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Third order Newton-like method in Riemannian manifolds

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Abstract In this paper, we study a semilocal convergence analysis of a Newton-like method in Riemannian manifolds. We present a semilocal convergence analysis of a Newton-like method under ω-continuity condition on the second order covariant derivative of a vector field. Using normal coordinates the order of convergence is derived. Finally, two numerical examples are provided.

1 Introduction

Many problems in engineering and technology field can be solved through nonlinear equation

$$\mathfrak{G}(x) = 0, \tag{1.1}$$

where $\mathfrak{G}$ is a nonlinear operator defined in an open convex subset Θ of a Banach space B into itself. The exact solution of these equations are difficult to find so that we use iterative methods to solve these equations. There are various methods to find the solution of a nonlinear equation. The most famous iterative method is Newton's method which converges quadratically. To improve the convergence order many higher order iterative methods have been presented. The most famous third order iterative methods are Halley method, Euler method, Chebyshev method and Chebyshev-Halley method etc. Many articles of numerical iterative method in Banach spaces can be found in [1, 2, 3, 11, 12]. The Newton-like method [14] in Banach space to solve (1.1) is defined as:

$$\left.\begin{aligned} y_n &= x_n - \mathfrak{G}'(x_n)^{-1}\mathfrak{G}(x_n), \\ x_{n+1} &= x_n - \Big(\frac{\mathfrak{G}'(x_n) + \mathfrak{G}'(y_n)}{2}\Big)^{-1}\mathfrak{G}(x_n), \text{ for each } n = 0, 1, 2, \dots, \end{aligned}\right\} \tag{1.2}$$

where $\mathfrak{G}'$ is first Fréchet derivative of $\mathfrak{G}$. The Newton-like method being of third order is important to study because it does not contain higher order derivatives. Recently, there has been a growing interest in studying iterative methods in Riemannian manifolds, since there are many numerical problems in manifolds that arise in many contexts [4, 5]. In this paper, we study a semilocal convergence analysis of a Newton-like method (1.2) in Riemannian manifolds to find the singular point of a vector field and establish the convergence analysis under ω-continuity condition on the second order covariant derivative of a vector field.

The paper is organised as follows: Section 1 is the introduction. In Section 2, we introduce some basic results of differential geometry. In Section 3, we present a semilocal convergence analysis of a Newton-like method in Riemannian manifolds under ω-continuity condition on the second order covariant derivative of a vector field. In Section 4, the order of convergence of the

Newton-like method is derived. In Section 5, two examples are given. In Section 6, conclusion is given.

2 Preliminaries

In this section, we discuss some basic results of differential geometry (see [9] and the references therein).

Definition 2.1. Let $\mathbb{Z}$ be a real n dimensional Riemannian manifold. The tangent space of $\mathbb{Z}$ at a is denoted by $T_a\mathbb{Z}$. The inner product $\langle\, .,.\rangle_a$ on $T_a\mathbb{Z}$ induces the norm $\|.\|_a$. The tangent bundle of $\mathbb{Z}$ is denoted by $T\mathbb{Z}$ and is defined by

$$T\mathbb{Z} := \{(a,v); a \in \mathbb{Z} \text{ and } v \in T_a\mathbb{Z}\} = \bigcup_{a\in\mathbb{Z}} T_a\mathbb{Z}.$$

Let $a,t \in \mathbb{Z}$, and let $\rho : [0,1] \to \mathbb{Z}$ be a piecewise smooth curve joining the points a and t. The arc length of ρ is defined by

$$l(\rho) = \int_0^1 \|\rho'(x)\| dx = \int_0^1 \left\langle \frac{d\rho}{dx}, \frac{d\rho}{dx} \right\rangle^{\frac{1}{2}} dx$$

and the Riemannian distance from a to t is defined by

$$d(a,t) = \inf_{\rho} l(\rho),$$

where the infimum is taken over all the piecewise smooth curves ρ connecting a and t.

Definition 2.2. Let $\chi(\mathbb{Z})$ be the set of all vector fields of class C^∞ on $\mathbb{Z}$ and $D(\mathbb{Z})$ be the ring of real-valued functions of class C^∞ defined on $\mathbb{Z}$. An affine connection ∇ on $\mathbb{Z}$ is a mapping

$$\nabla : \chi(\mathbb{Z}) \times \chi(\mathbb{Z}) \to \chi(\mathbb{Z})$$
$$(X, \mathfrak{N}) \mapsto \nabla_X \mathfrak{N}.$$

Definition 2.3. Let $\mathfrak{N}$ be a vector field of class C^1 on $\mathbb{Z}$, the covariant derivative of $\mathfrak{N}$ is determined by the connection ∇ which defines on each $a \in \mathbb{Z}$ a linear application of $T_a\mathbb{Z}$ itself

$$D\mathfrak{N}(a) : T_a\mathbb{Z} \to T_a\mathbb{Z}$$
$$v \mapsto D\mathfrak{N}(a)(v) = \nabla_X \mathfrak{N}(a),$$

where X is a vector field satisfying $X(a) = v$.

