Palestine Journal of Mathematics

Vol 14(3)(2025) , 625–636

Spectra of T-vertex join and T-edge join of two graphs

Shreekant Patil*, Sourav Mondal, Mallikarjun Mathapati, Ashok Rathod and Veena R. Desai

Communicated by Ambat Vijayakumar

MSC 2020 Classifications: 05C76, 05C50.

Keywords and phrases: Adjacency spectrum, Laplacian spectrum, Signless Laplacian spectrum, T-vertex join, T-edge join, Spanning tree, Krichhoff index.

Abstract Let G_1 and G_2 be two graphs, $T(G)$ denotes the total graph of G [2]. The T-vertex join of G_1 with G_2, denoted by $G_1 \odot G_2$, is the graph obtained from $T(G_1)$ and G_2 by joining every vertex of $V(G_1) \cap V(T(G_1))$ with every vertex of $V(G_2)$. The T-edge join of G_1 with G_2, denoted by $G_1 \ominus G_2$, is the graph obtained from $T(G_1)$ and G_2 by joining every vertex of $E(G_1) \cap V(T(G_1))$ with every vertex of $V(G_2)$. In this paper, we determine the adjacency spectra, the Laplacian spectra and the signless Laplacian spectra of $G_1 \odot G_2$ (respectively, $G_1 \ominus G_2$) for a regular graph G_1 and an arbitrary graph G_2, in terms of the corresponding spectra of G_1 and G_2. Further, we obtain the number of spanning trees and the Kirchhoff index of $G_1 \odot G_2$ and $G_1 \ominus G_2$ for regular graphs G_1 and G_2 as applications of Laplacian spectra of $G_1 \odot G_2$ and $G_1 \ominus G_2$.

1 Introduction

We consider simple graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The degree of vertex v_i in G is denoted by $d_G(v_i)$. The incidence matrix $R(G)$ of G is the $(V(G) \times E(G))$-matrix $\{q_{ij}\}$, where $q_{ij} = 1$ if vertex v_i is incident to edge e_j and $q_{ij} = 0$, otherwise. Let $A(G)$ be the $(V(G) \times V(G))$- matrix $\{a_{ij}\}$, where $a_{ij} = 1$ if $(v_i, v_j) \in E(G)$ and $a_{ij} = 0$, otherwise. Let $D(G)$ be the $(V(G) \times V(G))$-matrix $\{d_G(v_{ij})\}$, where $d_G(v_{ii}) = d_G(v_i)$ and $d_G(v_{ij}) = 0$ for $i \neq j$. The matrices $A(G)$, $D(G)$, $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the adjacency matrix, the degree matrix, the Laplacian matrix, and the signless Laplacian matrix of G, respectively. The characteristic polynomial A of G is defined as $f_G(A; x) = det(xI_n - A)$, where I_n is the identity matrix of order n. The roots of $f_G(A; x)$ are called the eigenvalues of G. It is denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and are called A - spectrum of G. The eigenvalues of $L(G)$ and $Q(G)$ are denoted by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ and $\nu_1(G) \leq \nu_2(G) \leq \cdots \leq \nu_n(G)$. They are called the L - spectrum and Q - spectrum of G. The total graph $T(G)$ or T-graph of a graph G [2] is the graph whose vertex set is $V(G) \cup E(G)$, such that vertices are adjacent if and only if they are either adjacent or incident in G.

Until now, many derived graph related operations such as subdivision-vertex join, subdivision-edge join, R-vertex join, R-edge join, the subdivision-vertex neighbourhood corona, the subdivision-edge neighbourhood corona, R-vertex corona and R-edge corona have been introduced, and their spectra are computed in [5, 9, 10, 12, 13, 16]. Motivated by these works, we define two new graph operations based on total graphs as follows.

Definition 1.1. The T-vertex join of two graphs G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained from $T(G_1)$ and G_2 by joining every vertex of $V(G_1) \cap V(T(G_1))$ with every vertex of $V(G_2)$.

Definition 1.2. The T-edge join of two graphs G_1 and G_2, denoted by $G_1 \ominus G_2$, is the graph obtained from $T(G_1)$ and G_2 by joining every vertex of $E(G_1) \cap V(T(G_1))$ with every vertex of $V(G_2)$.

Note that if G_i has n_i vertices and m_i edges for $i = 1,2$, then $G_1 \odot G_2$ and $G_1 \ominus G_2$ have $n_1 + n_2 + m_1$ vertices.

The M-coronal $\Gamma_M(x)$ of an $n \times n$ matrix M is defined [3, 14] to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is, $\Gamma_M(x) = 1_n^T(xI_n - M)^{-1}1_n$, where 1_n denotes the column vector of size n with all entries equal to one. It is known [3, Proposition 2] that if M is an $n \times n$ matrix with each row sum equal to a constant t, then

$$\Gamma_M(x) = \frac{n}{x-t}. \tag{1.1}$$

In this paper, we determine the A-spectra, the L-spectra and the Q-spectra of $G_1 \odot G_2$ (respectively, $G_1 \ominus G_2$) for a regular graph G_1 and an arbitrary graph G_2, in terms of the corresponding spectra of G_1 and G_2. Further, we obtain the number of spanning trees and the Kirchhoff index of $G_1 \odot G_2$ and $G_1 \ominus G_2$ for regular graphs G_1 and G_2 as applications of L-spectra of $G_1 \odot G_2$ and $G_1 \ominus G_2$. We refer to [8] for unexplained terminology and notation.

Firstly, we introduce two lemmas that need to be used in the proof process.

Lemma 1.3. *[4] Let G be an r-regular graph with an adjacency matrix A and an incidence matrix R. Let $Ł(G)$ be its line graph. Then $RR^T = A + rI$ and $R^TR = A(Ł(G)) + 2I$.*

Lemma 1.4. *[19] Let M_1, M_2, M_3 and M_4 be matrices with M_4 invertible. Then*

$$det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = det(M_4)det(M_1 - M_2M_4^{-1}M_3),$$

where $M_1 - M_2M_4^{-1}M_3$ is called the Schur complement of M_4.

