Morse morphism. As a consequence, z starts with 001.

Proof of Proposition 3.3. Since w is a subword of τ, there exists a prefix of the Thue-Morse sequence that contains w as a subword. Then, the thesis can be obtained by showing that for any $\alpha \geq 2$ and for any finite prefix $p \in \{0,1\}^*$ of the Thue-Morse sequence there exists a word $\tilde{y}$ that starts with p, $E(\tilde{y}) = \alpha$ and $E(\delta^n(\tilde{y})) = \alpha$ for any $n \in \mathbb{N}$. Indeed, if the previous statement holds, the word y can be selected by properly applying δ to $\tilde{y}$.

If $\alpha = 2$, this is trivial by considering $\tilde{y} = \tau$. Otherwise, $\tilde{y}$ can be constructed by choosing properly β_1 in the construction given in Remark 3.4. Indeed, let us choose $r_1 = 2$, $s_1 \geq 3$ such that $2^{s_1} > \ell(p)$ and $t_1 = 5$, so that $\delta^{t_1}\mu^{s_1}(0^{r_1})$ starts with 00. Moreover, s_1 has to be chosen so that $\beta_1 = r_1 - t_1/2^{s_1} < \alpha$. Denoting by $\tau_{(s_1)}$ the prefix of τ with length 2^{s_1}, we have that p is a prefix of $\tau_{(s_1)}$. Moreover, we have that $\mu^{s_1}(0^{r_1}) = \mu^{s_1}(00) = \tau_{(s_1)}\tau_{(s_1)}$, so that $\phi_1(\epsilon) = \delta^{t_1}\mu^{s_1}(00) = \delta^5(\tau_{(s_1)})\tau_{(s_1)}$. Therefore, choosing $\bar{n} = 2^{s_1} - t_1$, we have that $\delta^{\bar{n}}\phi_1(\epsilon) = \tau_{(s_1)}$. Let z be a word obtained through the construction of Remark 3.4 with $\beta_1 = r_1 - t_1/2^{s_1}$. Then we have that $\tilde{y} = \delta^{\bar{n}}(z)$ starts with p and, being a suffix of z, we also have that $E(\tilde{y}) = \alpha$ and $E(\delta^n(\tilde{y})) = \alpha$ for any $n \in \mathbb{N}$, as required. $\square$

While the validity of Conjecture 3.2 remains uncertain, we present the following weaker result, which will prove useful in our subsequent analysis.

Proposition 3.5. *For any $w \in \{0,1\}^*$, we have $[\ell(w), +\infty] \subset E(\mathcal{P}(w))$.*

Proof. The idea is to concatenate w with a finite word 0^n, where the value of n will be fixed and determined later, and then append to it an infinite word y obtained using the construction of Remark 3.4.

More formally, we proceed as follows. If $\ell(w) \leq 2$, then the thesis holds as a direct consequence of Proposition 3.3. Therefore, we assume $\ell(w) \geq 3$ and, without loss of generality, that the last digit of w is 0. Otherwise, we work with the bitwise negation of w and then we apply a bitwise negation to the final construction once again. Hence, w has the form $\tilde{w}0^k$, where $\tilde{w}$ is either the empty word or it ends with 1. Let $n = \ell(w) - k$, and consider the word

$$w0^n = \tilde{w}0^k\,0^n = \tilde{w}\underbrace{0\ldots0}_{\ell(w)}.$$

For any $\alpha \geq \ell(w)$, consider $y \in \{0,1\}^\omega$ to be a word such that $E(y) = \alpha$, where this word is obtained through the construction presented in Remark 3.4, hence it starts with 001 and we can write

$$y = 00\tilde{y},$$

where $\tilde{y}$ begins with 1. Moreover, since we are assuming $\ell(w) \geq 3$, we notice also that $0^{\ell(w)}$ appears only once in $\tilde{w}0^{\ell(w)}\tilde{y}$. Indeed, $\tilde{w}$ is either the empty word or it ends with 1 and $\ell(\tilde{w}) < \ell(w)$.