Definition 2.4. A parametrized curve $\rho : I \subseteq \mathbb{R} \to \mathbb{Z}$ is a geodesic $p_0 \in I$, if $\nabla_{\rho'(p)}\rho'(p) = 0$ at the point p_0. If ρ is a geodesic for all $p \in I$, then we say ρ is a geodesic. If $[x,y] \subseteq I$, then ρ is a geodesic segment joining $\rho(x)$ to $\rho(y)$. Since $\rho'(p)$ is parallel along $\rho(p)$ therefore $\|\rho'(p)\|$ is constant. Let $U(a,s)$ and $U[a,s]$ be an open and a closed geodesic ball with centre a and radius s respectively. By the Hopf-Rinow theorem, if $\mathbb{Z}$ is a complete metric space, then for any $a,t \in \mathbb{Z}$ there exists a geodesic ϱ called the minimizing geodesic joining a to t with

$$l(\rho) = d(a,t),$$

here we have assumed that $\mathbb{Z}$ is complete therefore if $v \in T_a\mathbb{Z}$, then there exists a unique minimizing geodesic ρ such that $\rho(0) = a$ and $\rho'(0) = v$. The point $\rho(1)$ is called the image of v by the exponential map at a, i.e.

$$\exp_a : T_a\mathbb{Z} \to \mathbb{Z},$$

such that $\exp_a(v) = \rho(1)$ and $\rho(p) = \exp_a(pv)$, for any $p \in [0,1]$.

Definition 2.5. Let ρ be a piecewise smooth curve. Then for any $x, y \in \mathbb{R}$, the parallel transport along ρ is denoted by $R_{\rho,.,.}$ and given by

$$\begin{aligned} R_{\rho,x,y} : T_{\rho(x)}\mathbb{Z} &\to T_{\rho(y)}\mathbb{Z} \\ v &\mapsto V(\rho(y)), \end{aligned}$$

where V is the unique vector field along ρ such that $\nabla_{\rho'(p)}V = 0$ and $V(\rho(x)) = v$.

Definition 2.6. Let $j \in \mathbb{N}$ and $\mathfrak{N}$ be a vector field of class C^k. The covariant derivative of order j of $\mathfrak{N}$ is denoted by $D^j\mathfrak{N}$ and defined as the multilinear map

$$D^j\mathfrak{N} : \underbrace{C^k(T\mathbb{Z}) \times C^k(T\mathbb{Z}) \times \cdots \times C^k(T\mathbb{Z})}_{\text{j-times}} \to C^{k-j}(T\mathbb{Z})$$

which is given by

$$\begin{aligned} D^j\mathfrak{N}(A_1, A_2, \ldots, A_{j-1}, A) &= \nabla_A D^{j-1}\mathfrak{N}(A_1, A_2, \ldots, A_{j-1}) \\ &\quad - \sum_{i=1}^{j-1} D^{j-1}\mathfrak{N}(A_1, A_2, \ldots, \nabla_A A_i, \ldots, A_{j-1}), \end{aligned} \tag{2.1}$$

for all $A_1, A_2, \ldots, A_{j-1} \in C^k(T\mathbb{Z})$.

Definition 2.7. Let $\mathbb{Z}$ be a Riemannian manifold, $\Omega \subseteq \mathbb{Z}$ be an open convex set and $\mathfrak{N} \in \chi(\mathbb{Z})$. The covariant derivative $D\mathfrak{N} = \nabla_{(.)}\mathfrak{N}$ is Lipschitz with constant $\mathbf{M} > 0$, if for any geodesic ρ and $x, y \in \mathbb{R}$ such that $\rho[x, y] \subseteq \Omega$, it satisfies the inequality

$$\|R_{\rho,y,x} D\mathfrak{N}(\rho(y)) R_{\rho,x,y} - D\mathfrak{N}(\rho(x))\| \leq \mathbf{M} \int_x^y \|\rho'(p)\| dp.$$

We will write $D\mathfrak{N} \in Lip_{\mathbf{M}}(\Omega)$. If $\mathbb{Z} = \mathbb{R}^n$, then the above definition is the same as the usual Lipschitz condition for the operator $D\mathfrak{N} : \mathbb{Z} \to \mathbb{Z}$.

Proposition 1. Let ρ be a curve in $\mathbb{Z}$ and $\mathfrak{N}$ be a C^1 vector field on $\mathbb{Z}$, the covariant derivative of $\mathfrak{N}$ in the direction of $\rho'(t)$ is defined as

$$D\mathfrak{N}(\rho(t))\rho'(t) = \nabla_{\rho'(t)}\mathfrak{N}_{\rho(t)} = \lim_{r \to 0} \frac{1}{r}\big(R_{\rho,t+r,t}\mathfrak{N}(\rho(t+r)) - \mathfrak{N}(\rho(t))\big).$$

If $\mathbb{Z} = \mathbb{R}^n$, then it is the same as the directional derivative in $\mathbb{R}^n$.

To prove the convergence analysis of our iterative method we need some theorems. The proofs are given in [7].