2 A-spectra of T-vertex join and T-edge join

Theorem 2.1. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then*
$f_{A(G_1\odot G_2)}(x) = f_{A(G_2)}(x)\cdot(x+2)^{m_1-n_1}\cdot\Big(x^2-\big(n_1\Gamma_{A(G_2)}(x)+3r_1-2\big)x-2\big(n_1(1-r_1)\Gamma_{A(G_2)}(x)+2r_1-r_1^2\big)\Big)\cdot\prod_{i=2}^{n_1}\Big(x^2-\big(2\lambda_i(G_1)+r_1-2\big)x-2\lambda_i(G_1)-\big(\lambda_i(G_1)+r_1\big)\big(1-\lambda_i(G_1)\big)\Big).$

Proof. Let R be the incidence matrix of G_1. Then, with a proper labeling of the vertices, the adjacency matrix of $G_1 \odot G_2$ can be written as

$$A(G_1 \odot G_2) = \begin{pmatrix} A(G_1) & R & J_{n_1\times n_2} \\ R^T & R^TR - 2I_{m_1} & O_{m_1\times n_2} \\ J_{n_2\times n_1} & O_{n_2\times m_1} & A(G_2) \end{pmatrix}.$$

Then the adjacency characteristic polynomial of $G_1 \odot G_2$ is given by

$$\begin{aligned} f_{G_1\odot G_2}(x) &= det\begin{pmatrix} xI_{n_1} - A(G_1) & -R & -J_{n_1\times n_2} \\ -R^T & (x+2)I_{m_1} - R^TR & O_{m_1\times n_2} \\ -J_{n_2\times n_1} & O_{n_2\times m_1} & xI_{n_2} - A(G_2) \end{pmatrix} \\ &= det(xI_{n_2} - A(G_2)) \cdot det(S) \\ &= f_{A(G_2)}(x) \cdot det(S), \end{aligned}$$

where

$$\begin{aligned} S &= \begin{pmatrix} xI_{n_1} - A(G_1) & -R \\ -R^T & (x+2)I_{m_1} - R^TR \end{pmatrix} \\ &\quad - \begin{pmatrix} -J_{n_1\times n_2} \\ O_{m_1\times n_2} \end{pmatrix}(xI_{n_2} - A(G_2))^{-1}\begin{pmatrix} -J_{n_2\times n_1} & O_{n_2\times m_1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)J_{n_1\times n_1} & -R \\ -R^T & (x+2)I_{m_1} - R^TR \end{pmatrix} \end{aligned}$$

is the Schur complement of $xI_{n_2} - A(G_2)$ obtained by Lemma 1.4. Add the first row left $-R^T$ of the block matrix S to the second row.

$$S = \begin{pmatrix} xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)J_{n_1\times n_1} & -R \\ -R^T((x+1)I_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)J_{n_1\times n_1}) & (x+2)I_{m_1} \end{pmatrix}.$$

Now,

$$\begin{aligned} det(S) &= det\big((x+2)I_{m_1}\big)\cdot det\Big(xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)J_{n_1\times n_1} \\ &\quad -R((x+2)I_{m_1})^{-1}R^T((x+1)I_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)J_{n_1\times n_1})\Big) \\ &= (x+2)^{m_1-n_1}det(B), \end{aligned}$$

where $B = (x+2)(xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)J_{n_1\times n_1}) - \big(A(G_1) + r_1 I_{n_1}\big)\big((x+1)I_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)J_{n_1\times n_1}\big)$. Here matrix B is a polynomial with two variables $A(G_1)$, $J_{n_1\times n_1}$ and real coefficients. Then $B(A(G_1), J_{n_1\times n_1})$ has the eigenvalues $\sigma_1 = B(r_1, n_1)$ and $\sigma_i = B(\lambda_i(G_1), 0)$ for $i = 2, 3, \ldots, n_1$. Hence,

$$\begin{aligned} \sigma_1 &= (x+2)(x - r_1 - n_1\Gamma_{A(G_2)}(x)) - (r_1 + r_1)(x + 1 - r_1 - n_1\Gamma_{A(G_2)}(x)) \\ &= x^2 - (n_1\Gamma_{A(G_2)}(x) + 3r_1 - 2)x - 2(n_1(1-r_1)\Gamma_{A(G_2)}(x) + 2r_1 - r_1^2) \ \text{ and} \\ \sigma_i &= (x+2)(x - \lambda_i(G_1)) - (\lambda_i(G_1) + r_1)(x + 1 - \lambda_i(G_1)) \\ &= x^2 - (2\lambda_i(G_1) + r_1 - 2)x - 2\lambda_i(G_1) - (\lambda_i(G_1) + r_1)(1 - \lambda_i(G_1)) \ \text{ for } \ i = 2, 3, \ldots n_1. \end{aligned}$$

Since $det(S) = (x+2)^{m_1-n_1}\prod_{i=1}^{n_1}\sigma_i$, The required result follows from $f_{A(G_1\odot G_2)}(x) = f_{A(G_2)}(x)\cdot det(S)$. □

Corollary 2.2. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices. Then the A-spectrum of $G_1 \odot G_2$ consists of:*

(a) -2, *repeated* $m_1 - n_1$ *times;*

(b) $\lambda_i(G_2)$ *for* $i = 2, 3, \ldots, n_2$*;*

(c) $\frac{(2\lambda_i(G_1)+r_1-2)\pm\sqrt{(2\lambda_i(G_1)+r_1-2)^2+4(2\lambda_i(G_1)+(\lambda_i(G_1)+r_1)(1-\lambda_i(G_1)))}}{2}$, *for* $i = 2, 3, \ldots, n_1$*;*

(d) *three roots of the equation* $x^3 - \big(r_2 + 3r_1 - 2\big)x^2 - \big(n_1n_2 - r_2(3r_1-2) + 2(2r_1 - r_1^2)\big)x + 2(r_2(2r_1 - r_1^2) - n_1n_2(1-r_1)) = 0.$

Proof. By Theorem 2.1, we obtain (a) readily. Since G_2 be an r_2-regular graph on n_2 vertices, by Eq. 1.1, we have $\Gamma_{A(G_2)}(x) = \frac{n_2}{x-r_2}$. The pole of $\Gamma_{A(G_2)}(x)$ is $x = r_2 = \lambda_1(G_2)$. Thus, by Theorem 2.1, (b) and (c) follows. The remaining 3 eigenvalues of $G_1 \odot G_2$ are obtained by solving

$$x^2 - \Big(n_1\frac{n_2}{x-r_2} + 3r_1 - 2\Big)x - 2\Big(n_1(1-r_1)\frac{n_2}{x-r_2} + 2r_1 - r_1^2\Big) = 0,$$

this yields the eigenvalues in (d). □

By substituting the well-known result [14] that $A(K_{p,q})$-coronal of complete bipartite with $p, q \geq 1$ vertices in the two parts of its bipartition is $\Gamma_{A(K_{p,q})}(x) = \frac{(p+q)x+2pq}{x^2-pq}$ in Theorem 2.1, we have the following corollary. The computations are routine as in the above Corollary 2.2, and hence we omit the proof.