Additionally, this implies

$$E(\tilde{w}0^{\ell(w)}) = \ell(w). \tag{3.3}$$

Since $\tilde{y}$ starts with 1 and it does not contain 000 as a subword, we have

$$E(0^{\ell(w)}\tilde{y}) = \alpha. \tag{3.4}$$

We claim

$$E(w0^n\tilde{y}) = E(\tilde{w}0^{\ell(w)}\tilde{y}) = \alpha,$$

thus obtaining the thesis due to the arbitrariness of α and the fact that n is independent of α. First, let us notice that if $\tilde{w} = \epsilon$, then the claim directly follows from (3.4). Hence, from now on let us assume $\tilde{w} \neq \epsilon$.

Since $E(00\tilde{y}) = \alpha$, we have that $E(\tilde{w}0^{\ell(w)}\tilde{y}) \geq \alpha$; therefore, looking for a contradiction, let us assume that $E(\tilde{w}0^{\ell(w)}\tilde{y}) > \alpha$. This means that there exists a subword p of $\tilde{w}0^{\ell(w)}\tilde{y}$ such that $p = v^t v'$, where v' is a prefix of v and $t + \frac{\ell(v')}{\ell(v)} > \alpha \geq \ell(w)$. Notice that in this case, since $\ell(w) \geq 3$, also t is greater than or equal to 3, so vvv is a subword of p.

By (3.3) and (3.4), we have that the first digit of p is in $\tilde{w}$ and the last is in $\tilde{y}$. The latter implies that v should contain the digit 1. Therefore, v cannot be a subword of $0^{\ell(w)}$. Moreover, since $t \geq 3$ and $0^{\ell(w)}$ appears only once in $\tilde{w}0^{\ell(w)}\tilde{y}$, we have that $0^{\ell(w)}$ cannot be a subword of v.

Since $0^{\ell(w)}$ is a subword of $p = v^t v'$ and v contains the digit 1, we deduce that $0^{\ell(w)}$ is a subword of vv. Using again the fact that v contains the digit 1, we obtain that vvv, which is a subword of p, contains at least twice $0^{\ell(w)}$, which is impossible. $\qquad\square$

4 The critical exponent functions and their properties

This section reviews the generic critical exponent functions, denoted by κ_n, and their analytical and dynamical properties with respect to the error of [1, Theorem 7]. In the following, we present the definition of the 2-critical exponential function, denoted by κ_2, which we consider as model case.

Definition 4.1. The critical exponent function $\kappa_2 \colon [0, 1] \to [0, 1]$ is defined as follows:

$$\kappa_2(x) \coloneqq \inf_{u \in \mathcal{L}(w_x)} \frac{1}{E(u)}.$$

Remark 4.2. Using the convention $1/\infty = 0$, we can write $\kappa_2(x) = 1/E(w_x)$. In other words, for $x \in [0, 1]$, $\kappa_2(x)$ is the inverse of the critical exponent of the binary expansion of x.

More generally, for each $x \in [0, 1]$, we can define the critical exponent function $\kappa_n \colon [0, 1] \to [0, 1]$ at x as the inverse of the critical exponent of the expansion of x in base n. For any $n \in \mathbb{N}$, $n \geq 2$, and $x \in [0, 1]$, we denote by $w_{x,n}$ the expansion of x in base n, namely $x = (0.w_{x,n})_n$. Hence, for every integer $n \geq 2$, the function $\kappa_n \colon [0, 1] \to [0, 1]$ is defined by:

$$\kappa_n(x) \coloneqq \inf_{u \in \mathcal{L}(w_{x,n})} \frac{1}{E(u)}.$$

4.1 The model case κ_2

Let us consider the properties of the function κ_2 as a model case for all the others functions κ_n. More precisely, we are going to study the results of [1] by showing how the properties of this function vary with respect to the observations made on [1, Theorem 7].