Theorem 2.8. *Let ρ be a geodesic in $\mathbb{Z}$ and let $\mathfrak{N}$ be a C^1-vector field on $\mathbb{Z}$. Then,*

$$R_{\rho,t,0}\mathfrak{N}(\rho(t)) = \mathfrak{N}(\rho(0)) + \int_0^t R_{\rho,\theta,0} D\mathfrak{N}(\rho(\theta))\rho'(\theta) d\theta.$$

Theorem 2.9. *Let ρ be a geodesic in $\mathbb{Z}$ and let $\mathfrak{N}$ be a C^2-vector field on $\mathbb{Z}$. Then,*

$$R_{\rho,t,0} D\mathfrak{N}(\rho(t))\rho'(t) = D\mathfrak{N}(\rho(0))\rho'(0) + \int_0^t R_{\rho,\theta,0} D^2\mathfrak{N}(\rho(\theta))(\rho'(\theta), \rho'(\theta)) d\theta.$$

Theorem 2.10. *Let ρ be a geodesic in $\mathbb{Z}$ such that $[0, 1] \subseteq Dom(\rho)$ and let $\mathfrak{N}$ be a C^2-vector field on $\mathbb{Z}$. Then,*

$$R_{\rho,1,0}\mathfrak{N}(\rho(1)) = \mathfrak{N}(\rho(0)) + D\mathfrak{N}(\rho(0))\rho'(0) + \int_0^1 (1-\theta) R_{\rho,\theta,0} D^2\mathfrak{N}(\rho(\theta))(\rho'(\theta), \rho'(\theta)) d\theta.$$

3 Semilocal convergence analysis

In this section, we shall study a semilocal convergence analysis of a Newton-like method in Riemannian manifolds. The Newton-like method (1.2) in $\mathbb{Z}$ can be formulated as:

$$\left.\begin{aligned} f_n &= -\Gamma_n \mathfrak{N}(a_n), \\ b_n &= \exp_{a_n}(f_n), \\ \Psi_n(t) &= \exp_{a_n}(tf_n), \\ g_n &= -\left(\frac{D\mathfrak{N}(a_n) + R_{\Psi_n,1,0} D\mathfrak{N}(b_n) R_{\Psi_n,0,1}}{2}\right)^{-1} \mathfrak{N}(a_n), \\ a_{n+1} &= \exp_{a_n}(g_n), \quad \text{for each } n = 0, 1, 2, \dots, \end{aligned}\right\} \tag{3.1}$$

where $D\mathfrak{N}(a_n) = \nabla_{(.)}\mathfrak{N}(a_n)$ and $\Gamma_n = D\mathfrak{N}(a_n)^{-1}$. Let $a_0 \in \Omega \subseteq \mathbb{Z}$ and assume that

(i) $\|\Gamma_0\| \leq \xi, \ \ \xi > 0$,

(ii) $\|\Gamma_0 \mathfrak{N}(a_0)\| \leq \zeta, \ \ \zeta > 0$,

(iii) $\|D^2\mathfrak{N}(a)\| \leq A, \ \ A > 0, \ \ \forall a \in \Omega$,

(iv) $\|R_{\varrho,b,a} D^2\mathfrak{N}(\varrho(b)) R^2_{\varrho,a,b} - D^2\mathfrak{N}(\varrho(a))\| \leq \omega(d(p,q))$,
where ϱ is a geodesic such that $\varrho(a) = q, \ \varrho(b) = p$, and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and increasing function such that $\omega(0) \geq 0$,

(v) There exists a continuous and increasing function $\mathcal{C} : [0,1] \to \mathbb{R}_+$ such that $\omega(ta) \leq \mathcal{C}(t)\omega(a)$ with $t \in [0,1]$ and $a \in \mathbb{R}_+$.

Let $x_0 = A\xi\zeta, \ \ y_0 = \xi\zeta\omega(\zeta)$ and for all $n \geq 1$, we define the sequences

$$z_n = \mathcal{A}(x_n)\mathcal{B}(x_n, y_n), \quad x_{n+1} = x_n \mathcal{A}(x_n) z_n, \quad y_{n+1} = y_n \mathcal{A}(x_n) z_n \mathcal{C}(z_n), \tag{3.2}$$

where

$$\mathcal{A}(a) = \frac{2-a}{2-3a}, \tag{3.3}$$

$$\mathcal{B}(a,b) = \left[\frac{a(4-a)(3-a)}{(2-a)^2} + Lb\right], \ \ L > 0. \tag{3.4}$$

Let $\mathbf{b}_0 = 0.1905960896\dots$ be the smallest positive zero of the polynomial $\mathcal{C}(a) = -a^3 + 16a^2 - 24a + 4$. Before studying the convergence of (3.1), we will first analyze some properties of the sequences given in (3.2). For this we will prove some lemmas.

Lemma 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be the real valued functions, which is given by* (3.3) *and* (3.4)*, then, for $a \in (0, \boldsymbol{b}_0]$*

(i) $\mathcal{A}$ is increasing and $\mathcal{A}(a) > 1$,

(ii) $\mathcal{B}$ is increasing in both arguments for $b > 0$,

(iii) $\mathcal{A}(\lambda a) < \mathcal{A}(a)$ and $\mathcal{B}(\lambda a, \lambda b) < \lambda \mathcal{B}(a,b)$, for $\lambda \in (0,1)$.

Proof. It is easy to prove and hence omitted. □

Lemma 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be the real valued functions, which is given by* (3.3) *and* (3.4) *and $\mathcal{C}(t) \leq 1, \ \forall t \in [0,1]$. Let us define a function*

$$\mathcal{D}(a) = \frac{(-a^3 + 16a^2 - 24a + 4)}{L(2-a)^2}. \tag{3.5}$$

If $x_0 \in (0, \boldsymbol{b}_0]$ and $0 \leq \boldsymbol{b}_0 \leq \mathcal{D}(x_0)$, then, *(i)* $z_n\mathcal{A}(x_n) \le 1$,

(ii) $\{x_n\},\ \{y_n\},\ \{z_n\}$ *are decreasing and* $x_n < 1,\ z_n < 1$ *for all* $n \ge 1$.