Corollary 2.3. *Let G be an r-regular graph on n vertices and m edges. Then the A-spectrum of $G \odot K_{p,q}$ consists of:*

(a) 0, *repeated* $p + q - 2$ *times;*

(b) -2, *repeated* $m - n$ *times;*

(c) $\frac{(2\lambda_i(G)+r-2)\pm\sqrt{(2\lambda_i(G)+r-2)^2+4(2\lambda_i(G)+(\lambda_i(G)+r)(1-\lambda_i(G)))}}{2}$, *for* $i = 2, 3, \ldots, n$*;*

(d) *four roots of the equation* $x^4 + (3r-2)x^3 - \big(pq + n(p+q) + 2(2r-r^2)\big)x^2 - \big(2npq - pq(3r-2) + 2n(1-r)(p+q)\big)x + 2pq\big((2r-r^2) - 2n(1-r)\big) = 0$.

Theorem 2.4. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then*
$f_{A(G_1 \ominus G_2)}(x) = f_{A(G_2)}(x) \cdot (x+2)^{m_1-n_1} \cdot \Big(x^2 + \big(2 - 3r_1 + \frac{n_1 r_1}{2}\Gamma_{A(G_2)}(x)\big)x - (2r_1(2-r_1) - \frac{n_1 r_1^2}{2}\Gamma_{A(G_2)}(x))\Big) \cdot \prod_{i=2}^{n_1} \Big(x^2 - (2\lambda_i(G_1) + r_1 - 2)x - 2\lambda_i(G_1) - (\lambda_i(G_1) + r_1)(1 - \lambda_i(G_1))\Big)$.

Proof. Let R be the incidence matrix of G_1. Then the adjacency matrix of $G_1 \ominus G_2$ can be written as

$$A(G_1 \ominus G_2) = \begin{pmatrix} A(G_1) & R & O_{n_1 \times n_2} \\ R^T & R^T R - 2I_{m_1} & J_{m_1 \times n_2} \\ O_{n_2 \times n_1} & J_{n_2 \times m_1} & A(G_2) \end{pmatrix}.$$

Then the adjacency characteristic polynomial of $G_1 \ominus G_2$ is given by

$$\begin{aligned} f_{G_1 \ominus G_2}(x) &= det \begin{pmatrix} xI_{n_1} - A(G_1) & -R & O_{n_1 \times n_2} \\ -R^T & (x+2)I_{m_1} - R^T R & -J_{m_1 \times n_2} \\ O_{n_2 \times n_1} & -J_{n_2 \times m_1} & xI_{n_2} - A(G_2) \end{pmatrix} \\ &= det(xI_{n_2} - A(G_2)) \cdot det(S) \\ &= f_{A(G_2)}(x) \cdot det(S), \end{aligned}$$

where

$$S = \begin{pmatrix} xI_{n_1} - A(G_1) & -R \\ -R^T & (x+2)I_{m_1} - R^T R - \Gamma_{A(G_2)}(x) J_{m_1 \times m_1} \end{pmatrix}$$

is the Schur complement of $xI_{n_2} - A(G_2)$ obtained by Lemma 1.4. Add the first row left $-R^T$ of the block matrix S to the second row. Since G_1 is the regular graph, then $J_{m_1 \times n_1} R = 2J_{m_1 \times m_1}$. Therefore,

$$S = \begin{pmatrix} xI_{n_1} - A(G_1) & -R \\ -R^T\big((x+1)I_{n_1} - A(G_1)\big) & (x+2)I_{m_1} - \frac{1}{2}\Gamma_{A(G_2)}(x) J_{m_1 \times n_1} R \end{pmatrix}.$$

Now, multiply the first row of this block matrix $-\frac{1}{2}\Gamma_{A(G_2)}(x) J_{m_1 n_1}$ and add to the second row.

$$S = \begin{pmatrix} xI_{n_1} - A(G_1) & -R \\ -R^T\big((x+1)I_{n_1} - A(G_1)\big) - \frac{1}{2}\Gamma_{A(G_2)}(x) J_{m_1 n_1}(xI_{n_1} - A(G_1)) & (x+2)I_{m_1} \end{pmatrix}.$$

Since G_1 is the r_1-regular graph, then $RJ_{m_1 \times n_1} = r_1 J_{n_1 \times n_1}$. By Lemma 1.4 and 1.3, we have

$$det(S) = (x+2)^{m_1-n_1} \cdot det(B),$$

where $B = (x+2)(xI_{n_1} - A(G_1)) - (A(G_1) + r_1 I_{n_1})((x+1)I_{n_1} - A(G_1)) + \frac{1}{2} r_1 \Gamma_{A(G_2)}(x) J_{n_1 \times n_1}(xI_{n_1} - A(G_1))$. Similar argument in the proof of Theorem 2.1. Therefore the eigenvalues of B are

$$\begin{aligned} \sigma_1 &= (x+2)(x-r_1) - (r_1+r_1)((x+1)-r_1) + \frac{n_1 r_1}{2}\Gamma_{A(G_2)}(x)(x-r_1) \\ &= x^2 + \big(2 - 3r_1 + \frac{n_1 r_1}{2}\Gamma_{A(G_2)}(x)\big)x - \big(2r_1(2-r_1) + \frac{n_1 r_1^2}{2}\Gamma_{A(G_2)}(x)\big) \text{ and} \\ \sigma_i &= (x+2)(x - \lambda_i(G_1)) - (\lambda_i(G_1) + r_1)(x + 1 - \lambda_i(G_1)) \\ &= x^2 - \big(2\lambda_i(G_1) + r_1 - 2\big)x - 2\lambda_i(G_1) - (\lambda_i(G_1) + r_1)(1 - \lambda_i(G_1)) \text{ for } i = 2, 3, \dots n_1. \end{aligned}$$

Since $det(S) = (x+2)^{m_1-n_1} \cdot \prod_{i=1}^{n_1} \sigma_i$, The required result follows from $f_{G_1 \ominus G_2}(x) = f_{A(G_2)}(x) \cdot det(S)$. □

Similar to corollaries 2.2 and 2.3, Theorem 2.4 implies the following results.