Let us denote the set of the zeros of κ_2 by C_2, and its complementary set by D_2, i.e.,

$$C_2 := \{x \in [0, 1] : \kappa_2 = 0\}, \qquad D_2 \coloneqq [0, 1] \setminus C_2.$$

Based on the presentation in [1], we have the following results, which do not rely on [1, Theorem 7]:

- the function κ_2 is upper semi-continuous, and therefore it is of Baire class 1 (cf. [1, Proposition 11]), i.e., it can be obtained as the pointwise limit of a sequence of continuous functions;

- the set of points of continuity of the function κ_2 coincides with the set C_2. (cf. [1, Proposition 12]);

- the set C_2 is co-meagre and it has full Lebesgue measure (cf. [1, Proposition 14]);

- both C_2 and D_2 are dense in $[0, 1]$, and they are neither closed nor open (cf. [1, Proposition 15 and Proposition 16]);

A result heavily dependent on [1, Theorem 7] is [1, Proposition 19], which states that if $\kappa_2(x) \neq 0$, then $[0, \kappa_2(x)[$ is a subset of the left or right range of κ_2 at x, which are the sets of values the function κ_2 assume on every open left or right neighbourhood of x, respectively. More formally, if we denote them by $R^-(\kappa_2, x)$ and $R^+(\kappa_2, x)$, they are defined as follows:

$$R^-(\kappa_2, x) := \left\{ \alpha \in \mathbb{R} : \kappa_2^{-1}(\alpha) \cap (x - \delta, x) \neq \emptyset, \forall \delta > 0 \right\},$$

$$R^+(\kappa_2, x) := \left\{ \alpha \in \mathbb{R} : \kappa_2^{-1}(\alpha) \cap (x, x + \delta) \neq \emptyset, \forall \delta > 0 \right\}.$$

With this notation, [1, Proposition 19] ensured that if $\kappa_2(x) = 0$, then both $R^-(\kappa_2, x)$ and $R^+(\kappa_2, x)$ contain only 0. On the other hand, if $\kappa_2(x) > 0$ then at least one of the following holds:

$$[0, \kappa_2(x)[\subset R^-(\kappa_2, x) \quad \text{or} \quad [0, \kappa_2(x)[\subset R^+(\kappa_2, x).$$

It remains uncertain if [1, Proposition 19] holds true or not, as Conjecture 3.2 could be sufficient to prove it. We remark that the result does not follow directly from the conjecture, since one has to ensure that for any (finite) prefix w of w_x and for any $\alpha \geq \mathcal{E}_w$ the infinite word y_α such that $E(wy_\alpha) = \alpha$ can be chosen so that $(0.wy_\alpha)_2 < x$ (or $(0.wy_\alpha)_2 > x$) for every α. Otherwise, in principle, some strange phenomena could occur, such as, for instance, that $R^-(\kappa_2, x) = [0, \kappa_2(x)] \cap \mathbb{Q}$ while $R^+(\kappa_2, x) = [0, \kappa_2(x)] \setminus \mathbb{Q}$.

While the full validity of [1, Proposition 19] remains uncertain, the result holds for the special case of x_τ, which is the number whose binary expansion is the Thue-Morse sequence, namely $x_\tau = (0.\tau)_2$. Indeed, we have the following result.

Lemma 4.3. *The interval $[0, 1/2)$ is a subset of $R^-(\kappa_2, x_\tau)$.*

Proof. Recalling that for any $n \in \mathbb{N}$ the $(2^n + 1)$-th digit of τ is 1 (see Remark 2.4), for any $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that $I_{\tau_{(m)}0} \subset (x_\tau - \epsilon, x_\tau)$, where $\tau_{(m)}$ is the prefix of length 2^m of the Thue-Morse sequence and I_w is the open interval characterized by w. Indeed, for any $\epsilon > 0$ there exists a positive integer m such that $I_{\tau_{(m)}} \subset (x_\tau - \epsilon, x_\tau + \epsilon)$; moreover if $y \in I_{\tau_{(m)}0} \subset I_{\tau_{(m)}}$, then

$$y = (0.\tau_{(m)}0\ldots)_2 < (0.\tau_{(m)}1\,\tau_{2^m+2}\tau_{2^m+3}\ldots)_2 = (0.\tau)_2 = x_\tau,$$

hence $I_{\tau_{(m)}0} \subset (x_\tau - \epsilon, x_\tau)$.