Proof. See [14]. □

Lemma 3.3. *Suppose* $x_0 \in (0, \boldsymbol{b}_0)$ *and* $0 < \boldsymbol{b}_0 < \mathcal{D}(x_0)$. *Define* $\varsigma = x_1/x_0$, *then for* $n \ge 1$, *we have*

(i) $x_n \le \varsigma^{2^{n-1}} x_{n-1} \le \varsigma^{2^n-1} x_0$, *for* $n \ge 2$,

(ii) $y_n \le \varsigma^{2^{n-1}} y_{n-1} < \varsigma^{2^{n-1}} y_0$,

(iii) $z_n \le \varsigma^{2^n}/\mathcal{A}(x_0)$.

Proof. See [14].

□

Lemma 3.4. *Let* $\mathfrak{N}$ *be a* C^2 *vector field on on* $\mathbb{Z}$ *and let* $\Psi_n(t)$ *be given as above. Then, for all* $n \ge 0$, *we have*

$$R_{\Psi_n,1,0}\mathfrak{N}(b_n) = \int_0^1 (1-t) R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t))(R_{\Psi_n,0,t} f_n, R_{\Psi_n,0,t} f_n)dt.$$

Proof. By Theorem 2.10 and (3.1), we have

$$\begin{aligned} P_{\Psi_n,1,0}\mathfrak{N}(b_n) &= \mathfrak{N}(a_n) + D\mathfrak{N}(a_n) f_n + \int_0^1 (1-t) R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t))(\Psi_n'(t), \Psi_n'(t))dt \\ &= \mathfrak{N}(a_n) + D\mathfrak{N}(a_n)(-D\mathfrak{N}(a_n)^{-1}\mathfrak{N}(a_n)) \\ &+ \int_0^1 (1-t) R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t))(\Psi_n'(t), \Psi_n'(t))dt \\ &= \int_0^1 (1-t) R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t))(\Psi_n'(t), \Psi_n'(t))dt. \end{aligned}$$

Since Ψ_n is the family of minimizing geodesics, then $\Psi_n'(t)$ is parallel and $\Psi_n'(t) = R_{\Psi_n,0,t}\Psi_n'(0)$, $\Psi_n'(0) = f_n$. We have

$$R_{\Psi_n,1,0}\mathfrak{N}(b_n) = \int_0^1 (1-t) R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t))(R_{\Psi_n,0,t} f_n, R_{\Psi_n,0,t} f_n)dt.$$

□

Lemma 3.5. *Let* $\mathfrak{N}$ *be a* C^2 *vector field on* $\mathbb{Z}$ *and let* $\mu(t)$ *be given as* $\mu(t) = \exp_{b_n}(tl_n)$, *where* $l_n = R_{\Psi_n,0,1}\Big[D\mathfrak{N}(a_n)^{-1}\mathfrak{N}(a_n) - 2\big(D\mathfrak{N}(a_n) + R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1}\big)^{-1}\mathfrak{N}(a_n)\Big]$, $\mu(0) = b_n$ *and* $\mu(1) = a_{n+1}$ *and let* $\Psi_n(t)$ *be given as above. Then, for all* $n \ge 0$, *we have*

$$\begin{aligned} R_{\mu,1,0}\mathfrak{N}(a_{n+1}) &= \int_0^1 (1-t) R_{\mu,t,0} D^2\mathfrak{N}(\mu(t)) R^2_{\mu,0,t}(l_n, l_n)dt \\ &+ \frac{1}{2} R_{\Psi_n,0,1} \int_0^1 R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t)) R^2_{\Psi_n,0,t}(f_n, R_{\Psi_n,1,0} l_n)dt \\ &+ R_{\Psi_n,0,1} \int_0^1 (1-t) R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t)) R^2_{\Psi_n,0,t}(f_n, f_n)dt \\ &- \frac{1}{2} R_{\Psi_n,0,1} \int_0^1 R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t)) R^2_{\Psi_n,0,t}(f_n, f_n)dt. \end{aligned}$$

Proof. Since $\Psi_n(0) = a_n$ and $\Psi_n(1) = b_n$, by Theorem 2.9, we have

$$\begin{aligned}
R_{\Psi_n,1,0} D\mathfrak{N}(b_n) l_n &= \frac{1}{2}[R_{\Psi_n,1,0} D\mathfrak{N}(b_n) R_{\Psi_n,0,1} - D\mathfrak{N}(a_n)] R_{\Psi_n,1,0} l_n \\
&+ \frac{1}{2}[R_{\Psi_n,1,0} D\mathfrak{N}(b_n) R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)] R_{\Psi_n,1,0} l_n \\
&= \frac{1}{2}\int_0^1 R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t)) R^2_{\Psi_n,0,t}(f_n, R_{\Psi_n,1,0} l_n) dt \\
&+ \frac{1}{2}[R_{\Psi_n,1,0} D\mathfrak{N}(b_n) R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)] \times R_{\Psi_n,1,0} R_{\Psi_n,0,1} \\
&\Big[D\mathfrak{N}(a_n)^{-1}\mathfrak{N}(a_n) - 2\big(D\mathfrak{N}(a_n) + R_{\Psi_n,1,0} D\mathfrak{N}(b_n) R_{\Psi_n,0,1}\big)^{-1}\mathfrak{N}(a_n)\Big] \\
&= \frac{1}{2}\int_0^1 R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t)) R^2_{\Psi_n,0,t}(f_n, R_{\Psi_n,1,0} l_n) dt \\
&+ \Big[\frac{1}{2}[R_{\Psi_n,1,0} D\mathfrak{N}(b_n) R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)] - D\mathfrak{N}(a_n)\Big] D\mathfrak{N}(a_n)^{-1}\mathfrak{N}(a_n) \\
&= \frac{1}{2}\int_0^1 R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t)) R^2_{\Psi_n,0,t}(f_n, R_{\Psi_n,1,0} l_n) dt \\
&- \frac{1}{2}\int_0^1 R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t)) R^2_{\alpha,0,t}(f_n, f_n) dt,
\end{aligned}$$