Corollary 2.5. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices. Then the A-spectrum of $G_1 \ominus G_2$ consists of:*

(a) -2, *repeated $m_1 - n_1$ times;*

(b) $\lambda_i(G_2)$ *for $i = 2, 3, \ldots, n_2$;*

(c) $\frac{(2\lambda_i(G_1)+r_1-2)\pm\sqrt{(2\lambda_i(G_1)+r_1-2)^2+4(2\lambda_i(G_1)+(\lambda_i(G_1)+r_1)(1-\lambda_i(G_1)))}}{2}$, *for $i = 2, 3, \ldots, n_1$;*

(d) *three roots of the equation $x^3 + \big(2-3r_1-r_2\big)x^2 - \big(r_2(2-3r_1) - \frac{1}{2}n_1n_2r_1 + 2r_1(2-r_1)\big)x + 2r_1r_2(2-r_1) - \frac{1}{2}n_1n_2r_1^2 = 0$.*

Corollary 2.6. *Let G be an r-regular graph on n vertices and m edges. Then the A-spectrum of $G \ominus K_{p,q}$ consists of:*

(a) 0, *repeated $p+q-2$ times;*

(b) -2, *repeated $m-n$ times;*

(c) $\frac{(2\lambda_i(G)+r-2)\pm\sqrt{(2\lambda_i(G)+r-2)^2+4(2\lambda_i(G)+(\lambda_i(G)+r)(1-\lambda_i(G)))}}{2}$, *for $i = 2, 3, \ldots, n$;*

(d) *four roots of the equation $x^4 + \big(2-3r\big)x^3 - \big(pq - \frac{1}{2}nr(p+q) + 2r(2-r)\big)x^2 - \big(pq(2-3r) - nrpq - \frac{1}{2}nr^2(p+q)\big)x + rpq\big(2(2-r) - nr\big) = 0$.*

3 L-spectra of T-vertex join and T-edge join

Theorem 3.1. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then*
$f_{L(G_1 \odot G_2)}(x) = f_{L(G_2)}(x-n_1)\cdot(x-2r_1-2)^{m_1-n_1}\cdot\Big(x^2 - \big(r_1+n_2+2+n_1\Gamma_{L(G_2)}(x-n_1)\big)x + 2(r_1+n_2+n_1\Gamma_{L(G_2)}(x-n_1)(r_1+1)) + 2r_1(r_1+n_2+1+n_1\Gamma_{L(G_2)}(x-n_1))\Big)\cdot\prod_{i=1}^{n_1-1}\Big(x^2 - \big(r_1+n_2+2(\mu_i(G_1)+1)\big)x - \big(2n_2+\mu_i(G_1)(r_1+n_2+3+\mu_i(G_1))\big)\Big)$.

Proof. Let R be the incidence matrix of G_1. Then, with a proper labeling of the vertices, the degree matrix of $G_1 \odot G_2$ can be written as

$$D(G_1 \odot G_2) = \begin{pmatrix} (2r_1+n_2)I_{n_1} & O_{n_1\times m_1} & O_{n_1\times n_2} \\ O_{m_1\times n_1} & 2r_1I_{m_1} & O_{m_1\times n_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & n_1I_{n_2}+D(G_2) \end{pmatrix}.$$

Then using adjacency matrix given in Theorem 2.1, Laplacian matrix of $G_1 \odot G_2$ is given by,

$$L(G_1 \odot G_2) = \begin{pmatrix} (2r_1+n_2)I_{n_1} - A(G_1) & -R & -J_{n_1\times n_2} \\ -R^T & (2r_1+2)I_{m_1} - R^TR & O_{m_1\times n_2} \\ -J_{n_2\times n_1} & O_{n_2\times m_1} & n_1I_{n_2}+L(G_2) \end{pmatrix}$$

and the Laplacian characteristic polynomial of $G_1 \odot G_2$ is

$$\begin{aligned} f_{L(G_1\odot G_2)}(x) &= det\begin{pmatrix} (x-2r_1-n_2)I_{n_1}+A(G_1) & R & J_{n_1\times n_2} \\ R^T & (x-2r_1-2)I_{m_1}+R^TR & O_{m_1\times n_2} \\ J_{n_2\times n_1} & O_{n_2\times m_1} & (x-n_1)I_{n_2}-L(G_2) \end{pmatrix} \\ &= det\big((x-n_1)I_{n_2}-L(G_2)\big)\cdot det(S) \\ &= f_{L(G_2)}(x-n_1)\cdot det(S), \end{aligned}$$

where

$$\begin{aligned} S &= \begin{pmatrix} (x-2r_1-n_2)I_{n_1}+A(G_1) & R \\ R^T & (x-2r_1-2)I_{m_1}+R^TR \end{pmatrix} - \begin{pmatrix} J_{n_1\times n_2} \\ O_{m_1\times n_2} \end{pmatrix} \\ &\cdot\Big((x-n_1)I_{n_2}-L(G_2)\Big)^{-1} \begin{pmatrix} J_{n_2\times n_1} & O_{n_2\times m_1} \end{pmatrix} \\ &= \begin{pmatrix} (x-2r_1-n_2)I_{n_1}+A(G_1)-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1} & R \\ R^T & (x-2r_1-2)I_{m_1}+R^TR \end{pmatrix} \end{aligned}$$

is the Schur complement of $(x-n_1)I_{n_2}-L(G_2)$ obtained by Lemma 1.4. Add the first row left $-R^T$ of the block matrix S to the second row.

$$S = \begin{pmatrix} (x-2r_1-n_2)I_{n_1}+A(G_1)-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1} & R \\ -R^T(x-2r_1-n_2-1)I_{n_1}+A(G_1)-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1} & (x-2r_1-2)I_{m_1} \end{pmatrix}.$$

By 1.4, hence,

$$\begin{aligned} det(S) &= (x-2r_1-2)^{m_1}\cdot det\Big((x-2r_1-n_2)I_{n_1}+A(G_1)-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1} \\ &\quad +\frac{RR^T}{(x-2r_1-2)}\big((x-2r_1-n_2-1)I_{n_1}+A(G_1)-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1}\big) \\ &\quad -A(G_1)-\Gamma_{A(G_2)}(x)J_{n_1\times n_1})\Big) \\ &= (x-2r_1-2)^{m_1-n_1}det(B), \end{aligned}$$

where, $B=\big((x-2r_1-n_2)I_{n_1}+A(G_1)-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1}\big)\cdot(x-2r_1-2)+(A(G_1)+r_1I_{n_1})((x-2r_1-n_2-1)I_{n_1}+A(G_1)-\Gamma_{L(G_2)}(x-n_1)J_{n_1\times n_1})$. Here matrix B is a polynomial with two variables $A(G_1)$, $J_{n_1\times n_1}$ and real coefficients. Then $B(A(G_1),J_{n_1\times n_1})$ has the eigenvalues $\sigma_{n_1}=B(r_1,n_1)$ and $\sigma_i=B(r_1-\mu_i(G_1),0)$ for $i=1,2,\ldots,n_1-1$. Hence,

$$\begin{aligned} \sigma_{n_1} &= (x-2r_1-n_2+r_1-n_1\Gamma_{L(G_2)}(x-n_1))(x-2r_1-2)+(r_1+r_1)(x-2r_1-n_2-1 \\ &\quad +r_1-n_1\Gamma_{L(G_2)}(x-n_1)) \\ &= x^2-\big[r_1+n_2+2+n_1\Gamma_{L(G_2)}(x-n_1)\big]x+\big[2(r_1+1)(r_1+n_2+n_1\Gamma_{L(G_2}(x-n_1)) \\ &\quad -2r_1(r_1+n_2+1+n_1\Gamma_{L(G_2)}(x-n_1))\big] \text{ and} \\ \sigma_i &= (x-2r_1-n_2+r_1-\mu_i(G_1))(x-2r_1-2)+(r_1+r_1-\mu_i(G_1)) \\ &\quad (x-2r_1-n_2-1+r_1-\mu_i(G_1)) \\ &= x^2-\big[r_1+n_2+2\mu_i(G_1)+2\big]x+\big[2n_2+\mu_i(G_1)(r_1+n_2+3+\mu_i(G_1))\big], \\ &\quad \text{for } i=1,2,\ldots,n_1-1. \end{aligned}$$