By Proposition 3.3, it suffices to show that $\tau_{(m)}0$ is a subword of τ. Using again the construction of τ given in Remark 2.4, and denoting by $\overline{\tau_{(m)}}$ the bitwise negation of $\tau_{(m)}$, we have that τ starts as follows:

$$\tau = \tau_{(m)} \overline{\tau_{(m)}} \overline{\tau_{(m)}} \tau_{(m)} \overline{\tau_{(m)}} \tau_{(m)} \tau_{(m)} \overline{\tau_{(m)}} \cdots .$$

As a consequence, $\tau_{(m)}\tau_{(m)}$ is a subword of τ, ans since $\tau_{(m)}$ starts with 0, also $\tau_{(m)}0$ is a subword of τ, and this ends the proof. □

Lemma 4.3 is sufficient to restore some properties that in [1] were proved using [1, Proposition 19], such as the existence of almost fixed points and fixed points of κ_2. We recall that an almost fixed point of a function is a point that belongs to the interior of the left or right range of the function at that point. More formally, we have the following definition.

Definition 4.4. A point $x \in [0, 1]$ is an *almost fixed point* of κ_2 if and only if

$$x \in \text{Int}(R^-(\kappa_2, x)) \cup \text{Int}(R^+(\kappa_2, x)).$$

We denote the set of almost fixed points of κ_2 as $\text{aFix}(\kappa_2)$.

In [1], a characterization of fixed points was given, namely $x \in \mathrm{aFix}(\kappa_2)$ if and only if $\kappa_2(x) > x$ (cf. [1, Corollary 24]). However, this result relies on [1, Proposition 19], hence its validity is no longer ensured. As a consequence, the uncountability of $\mathrm{aFix}(\kappa_2)$ can't be ensured (cf. [1, Corollary 25]). The only thing we can conclude, by employing the upper semi-continuity of κ_2, is that if $x \in [0,1]$ is an almost fixed point of κ_2, then $\kappa_2(x) > x$. However, based on Lemma 4.3, we can establish that $x_\tau \in \mathrm{aFix}(\kappa_2)$. Indeed, since $x_\tau < \kappa_2(x_\tau) = 1/2$, we have that $x_\tau \in (0, 1/2) \subset \mathrm{Int}(R^-(\kappa_2, x))$. This is also sufficient to show the existence of a fixed point of κ_2. Indeed, [1, Corollary 26] stated that, for any $x_0 \in \mathrm{aFix}(\kappa_2)$ and $\epsilon > 0$, there exists a fixed point of κ_2 in $(x_0 - \epsilon, x_0 + \epsilon)$. This is not correct, but the same proof of [1, Corollary 26] shows the following result.

Corollary 4.5. *For any $\epsilon > 0$, there exists $z \in (x_\tau - \epsilon, x_\tau)$ such that $\kappa_2(z) = z$.*

This result, which may appear a bit surprising, ensures the existence of points that coincide with the inverse of the critical exponent of their binary expansion, and they are (at least) countably many, allowing [1, Corollary 27] to still hold true.

The analysis in [1] continued by considering the topological entropy of κ_2. We recall that the topological entropy h of an interval map f can be defined as follows (see, e.g., [12, 13]):

$$h(f) = \lim_{\varepsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log |(n, \varepsilon)| \right), \tag{4.1}$$

where $|(n, \varepsilon)|$ is the maximum cardinality of an ε-separated set in the following metric d_n:

$$d_n(x_1, x_2) := \max \left\{ \left| f^i(x_1) - f^i(x_2) \right| : 0 \le i \le n \right\}. \tag{4.2}$$

In other words, the topological entropy can be interpreted as an estimate of the exponential increase rate of the number of topologically distinguishable orbits when the resolution power diverges.

Specifically, in [1, Proposition 31] it was shown that κ_2 has infinite topological entropy (cf. [1, Proposition 31]). This is a direct consequence of the existence of horseshoes of every order (cf. [1, Proposition 30]). The latter result that can be recovered by using Proposition 3.3, instead of the (now uncertain) [1, Proposition 19]. For a clearer understanding, we report here the revised proof.