implies that

$$\begin{aligned}
D\mathfrak{N}(b_n) l_n &= \frac{1}{2} R_{\Psi_n,0,1} \int_0^1 R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t)) R^2_{\Psi_n,0,t}(f_n, R_{\Psi_n,1,0} l_n) dt \\
&- \frac{1}{2} R_{\Psi_n,0,1} \int_0^1 R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t)) R^2_{\Psi_n,0,t}(f_n, f_n) dt.
\end{aligned}$$

By Theorem 2.10 and Lemma 3.4, we have

$$\begin{aligned}
R_{\mu,1,0}\mathfrak{N}(a_{n+1}) &= R_{\mu,1,0}\mathfrak{N}(a_{n+1}) - \mathfrak{N}(b_n) - D\mathfrak{N}(b_n) l_n + \mathfrak{N}(b_n) + D\mathfrak{N}(b_n) l_n \\
&= \int_0^1 (1-t) R_{\mu,t,0} D^2\mathfrak{N}(\mu(t)) R^2_{\mu,0,t}(l_n, l_n) dt \\
&+ \frac{1}{2} R_{\Psi_n,0,1} \int_0^1 R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t)) R^2_{\Psi_n,0,t}(f_n, R_{\Psi_n,1,0} l_n) dt \\
&+ R_{\Psi_n,0,1} \int_0^1 (1-t) R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t)) R^2_{\Psi_n,0,t}(f_n, f_n) dt \\
&- \frac{1}{2} R_{\Psi_n,0,1} \int_0^1 R_{\Psi_n,t,0} D^2\mathfrak{N}(\Psi_n(t)) R^2_{\Psi_n,0,t}(f_n, f_n) dt.
\end{aligned}$$

□

Theorem 3.6. *Let $\mathbb{Z}$ be a complete Riemannian manifold, $\boldsymbol{\Omega} \subseteq \mathbb{Z}$ be an open convex set, and $\mathfrak{N} \in \chi(\mathbb{Z})$ satisfies the conditions $(1)-(3)$ with:*

$$0 < x_0 \leq \boldsymbol{b}_0, \ \ \mathcal{D}(x_0) \geq 0, \ \ M\xi A\zeta < \frac{1}{2}, \ V(a_0, M\zeta) \subset \boldsymbol{\Omega},$$

where $\nabla = \frac{1}{\mathcal{A}(x_0)}$ and $M = \frac{2}{(2-x_0)(1-\varsigma\nabla)}$. Then, the method given by (3.1) converges to a singular point a^* of the vector field $\mathfrak{N}$, and the solution a^*, the iterations a_n, b_n belong to $V(a_0, M\zeta)$, and a^* is unique in $V(a_0, M\zeta)$.

Proof. First, we shall prove that the following inequalities hold for $n \geq 1$, by mathematical induction:

$(I)\ \|\Gamma_n\| \leq \mathcal{A}(x_{n-1})\|\Gamma_{n-1}\|$,
$(II)\ d(b_n, a_n) \leq z_{n-1} d(b_{n-1}, a_{n-1})$,
$(III)\ A\|\Gamma_n\| d(b_n, a_n) \leq x_n$,
$(IV)\ d(a_{n+1}, a_n) \leq 2/(2 - x_n) d(b_n, a_n)$,
$(V)\ d(a_{n+1}, b_n) \leq (4 - x_n)/(2 - x_n) d(b_n, a_n)$,
$(VI) \|\Gamma_n\| d(b_n, a_n) \omega(d(b_n, a_n)) \leq y_n$.

Before proving the inequalities $(I) - (VI)$, we will prove some results and later will be used to prove $(I) - (VI)$. As Γ_0 exists. Therefore $A\|\Gamma_0\| d(b_0, a_0) \leq A\xi\zeta = x_0$ and we have

$$\left\| I_{a_0} - \Gamma_0 \frac{R_{\Psi_0,1,0} D\mathfrak{N}(b_0) R_{\Psi_0,0,1} + D\mathfrak{N}(a_0)}{2} \right\| = \left\| \Gamma_0 \frac{D\mathfrak{N}(a_0) - R_{\Psi_0,1,0} D\mathfrak{N}(b_0) R_{\Psi_0,0,1}}{2} \right\|$$
$$\leq \frac{1}{2} A\|\Gamma_0\| d(b_0, a_0) \leq \frac{x_0}{2} < 1.$$