Since $det(S)=(x-2r_1-2)^{m_1-n_1}\prod_{i=1}^{n_1}\sigma_i$, The required result follows from $f_{L(G_1\odot G_2)}(x)=f_{L(G_2)}(x-n_1)\cdot det(S)$. □

Note that, for regular graph G_2, we have $\Gamma_{L(G_2)}(x-n_1)=\frac{n_2}{x-n_1}$. Theorem 3.1 implies the following result.

Corollary 3.2. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices. Then the L-spectrum of $G_1\odot G_2$ consists of:*

(a) *0;*

(b) $2(r_1+1)$*, repeated* m_1-n_1 *times;*

(c) $\mu_i(G_2)+n_1$ *for* $i=1,2,\ldots,n_2-1$*;*

(d) *two roots* x_1,x_2 *of the equation* $x^2-(n_1+n_2+r_1+2)x-\Big(n_1(r_1+2)+2n_2\Big)=0$*;*

(e) *two roots* $x_{i,1}, x_{i,2}$ *of the equation* $x^2 - \big(r_1 + n_2 + 2(\mu_i(G_1) + 1)\big)x - \big(2n_2 + \mu_i(G_1)(r_1 + n_2 + 3 + \mu_i(G_1))\big) = 0$, *for* $i = 1, 2, \ldots, n_1 - 1$.

Next we consider the Laplacian spectra of T-edge join.

Theorem 3.3. *Let* G_1 *be an* r_1*-regular graph on* n_1 *vertices and* m_1 *edges, and* G_2 *an arbitrary graph on* n_2 *vertices. Then*
$f_{L(G_1 \ominus G_2)}(x) = f_{L(G_2)}(x - m_1) \cdot (x - 2r_1 - n_2 - 2)^{m_1 - n_1} \cdot \Big(x^2 - \big(r_1 + n_2 + 2 + m_1\Gamma_{L(G_2)}(x - m_1)\big)x + \big(r_1 n_2 + r_1 m_1 \Gamma_{L(G_2)}(x - m_1)\big)\Big) \cdot \prod_{i=1}^{n_1-1} \Big(x^2 - \big(r_1 + n_2 + 2 + 2\mu_i(G_1)\big)x + (2r_1 + n_2 + 2)(r_1 + \mu_i(G_1)) - (2r_1 - \mu_i(G_1)) \cdot (r_1 + 1 + \mu_i(G_1))\Big)$.

Proof. Let R be the incidence matrix of G_1. Then, with a proper labeling of the vertices, the degree matrix of $G_1 \ominus G_2$ can be written as

$$D(G_1 \ominus G_2) = \begin{pmatrix} 2r_1 I_{n_1} & O_{n_1 \times m_1} & O_{n_1 \times n_2} \\ O_{m_1 \times n_1} & (2r_1 + n_2) I_{m_1} & O_{m_1 \times n_2} \\ O_{n_2 \times n_1} & O_{n_2 \times m_1} & m_1 I_{n_2} + D(G_2) \end{pmatrix}.$$

Then using adjacency matrix given in Theorem 2.4, Laplacian matrix of $G_1 \ominus G_2$ is given by,

$$L(G_1 \ominus G_2) = \begin{pmatrix} 2r_1 I_{n_1} - A(G_1) & -R & O_{n_1 \times n_2} \\ -R^T & (2r_1 + n_2 + 2) I_{m_1} - R^T R & -J_{m_1 \times n_2} \\ O_{n_2 \times n_1} & -J_{n_2 \times m_1} & m_1 I_{n_2} + L(G_2) \end{pmatrix}$$

and the Laplacian characteristic polynomial of $G_1 \ominus G_2$ is

$$\begin{aligned} f_{L(G_1 \ominus G_2)}(x) &= det \begin{pmatrix} (x - 2r_1) I_{n_1} + A(G_1) & R & O_{n_1 \times n_2} \\ R^T & (x - 2r_1 - n_2 - 2) I_{m_1} + R^T R & J_{m_1 \times n_2} \\ O_{n_2 \times n_1} & J_{n_2 \times m_1} & (x - m_1) I_{n_2} - L(G_2) \end{pmatrix} \\ &= det\big((x - m_1) I_{n_2} - L(G_2)\big) \cdot det(S) \\ &= f_{L(G_2)}(x - m_1) \cdot det(S), \end{aligned}$$

where,

$$\begin{aligned} S &= \begin{pmatrix} (x - 2r_1) I_{n_1} + A(G_1) & R \\ R^T & (x - 2r_1 - n_2 - 2) I_{m_1} + R^T R \end{pmatrix} - \begin{pmatrix} O_{n_1 \times n_2} \\ J_{m_1 \times n_2} \end{pmatrix} \\ &\quad \cdot \big((x - m_1) I_{n_2} - L(G_2)\big)^{-1} \begin{pmatrix} O_{n_2 \times n_1} & J_{n_2 \times m_1} \end{pmatrix} \\ &= \begin{pmatrix} (x - 2r_1) I_{n_1} + A(G_1) & R \\ R^T & (x - 2r_1 - n_2 - 2) I_{m_1} + R^T R - \Gamma_{L(G_2)}(x - m_1) J_{m_1 \times m_1} \end{pmatrix} \end{aligned}$$

is the Schur complement of $(x - m_1) I_{n_2} - L(G_2)$ obtained by Lemma 1.4. Add the first row left $-R^T$ of the block matrix S to the second row. Since G_1 is the regular graph, then $J_{m_1 \times n_1} R = 2J_{m_1 \times m_1}$. Therefore,

$$S = \begin{pmatrix} (x - 2r_1) I_{n_1} + A(G_1) & R \\ -R^T\big((x - 2r_1 - 1) I_{n_1} + A(G_1)\big) & (x - 2r_1 - n_2 - 2) I_{m_1} - \Gamma_{L(G_2)}(x - m_1) \frac{1}{2} J_{m_1 \times n_1} R \end{pmatrix}.$$

Now, multiply the first row of this block matrix $\frac{1}{2}\Gamma_{L(G_2)}(x - m_1) J_{m_1 n_1}$ and add to the second row.