Proposition 4.6 (Proposition 30 of [1]). *The function κ_2 has an m–horseshoe for any $m \ge 2$, i.e., there exists an interval $J \subset [0,1]$ and a family of pairwise closed disjoint intervals $I_1, \ldots, I_m \subset J$ such that $J \subset \kappa_2(I_k)$, for all $k = 1, \ldots, m$.*

Proof. For any $k = 1, \ldots, m$, let $\tau_{(k)}$ be the prefix of length 2^k of the Thue-Morse sequence, and let I_k be the closed cylinder set characterized by $\tau_{(k+1)}0$, i.e.,

$$I_k = [(0.\tau_{(k+1)}0)_2, (0.\tau_{(k+1)}1)_2].$$

Therefore, we have

$$\begin{aligned}
&\text{if } x \in I_1, \text{then } x = (0.01100\ldots)_2, \\
&\text{if } x \in I_2, \text{then } x = (0.011010010\ldots)_2, \\
&\text{if } x \in I_3, \text{then } x = (0.01101001100101100\ldots)_2,
\end{aligned}$$

and so forth, recalling that, if x is a dyadic rational, we consider its binary expansion whose digits are ultimately 1. Since the $(2^k + 1)$-th digit of the Thue-Morse sequence is 1 for any $k \ge 1$, the intervals I_k are pairwise disjoint and if $x \in I_k$ then $x < x_\tau$. For any $k \ge 1$, the word $\tau_{(k+1)}0$ is a subword of $\tau_{(k+1)}\tau_{(k+1)}$, which is a subword of the Thue-Morse sequence, as one can infer from the bitwise negation construction of τ given in Remark 2.4. According to Proposition 3.3, $\kappa_2(I_k) = [0, 1/2)$ for any $k = 1, \ldots, m$. Therefore, by setting $J = [0, x_\tau]$, we obtain the thesis, given the generality of m. $\qquad \square$

We end the exploration of κ_2 by recovering some results on its dynamical properties. First, we can ensure the following result (cf. [1, Lemma 32]), whose original proof relied on [1, Theorem 7], but a minor revision of it is sufficient to demonstrate the claim.

Lemma 4.7. *For any $n \geq 1$ and $y \in [0, 1/2^n]$, there exists $x \in [0, 1/2^{2^n}]$ such that $\kappa_2(x) = y$.*

Proof. If $y = 0$, then $x = 0$. If $y \in (0, 1/2^n]$, then $1/y \geq 2^n$. By Proposition 3.5, we have that $[2^n, +\infty] \subset E(\mathcal{P}(0^{2^n}))$. Then, there exists $\alpha_y \in \{0, 1\}^\omega$ such that $E(0^{2^n}\alpha_y) = 1/y$. As a consequence, the number $x = (0.0^{2^n}\alpha_y)_2$ is such that $\kappa_2(x) = y$ and $x \in [0, 1/2^{2^n}]$. $\square$

Lemma 4.7 and Proposition 3.5 lead to the topological mixing property of $\bar{\kappa}_2 := \kappa_2|_{[0,1/2]}$. Indeed, they are sufficient to prove that for any non-empty open interval $I \subset [0, 1]$ there exists $n \in \mathbb{N}$ such that $\kappa_2^n(I) = [0, 1/2]$ (cf. [1, Proposition 33]), from which the topological mixing property directly follows.

The last result about the properties of κ_2 was [1, Lemma 36], which is about a kind of "finite-type transitivity" and "finite-type Li–Yorke pairs". We recall that a point is called *transitive* if its orbit is dense, and that a *Li-Yorke pair* is a pair of points (x, y) such that

$$\limsup_{n \to \infty} |\kappa_2^n(x) - \kappa_2^n(y)| > 0 \quad \text{and} \quad \liminf_{n \to \infty} |\kappa_2^n(x) - \kappa_2^n(y)| = 0.$$

In [1], different steps of the proof of [1, Lemma 36] are based on [1, Theorem 7], but we are going to show that they can be recovered by using Proposition 3.3 and Proposition 3.5.

Lemma 4.8 (Lemma 36 of [1]). *For any $w_{(1)}, w_{(2)} \in \{0, 1\}^+$ such that both $(0.w_{(1)})_2$ and $(0.w_{(2)})_2$ are less than $1/2$, there exist $x_1, x_2 \in I_{w_{(1)}}$ and a number $n \in \mathbb{N}$ such that:*

(i) $\kappa_2^{n+2}(x_1), \kappa_2^{n+2}(x_2) \in I_{w_{(2)}}$;

(ii) $\kappa_2^n(x_2) - \kappa_2^n(x_1) > 1/8$.