By Banach's lemma, $\left(\frac{R_{\Psi_0,1,0} D\mathfrak{N}(b_0) R_{\Psi_0,0,1} + D\mathfrak{N}(a_0)}{2} \right)^{-1} D\mathfrak{N}(a_0)$ exists and

$$\left\| \left(\frac{R_{\Psi_0,1,0} D\mathfrak{N}(b_0) R_{\Psi_0,0,1} + D\mathfrak{N}(a_0)}{2} \right)^{-1} D\mathfrak{N}(a_0) \right\| \leq \frac{2}{2 - x_0}.$$

Thus, we have

$$d(a_1, a_0) \leq \left\| \left(\frac{R_{\Psi_0,1,0} D\mathfrak{N}(b_0) R_{\Psi_0,0,1} + D\mathfrak{N}(a_0)}{2} \right)^{-1} D\mathfrak{N}(a_0) \right\| \|D\mathfrak{N}(a_0)^{-1} \mathfrak{N}(a_0)\|$$
$$\leq \frac{2}{2 - x_0} d(b_0, a_0)$$

and

$$d(a_1, b_0) \leq d(a_1, a_0) + d(a_0, b_0) \leq \frac{4 - x_0}{2 - x_0} d(b_0, a_0).$$

Also

$$\|\Gamma_0\| d(b_0, a_0) \omega(d(b_0, a_0)) \leq \xi\zeta\omega(\zeta) = y_0.$$

Now, we will prove the inequalities $(I) - (VI)$ for $n \geq 1$. We have

$$\|I_{a_0} - \Gamma_0 R_{\Upsilon_0,1,0} D\mathfrak{N}(a_1) R_{\Upsilon_0,0,1}\| = \|\Gamma_0\| \, \|D\mathfrak{N}(a_0) - R_{\Upsilon_0,1,0} D\mathfrak{N}(a_1) R_{\Upsilon_0,0,1}\| \leq A\|\Gamma_0\| d(a_1, a_0)$$
$$\leq \frac{2}{2 - x_0} A\xi d(b_0, a_0) \leq \frac{2x_0}{2 - x_0} < 1,$$

where Υ_n is the family of minimizing geodesics such that $\Upsilon_n(0) = a_n$, $\Upsilon_n(1) = a_{n+1}$ for each $n = 0, 1, 2 \ldots$. Therefore by Banach's lemma Γ_1 exists and

$$\|R_{\Upsilon_0,1,0} \Gamma_1 R_{\Upsilon_0,0,1}\| = \|\Gamma_1\| \leq \frac{\|\Gamma_0\|}{1 - A\|\Gamma_0\| d(a_1, a_0)} \leq \frac{\|\Gamma_0\|}{1 - 2x_0/(2 - x_0)} = \mathcal{A}(x_0)\|\Gamma_0\|.$$

Also

$$\left\|\left(\int_0^1 R_{\Psi_0,t,0}D^2\mathfrak{N}(\Psi_0(t))R^2_{\Psi_0,0,t}(1-t)dt(f_0,f_0)-\frac{1}{2}\int_0^1 R_{\Psi_0,t,0}D^2\mathfrak{N}(\Psi_0(t))R^2_{\Psi_0,0,t}dt(f_0,f_0)\right)\right\|$$

$$\le\left\|\int_0^1\Big(R_{\Psi_0,t,0}D^2\mathfrak{N}(\Psi_0(t))R^2_{\Psi_0,0,t}-D^2\mathfrak{N}(a_0)\Big)(1-t)dt\right\|d(b_0,a_0)^2$$

$$+\left\|\frac{1}{2}\int_0^1\Big(R_{\Psi_0,t,0}D^2\mathfrak{N}(\Psi_0(t))R^2_{\Psi_0,0,t}-D^2\mathfrak{N}(a_0)\Big)dt\right\|d(b_0,a_0)^2$$

$$\le\int_0^1\omega(td(b_0,a_0))(1-t)dtd(b_0,a_0)^2+\frac{1}{2}\int_0^1\omega(td(b_0,a_0))dtd(b_0,a_0)^2$$

$$\le\int_0^1\mathcal{C}(t)(1-t)dt\omega(d(b_0,a_0))d(b_0,a_0)^2+\frac{1}{2}\int_0^1\mathcal{C}(t)dt\omega(d(b_0,a_0))d(b_0,a_0)^2$$

$$=L\omega(d(b_0,a_0))d(b_0,a_0)^2,$$

where

$$L=\int_0^1\mathcal{C}(t)(1-t)dt+\frac{1}{2}\int_0^1\mathcal{C}(t)dt.$$

Now by Lemma 3.5, we have

$$\begin{aligned}\|R_{\mu,1,0}\mathfrak{N}(a_1)\|\le&\frac{A}{2}d(a_1,b_0)^2+\frac{A}{2}d(a_1,b_0)d(b_0,a_0)+L\omega(d(b_0,a_0))d(b_0,a_0)^2\\ \le&A\frac{(4-x_0)(3-x_0)}{(2-x_0)^2}d(b_0,a_0)^2+L\omega(d(b_0,a_0))d(b_0,a_0)^2,\end{aligned}$$

$$\begin{aligned}\|\Gamma_1R_{\mu,1,0}\mathfrak{N}(a_1)\|\le&\|\Gamma_1\|\|\mathfrak{N}(a_1)\|\\ \le&\mathcal{A}(x_0)\|\Gamma_0\|\left[A\frac{(4-x_0)(3-x_0)}{(2-x_0)^2}d(b_0,a_0)^2+L\omega(d(b_0,a_0))d(b_0,a_0)^2\right]\\ \le&\mathcal{A}(x_0)\left[\frac{x_0(4-x_0)(3-x_0)}{(2-x_0)^2}+Ly_0\right]d(b_0,a_0)\\ =&\mathcal{A}(x_0)\mathcal{A}(x_0,y_0)d(b_0,a_0)=z_0d(b_0,a_0).\end{aligned}$$