$$S = \begin{pmatrix} (x - 2r_1) I_{n_1} + A(G_1) & R \\ C & (x - 2r_1 - n_2 - 2) I_{m_1} \end{pmatrix}$$

where $C = -R^T\big((x-2r_1-1)I_{n_1}+A(G_1)\big)+\frac{1}{2}\Gamma_{L(G_2)}(x-m_1)J_{m_1\times n_1}((x-2r_1)I_{n_1}+A(G_1))$.
Since G_1 is the r_1-regular graph, then $RJ_{m_1\times n_1} = r_1J_{n_1\times n_1}$. By lemmas 1.3 and 1.4, we have

$$det(S) = (x-2r_1-n_2-2)^{m_1-n_1}\cdot det(B),$$

where $B = \big((x-2r_1)I_{n_1}+A(G_1)\big)\cdot(x-2r_1-n_2-2)+(A(G_1)+r_1I_{n_1})((x-2r_1-1)I_{n_1}+A(G_1))-\frac{1}{2}\Gamma_{L(G_2)}(x-m_1)r_1J_{n_1\times n_1}((x-2r_1I_{n_1}+A(G_1)))$. Similar argument in the proof of Theorem 3.1. Therefore the eigenvalues of B are

$$\begin{aligned}
\sigma_{n_1} &= (x-2r_1+r_1)(x-2r_1-n_2-2)+(r_1+r_1)(x-2r_1-1+r_1)\\
&\quad -\frac{1}{2}r_1n_1\Gamma_{L(G_2)}(x-m_1)(x-2r_1+r_1)\\
&= x^2-\big(r_1+n_2+2+m_1\Gamma_{L(G_2)}(x-m_1)\big)x+\big(r_1n_2+r_1m_1\Gamma_{L(G_2)}(x-m_1)\big) \text{ and}\\
\sigma_i &= (x-2r_1+r_1-\mu_i(G_1))(x-2r_1-n_2-2)+(r_1+r_1-\mu_i(G_1))(x-2r_1-1+r_1-\mu_i(G_1))\\
&= x^2-\big(r_1+n_2+2+2\mu_i(G_1)\big)x+\big((2r_1+n_2+2)(r_1+\mu_i(G_1))-(2r_1-\mu_i(G_1))\\
&\quad \cdot(r_1+1+\mu_i(G_1))\big) \text{ for } i=1,2,\dots n_1-1.
\end{aligned}$$

Since $det(S) = (x-2r_1-n_2-2)^{m_1-n_1}\cdot\prod_{i=1}^{n_1-1}\sigma_i$, The required result follows from $f_{L(G_1\ominus G_2)}(x) = f_{L(G_2)}(x)\cdot det(S)$. □

Corollary 3.4. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices. Then the L-spectrum of $G_1\ominus G_2$ consists of:*

(a) *0;*

(b) $2r_1+n_2+2$*, repeated* m_1-n_1 *times;*

(c) $\mu_i(G_2)+m_1$ *for* $i=1,2,\dots,n_2-1$*;*

(d) *two roots* $x_{i,1},x_{i,2}$ *of the equation* $x^2-\big(r_1+n_2+2+2\mu_i(G_1)\big)x+\big((2r_1+n_2+2)(r_1+\mu_i(G_1))-(2r_1-\mu_i(G_1))\cdot(r_1+1+\mu_i(G_1))\big)=0$*, for* $i=1,2,\dots,n_1-1$*;*

(e) *two roots* x_1,x_2 *of the equation* $x^2-\big(m_1+n_2+r_1+2\big)x+\big((r_1+2)m_1+r_1n_2\big)=0$*.*

4 Q-spectra of T-vertex join and T-edge join

Theorem 4.1. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then*
$f_{Q(G_1\odot G_2)}(x) = f_{Q(G_2)}(x-n_1)\cdot(x-2r_1-2)^{m_1-n_1}\cdot\Big(x^2-\big(9r_1+n_2+2+n_1\Gamma_{Q(G_2)}(x-n_1)\big)x+(5r_1+2)(4r_1+n_2+n_1\Gamma_{Q(G_2)}(x-n_1))-3r_1\Big)\cdot\prod_{i=1}^{n_1-1}\Big(x^2-\big(3r_1+n_2+2(\nu_i(G_1)+1)\big)x+2(r_1+1)(r_1+n_2+\nu_i(G_1))+\nu_i(G_1)(r_1+n_2-1+\nu_i(G_1))\Big)$.

Proof. Let R be the incidence matrix of G_1. Then, with a proper labeling of the vertices, the degree matrix of $G_1\odot G_2$ can be written as

$$D(G_1\odot G_2) = \begin{pmatrix} (2r_1+n_2)I_{n_1} & O_{n_1\times m_1} & O_{n_1\times n_2}\\ O_{m_1}\times n_1 & 2r_1I_{m_1} & O_{m_1\times n_2}\\ O_{n_2\times n_1} & O_{n_2\times m_1} & n_1I_{n_2}+D(G_2)\end{pmatrix}.$$

Then using adjacency matrix given in Theorem 3.1, signless Laplacian matrix of $G_1\odot G_2$ is given by,

$$Q(G_1\odot G_2) = \begin{pmatrix} (2r_1+n_2)I_{n_1}+A(G_1) & R & J_{n_1\times n_2}\\ R^T & (2r_1-2)I_{m_1}+R^TR & O_{m_1\times n_2}\\ J_{n_2\times n_1} & O_{n_2\times m_1} & n_1I_{n_2}+Q(G_2)\end{pmatrix}.$$

The rest of the proof is similar to that of Theorem 3.1 and hence we omit details. □

Note that, for regular graph G_2, we have $\Gamma_{Q(G_2)}(x) = \frac{n_2}{x-2r_2}$. Theorem 4.1 implies the following result.