Proof. Since $(0.w_{(2)})_2 < 1/2$, and both 00 and 01 are subwords of the Thue-Morse sequence, by Proposition 3.3 there exist $\alpha_1, \alpha_2 \in \{0, 1\}^\omega$ such that, setting $z_1 = (0.00\alpha_1)_2$ and $z_2 = (0.01\alpha_2)_2$, we have $\kappa_2(z_1), \kappa_2(z_2) \in I_{w_{(2)}}$. Since $z_1 < 1/4$, by Lemma 4.7 there exists $y_1 < 1/16$ such that $\kappa_2(y_1) = z_1$. By using again Proposition 3.3, there exists $y_2 = (0.010\ldots)_2$ such that $\kappa_2(y_2) = z_2$. Therefore, we have $\kappa_2^2(y_1), \kappa_2^2(y_2) \in I_{w_{(2)}}$ and $y_2 - y_1 > 1/8$, and the proof is complete by showing the existence of a positive integer n such that $\kappa_2^n(I_{w_{(1)}}) = [0, 1/2]$. By applying Proposition 3.5 to $w = w_{(1)}$, we have $\kappa_2(I_{w_{(1)}}) = [0, 1/\ell(w_{(1)})]$. Then, through an iterative procedure involving Lemma 4.7, we can identify an $n \in \mathbb{N}$ such that $\kappa_2^n(I_{w_{(1)}}) = [0, 1/2]$, thus concluding our argument. $\square$

Summarizing, after reviewing all the properties of κ_2 studied in [1], we find that all of them hold true except for the following ones:

- The property of being left- or right-Darboux at every point, as stated in [1, Proposition 19], remains unclear if this property holds or not;

- The characterization of the almost fixed points of κ_2, specifically in [1, Corollary 24], and the subsequent uncountability of them (cf. [1, Corollary 25]);

- The existence of the fixed point can be granted only on any left-neighbourhood of x_τ, not in a neighbourhood of any almost fixed point (cf. [1, Corollary 26] and 4.5).

4.2 On the functions κ_n

In [1], several properties of the general function κ_n, with $n \geq 3$, were explored by examining its restriction to the set

$$A_2^n := \{x \in [0, 1] : w_{x,n} \in \{0, 1\}^\omega\},$$

where $w_{x,n}$ is the n–base expansion of x, and by employing [1, Theorem 7]. In particular, [1, Corollary 42], which plays an analogous role of [1, Proposition 19], is no longer granted. However, as for the case of κ_2, most of the properties can be regained by using Proposition 3.3 and

Proposition 3.5. In particular, we have that $x_{\tau,n} = (0.01101001\ldots)_n$ is an almost fixed point for κ_n and for any $\epsilon > 0$ there exists a fixed point in the interval $(x_{\tau,n} - \epsilon, x_{\tau,n})$. Thus, there are (at least) countably many fixed points for κ_n. Moreover, with some minor adjustments on the proof, it can be showed that κ_n admits horseshoes of every order (cf. Proposition 4.6), so it has infinite topological entropy,

4.3   On the function κ

Finally, in [1] the properties of the function $\kappa\colon [0,1] \to [0,1]$, which was defined by considering the supremum of κ_n among all $n \geq 2$, were studied. More precisely,

Definition 4.9. We indicate by $\kappa\colon [0,1] \to [0,1]$ the function given by

$$\kappa(x) := \sup\left\{\kappa_n(x) : n \in \mathbb{N}, n \geq 2\right\}.$$

Since [1, Theorem 7] and strictly related results were never employed in the analysis of κ, all the properties presented in [1] for κ hold true. The most important ones are:

- κ is 0 a.e., hence Lebesgue-measurable;

- the points where κ is different from 0 are uncountably many;

- κ is discontinuous at any point $x \in [0,1]$;

- the function κ belongs to the Baire class 2 (but not to the Baire class 1), i.e., it can be obtained as the pointwise limit of a sequence of functions of Baire class 1.

Additional dynamical properties of the map κ seem an interesting subject for further studies. For instance, the structure of the ω-limit sets determined by κ is probably far from trivial. Moreover, the related concept of statistical ω-limit and statistical ω-cluster ([14, 15]), which seem well-suited for irregular maps, can be investigated.

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