Also

$$A\|\Gamma_1\|d(b_1,a_1)\le A\|\Gamma_0\|\mathcal{A}(x_0)z_0d(b_0,a_0)\le x_0\mathcal{A}(x_0)z_0=x_1.$$

Again as Γ_1 exists , therefore

$$\begin{aligned}\left\|I_{a_1}-\Gamma_1\frac{R_{\Psi_1,1,0}D\mathfrak{N}(b_1)R_{\Psi_1,0,1}+D\mathfrak{N}(a_1)}{2}\right\|=&\left\|\Gamma_1\frac{D\mathfrak{N}(a_1)-R_{\Psi_1,1,0}D\mathfrak{N}(b_1)R_{\Psi_1,0,1}}{2}\right\|\\ \le&\frac{1}{2}A\|\Gamma_1\|d(b_1,a_1)\le\frac{x_1}{2}<1.\end{aligned}$$

By Banach's lemma, $\left(\frac{R_{\Psi_1,1,0}D\mathfrak{N}(b_1)R_{\Psi_1,0,1}+D\mathfrak{N}(a_1)}{2}\right)^{-1}D\mathfrak{N}(a_1)$ exists and

$$\left\|\left(\frac{R_{\Psi_1,1,0}D\mathfrak{N}(b_1)R_{\Psi_1,0,1}+D\mathfrak{N}(a_1)}{2}\right)^{-1}D\mathfrak{N}(a_1)\right\|\le\frac{2}{2-x_1}.$$

We have

$$d(a_2,a_1) \leq \left\| \left(\frac{R_{\Psi_1,1,0} D\mathfrak{N}(b_1) R_{\Psi_1,0,1} + D\mathfrak{N}(a_1)}{2} \right)^{-1} D\mathfrak{N}(a_1) \right\| \left\| D\mathfrak{N}(a_1)^{-1}\mathfrak{N}(a_1) \right\|$$
$$\leq \frac{2}{2-x_1} d(b_1,a_1)$$

and

$$d(a_2,b_1) \leq d(a_2,a_1) + d(a_1,b_1) \leq \frac{4-x_1}{2-x_1} d(b_1,a_1).$$

Again

$$\begin{aligned} \|\Gamma_1\| d(b_1,a_1)\omega(d(b_1,a_1)) &\leq \|\Gamma_0\| \mathcal{A}(x_0) z_0 d(b_0,a_0)\omega(z_0 d(b_0,a_0)) \\ &\leq \mathcal{A}(x_0) z_0 \mathcal{C}(z_0) \|\Gamma_0\| d(b_0,a_0))\omega(d(b_0,a_0)) \\ &\leq y_0 \mathcal{A}(x_0) z_0 \mathcal{C}(z_0) = y_1. \end{aligned}$$

Therefore all the inequalities $(I)-(VI)$ for $n=1$ hold. Suppose that all the inequalities hold for $n = 2,3,4,\dots,k$. Then proceeding similarly as above we can show that all the inequalities hold for $n=k+1$. Hence by mathematical induction all the inequalities hold for all n. Now, to prove the sequence $\{a_n\}$ is convergent, it will be sufficient to show that it is Cauchy sequence. For $x_0 = \mathbf{b}_0,\ \mathcal{D}(x_0) = 0$ and $\mathcal{A}(x_0)z_0 = 1$. From (3.2), we have $x_n = x_{n-1} = \cdots = x_0,\ z_n = z_{n-1} = \cdots = z_0$, and $y_n = y_{n-1} = \cdots = y_0 = 0$. From inequality (II), we have

$$d(b_n,a_n) \leq z_{n-1} d(b_{n-1},a_{n-1}) = z_0 d(b_{n-1},a_{n-1}) \leq \cdots \leq z_0^n d(b_0,a_0) = \nabla^n \zeta$$

and

$$d(a_{n+1},a_n) \leq \frac{2}{2-x_n} d(b_n,a_n) \leq \frac{2}{2-x_0} \nabla^n \zeta.$$

Thus, we have

$$\begin{aligned} d(a_{m+n},a_m) \leq & d(a_{m+n},a_{m+n-1}) + \cdots + d(a_{m+1},a_m) \\ \leq & \frac{2}{2-x_0}\left[\nabla^{m+n-1} + \cdots + \nabla^m\right]\zeta = \frac{2\nabla^m}{2-x_0}\left(\frac{1-\nabla^n}{1-\nabla}\right)\zeta. \end{aligned} \tag{3.6}$$