Corollary 4.2. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices. Then the Q-spectrum of $G_1 \odot G_2$ consists of:*

(a) $2(r_1+1)$, *repeated* $m_1 - n_1$ *times;*

(b) $\nu_i(G_2) + n_1$ *for* $i = 2, 3, \ldots, n_2$*;*

(c) $\frac{\left(3r_1+n_2+2(\nu_i(G_1)+1)\right) \pm \sqrt{\left(3r_1+n_2+2(\nu_i(G_1)+1)\right)^2 - 4\left(2(r_1+1)(r_1+n_2+\nu_i(G_1)) + \nu_i(G_1)(r_1+n_2-1+\nu_i(G_1))\right)}}{2}$, *for* $i = 1, 2, \ldots, n_1 - 1$*;*

(d) *three roots of the equation* $x^3 - (9r_1 + n_1 + n_2 + 2r_2 + 2)x^2 + \left((2r_2 + n_1)(9r_1 + n_2 + 2) + (5r_1+2)(4r_1+n_2) - n_1n_2\right)x - \left((5r_1+2)(4r_1+2)(n_1+2r_2) - n_1n_2(5r_1+2) + 3r_1\right) = 0.$

Theorem 4.3. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then*

$f_{Q(G_1 \ominus G_2)}(x) = f_{Q(G_2)}(x-m_1) \cdot (x-2r_1-n_2+2)^{m_1-n_1} \cdot \Big(x^2 - \big(9r_1+n_2-2-\frac{1}{2}r_1n_1\Gamma_{Q(G_2)}(x-m_1)\big)x + r_1\big(4(2r_1+n_2-2) + 3(4r_1-1) - 2r_1n_1\Gamma_{Q(G_2)}(x-m_1)\big)\Big) \cdot \prod_{i=1}^{n_1-1} \Big(x^2 - \big(3r_1 + n_2 - 2 + 2\nu_i(G_1)\big)x + (2r_1+n_2-2)(r_1+\nu_i(G_1)) + \nu_i(G_1)(r_1+\nu_i(G_1)-1)\Big).$

Proof. Let R be the incidence matrix of G_1. Then, with a proper labeling of the vertices, the degree matrix of $G_1 \ominus G_2$ can be written as

$$D(G_1 \ominus G_2) = \begin{pmatrix} 2r_1I_{n_1} & O_{n_1\times m_1} & O_{n_1\times n_2} \\ O_{m_1\times n_1} & (2r_1+n_2)I_{m_1} & O_{m_1\times n_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & m_1I_{n_2} + D(G_2) \end{pmatrix}.$$

Then using adjacency matrix given in Theorem 3.3, signless Laplacian matrix of $G_1 \ominus G_2$ is given by,

$$Q(G_1 \ominus G_2) = \begin{pmatrix} 2r_1I_{n_1} + A(G_1) & R & O_{n_1\times n_2} \\ R^T & (2r_1+n_2-2)I_{m_1} + R^TR & J_{m_1\times n_2} \\ O_{n_2\times n_1} & J_{n_2\times m_1} & m_1I_{n_2} + Q(G_2) \end{pmatrix}.$$

The rest of the proof is similar to that of Theorem 3.3 and hence we omit details. □

Corollary 4.4. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices. Then the Q-spectrum of $G_1 \ominus G_2$ consists of:*

(a) $2r_1 + n_2 - 1$, *repeated* $m_1 - n_1$ *times;*

(b) $\nu_i(G_2) + m_1$ *for* $i = 2, 3, \ldots, n_2$*;*

(c) $\frac{\left(3r_1+n_2-2+2\mu_i(G_1)\right) \pm \sqrt{\left(3r_1+n_2-2+2\mu_i(G_1)\right)^2 - 4\left((2r_1+n_2-2)(r_1+\nu_i(G_1)) + \nu_i(G_1)(r_1+\nu_i(G_1)-1)\right)}}{2}$, *for* $i = 1, 2, \ldots, n_1 - 1$*;*

(d) *three roots of the equation* $x^3 - (m_1+11r_1+n_2-2)x^2 + \Big((m_1+2r_1)(9r_1+n_2-2) + \frac{1}{2}r_1n_1n_2 + r_1\big(4(2r_1+n_2-2)+3(4r_1-1)\big)\Big)x - r_1\Big((m_1+2r_1)\big(4(2r_1+n_2-2)+3(4r_1-1)\big) + 2r_1n_1n_2\Big) = 0.$

5 Applications

The Laplacian matrix of a graph has many interesting properties. In sub sections, we have used two properties to find the number of spanning trees and the Kirchhoff index of T-vertex join and T-edge join of graphs. Recently, computing the number of spanning trees and the Kirchhoff index of many graph products and circulant graphs attracted many researchers' attention (see, [1, 6, 11, 12, 15, 17, 18, 20]).

5.1 The number of spanning trees

Let $t(G)$ be the number of spanning trees of a connected graph G on n vertices with Laplacian spectrum $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$. Then [4]

$$t(G) = \frac{\mu_1(G)\mu_2(G)\cdots\mu_{n-1}(G)}{n}. \tag{5.1}$$

Theorem 5.1. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices. Then*

$$\begin{aligned} t(G_1 \odot G_2) &= \frac{1}{n_1+m_1+n_2}(2(r_1+1))^{m_1-n_1}\big(n_1(r_1+2)+2n_2\big)\prod_{i=1}^{n_2-1}(\mu_i(G_2)+n_1) \\ &\cdot \prod_{i=1}^{n_1-1}\big[2n_2+\mu_i(G_1)(r_1+n_2+3+\mu_i(G_1))\big]. \\ t(G_1 \ominus G_2) &= \frac{1}{n_1+m_1+n_2}(2(r_1+1)+n_2)^{m_1-n_1}\cdot\big((r_1+2)m_1+r_1n_2\big)\cdot\prod_{i=1}^{n_2-1}(\mu_i(G_2)+m_1) \\ &\cdot \prod_{i=1}^{n_1-1}\big[(2r_1+n_2+2)(r_1+\mu_i(G_1))-(2r_1-\mu_i(G_1))\cdot(r_1+1+\mu_i(G_1))\big]. \end{aligned}$$

Proof. Notice that the roots of $f_{L(G_1 \odot G_2)}(x)$ are given in Corollary 3.2. By the relation between coefficients and roots of the quadratic equation, we have $x_1x_2 = n_1(r_1+2)+2n_2$ and $x_{i,1}x_{i,2} = 2n_2+\mu_i(G_1)(r_1+n_2+3+\mu_i(G_1))$ for $i = 1, 2, \cdots, n_1-1$. Then $t(G_1 \odot G_2)$ is obtained by Eq. 5.1. Similarly, $t(G_1 \ominus G_2)$ follows from Corollary 3.4. □

5.2 The Kirchhoff index

Gutman and Mohar [7] and Zhu et al. [21] obtained the Kirchhoff index of a graph in terms of Laplacian eigenvalues as follows:

Lemma 5.2. *[7, 21] Let G be a connected graph with $n \geq 2$ vertices. Then*

$$Kf(G) = n\cdot\sum_{i=1}^{n-1}\frac{1}{\mu_i(G)}, \tag{5.2}$$

where $\mu_1(G), \mu_2(G), \cdots, \mu_{n-1}(G)$ are the nonzero Laplacian eigenvalues of G.

Theorem 5.3. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices. Then*

$$\begin{aligned} Kf(G_1 \odot G_2) &= (n_1+m_1+n_2)\Big[\frac{m_1-n_1}{2(r_1+1)}+\sum_{i=1}^{n_2-1}\frac{1}{\mu_i(G_2)+n_1}+\frac{n_1+n_2+r_1+2}{n_1(r_1+2)+2n_2} \\ &+\sum_{i=1}^{n_1-1}\frac{r_1+n_2+2(\mu_i(G_1)+1))}{2n_2+\mu_i(G_1)(r_1+n_2+3+\mu_i(G_1))}\Big]. \\ Kf(G_1 \ominus G_2) &= (n_1+m_1+n_2)\Big[\frac{m_1-n_1}{2(r_1+1)+n_2}+\sum_{i=1}^{n_2-1}\frac{1}{\mu_i(G_2)+m_1}+\frac{r_1+n_2+m_1+2}{(r_1+2)m_1+r_1n_2} \\ &+\sum_{i=1}^{n_1-1}\frac{r_1+n_2+2+2\mu_i(G_1)}{(2r_1+n_2+2)(r_1+\mu_i(G_1))-(2r_1-\mu_i(G_1))\cdot(r_1+1+\mu_i(G_1))}\Big]. \end{aligned}$$

Proof. Consider the roots in the Corollary 3.2 of $f_{L(G_1 \odot G_2)}(x)$. By the relation between coefficients and roots of the quadratic equation, we have $x_1 + x_2 = n_1 + n_2 + r_1 + 2$ and

$x_{i,1}x_{i,2} = r_1 + n_2 + 2(\mu_i(G_1) + 1)$ for $i = 1, 2, \cdots, n_1 - 1$. Then

$$\frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1 + x_2}{x_1 x_2} = \frac{n_1 + n_2 + r_1 + 2}{n_1(r_1 + 2) + 2n_2}$$

and

$$\frac{1}{x_{i,1}} + \frac{1}{x_{i,2}} = \frac{x_{i,1} + x_{i,2}}{x_{i,1}x_{i,2}} = \sum_{i=1}^{n_1-1} \frac{r_1 + n_2 + 2(\mu_i(G_1) + 1))}{2n_2 + \mu_i(G_1)(r_1 + n_2 + 3 + \mu_i(G_1))},$$

for $i = 1, 2, \cdots, n_1 - 1$. So $Kf(G_1 \odot G_2)$ is obtained by Eq. 5.2. Similarly, $Kf\big(G_1 \ominus G_2\big)$ follows from Corollary 3.4. □

References

[1] T. Atajan, X. Yong, H. Inaba, *Further analysis of the number of spanning trees in circulant graphs*, Discrete math., **306**, 2817–2827, (2006).

[2] M. Behzad, *A criterion for the planarity of a total graph*, Pro. Cambridge Philos. Soc., **63**, 697–681, (1967).

[3] S. Y. Cui, G. X. Tian, *The spectrum and the signless Laplacian spectrum of coronae*, Linear Algebra Appl., **437**, 1692–1703, (2012).

[4] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs–Theory and Applications*, 3rd Edition, Johann Ambrosius Barth, Heidelberg, (1995).

[5] A. Das, P. Panigrahi, *Spectra of R-Vertex join and R-edge join of two graphs*, Discuss. Math., **38**, 19–31, (2018).

[6] X. Gao, Y. Luo, W. Liu, *Kirchhoff index in line, subdivision and total graphs of a regular graph*, Discrete Appl. Math., **160**, 560–565, (2012).

[7] I. Gutman, B. Mohar, *The quasi-Wiener and the Kirchhoff indices coincide*, J. Chem. Inf. Comput. Sci., **36**, 982–985, (1996).

[8] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass, (1969).

[9] G. Indulal, *Spectra of two new joins of graphs and infinite families of integral graphs*, Kragujevac J. Math., **36**, 133–139, (2012).

[10] J. Lan, B. Zhou, *Spectra of graph operations based on R-graph*, Linear Multilinear Algebra, **63**, 1401–1422, (2015).

[11] X. Liu, Z. Zhang, *Spectra of subdivision-vertex and subdivision-edge join of two graphs*, Bull. Malays. Math. Sci. Soc., **45**, 15–31, (2019).

[12] X. Liu, J. Zhou, C. Bu, *Resistance distance and Kirchhoff index of R-vertex join and R-edge join of two graphs*, Discrete Appl. Math., **187**, 130–139, (2015).

[13] X. Liu, P. Lu, *Spectra of subdivision-vertex and subdivision-edge neighbourhood coronae*, Linear Algebra Appl., **438**, 3547–3559, (2013).

[14] C. McLeman, E. McNicholas, *Spectra of coronae*, Linear Algebra Appl., **435**, 998–1007, (2011).

[15] K. Ozeki, T. Yamashita, *Spanning trees, A survey*, Graphs Comb., **27**, 1–26, (2011).

[16] S. Patil, M. Mathapati, *Spectra of Indu-Bala product of graphs and some new pairs of cospectral graphs*, Disc. Math. Algo. Appl., **11(5)**, 1950056, (2019).

[17] W. Wang, D. Yang, Y. Luo, *The Laplacian polynomial and Kirchhoff index of graphs derived from regular graphs*, Discrete Appl. Math., **161**, 3063–3071, (2013).

[18] H. Zhang, Y. Yang, C. Li, *Kirchhoff index of composite graphs*, Discrete Appl. Math., **157**, 2918–2927, (2009).

[19] F. Z. Zhang, *The Schur Complement and its Applications*, Springer, New York, (2005).

[20] Y. Zhang, X. Yong, M. J. Golin, *The number of spanning trees in circulant graphs*, Discrete Math., **223**, 337–350, (2000).

[21] H. Y. Zhu, D. J. Klein, I. Lukovits, *Extensions of the Wiener number*, J. Chem. Inf. Comput. Sci., **36**, 420–428, (1996).

Author information

Shreekant Patil*, Department of Mathematics, BLDEA's S.B. Arts and K.C.P. Science College, Vijayapur - 586 103, Karnataka, India.
E-mail: shreekantpatil949@gmail.com

Sourav Mondal, Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur - 603 203, Tamil Nadu, India.
E-mail: souravmath94@gmail.com

Mallikarjun Mathapati, Department of Mathematics, BLDEA's V.P. Dr P.G.H. College of Engineering and Technology, Vijayapur - 586 103, Karnataka, India.
E-mail: mmathapati69@gmail.com

Ashok Rathod, Department of Mathematics, KLE's G.I. Bagewadi Arts, Science and Commerce College, Nipani - 591 237, Karnataka, India.
E-mail: ashokmrmaths@gmail.com

Veena R. Desai, Department of Mathematics, BLDEA's S.B. Arts and K.C.P. Science College, Vijayapur - 586 103, Karnataka, India.
E-mail: veenardesai6f@gmail.com

Received: 2024-09-27
Accepted: 2025-04-23.