From (3.6), $\{a_n\}$ is a Cauchy sequence as $\nabla < 1$. Let $0 < x_0 < \mathbf{b}_0$ and $\mathcal{D}(x_0) > y_0$. Now from the inequalities $(I)-(VI)$ and Lemma 3.3(2), for $n \geq 1$, we have

$$d(b_n,a_n) \leq z_{n-1} d(b_{n-1},a_{n-1}) \leq \cdots \leq \prod_{i=0}^{n-1}(z_i) d(b_0,a_0) < \prod_{i=0}^{n-1}(\varsigma^{2^i}\nabla)\zeta = \varsigma^{2^n-1}\nabla^n\zeta,$$

where $\varsigma = x_1/x_0 < 1$ and $\nabla = 1/\mathcal{A}(x_0) < 1$. We obtain that

$$\begin{aligned} d(a_{n+m},a_m) \leq & d(a_{n+m},a_{m+n-1}) + \cdots + d(a_{m+1},a_m) \\ \leq & \frac{2}{2-x_{m+n-1}} d(b_{n+m-1},a_{m+n-1}) + \cdots + \frac{2}{2-x_m} d(b_m,a_m) \\ < & \frac{2}{2-x_{m+n-1}} \varsigma^{2^{m+n-1}-1}\nabla^{m+n-1}\zeta + \cdots + \frac{2}{2-x_m}\varsigma^{2^m-1}\nabla^m\zeta \\ < & \frac{2\nabla^m}{2-x_m}\left[\varsigma^{2^{m+n-1}-1}\nabla^{n-1} + \cdots + \varsigma^{2^m-1}\right]\zeta \\ < & \frac{2\varsigma^{2^m-1}\nabla^m}{2-\varsigma^{2^m-1}x_0}\left[\varsigma^{2^m[2^{n-1}-1]}\nabla^{n-1} + \cdots + \varsigma^{2^m[2-1]}\nabla + 1\right]\zeta. \end{aligned}$$

By Bernoulli's inequality, we have

$$d(a_{n+m}, a_m) < \frac{2\varsigma^{2^m-1}\nabla^m}{2-\varsigma^{2^m-1}x_0} \times \frac{1-\varsigma^{2^m n}\nabla^n}{1-\varsigma^{2^m}\nabla}\zeta. \tag{3.7}$$

Put $m = 0$ in (3.7), we have

$$d(a_n, a_0) < \frac{2}{2-x_0} \times \frac{1-\varsigma^n\nabla^n}{1-\varsigma\nabla}\zeta < M\zeta. \tag{3.8}$$

Therefore $a_n \in V(a_0, M\zeta)$. Also $b_n \in V(a_0, M\zeta)$, as

$$\begin{aligned} d(b_{n+1}, a_0) \leq & d(b_{n+1}, a_{n+1}) + d(a_{n+1}, a_n) + \cdots + d(a_1, a_0) \\ \leq & d(b_{n+1}, a_{n+1}) + \frac{2}{2-x_n} d(b_n, a_n) + \cdots + \frac{2}{2-x_0} d(b_0, a_0) \\ < & \frac{2}{2-x_{n+1}} d(b_{n+1}, a_{n+1}) + \cdots + \frac{2}{2-x_0} d(b_0, a_0) \\ < & \cdots < \frac{2}{2-x_0} \frac{1-\varsigma^{n+1}\nabla^{n+1}}{1-\varsigma\nabla}\zeta < M\zeta. \end{aligned}$$

Taking $n \to \infty$ in (3.6) or (3.8), we get $a^* \in V(a_0, M\zeta)$. Now to prove a^* is a singular point of $\mathfrak{N}(a) = 0$, we find the bounds of $\|D\mathfrak{N}(a_n)\|$. For this let δ be a minimizing geodesic such that $\delta(0) = a_0,\ \delta(1) = a_n$, and $\|\delta'(0)\| = d(a_n, a_0)$, then

$$\begin{aligned} \|D\mathfrak{N}(a_n)\| = & \|R_{\delta,1,0} D\mathfrak{N}(a_n) R_{\delta,0,1} + D\mathfrak{N}(a_0) - D\mathfrak{N}(a_0)\| \\ \leq & \|R_{\delta,1,0} D\mathfrak{N}(a_n) R_{\delta,0,1} - D\mathfrak{N}(a_0)\| + \|D\mathfrak{N}(a_0)\| \\ \leq & \|D\mathfrak{N}(a_0)\| + AM\zeta. \end{aligned}$$

From (3.1), we have

$$\|\mathfrak{N}(a_n)\| \leq \|D\mathfrak{N}(a_n)\| d(b_n, a_n).$$

Since $\|D\mathfrak{N}(a_n)\|$ is bounded and $d(b_n, a_n) \to 0$, when $n \to \infty$, we obtain that $\|\mathfrak{N}(a^*)\| \leq 0$, thus $\mathfrak{N}(a^*) = 0$. Now, we will prove the singularity is unique. Before that we will find the bounds of $\|\Gamma_n\|$. We have

$$\|R_{\delta,1,0} D\mathfrak{N}(a_n) R_{\delta,0,1} - D\mathfrak{N}(a_0)\| \leq A \int_0^1 \|\delta'(0))\| ds = Ad(a_n, a_0) < AM\zeta$$

and

$$\|\Gamma_0\| \|R_{\delta,1,0} D\mathfrak{N}(a_n) R_{\delta,0,1} - D\mathfrak{N}(a_0)\| \leq \xi AM\zeta < 1.$$

So that $R_{\delta,1,0} D\mathfrak{N}(a_n) R_{\delta,0,1}$ is invertible by Banach's lemma and

$$\begin{aligned} \|\Gamma_n\| = & \|R_{\delta,1,0} \Gamma_n R_{\delta,0,1}\| \\ \leq & \frac{\|\Gamma_0\|}{1 - \|\Gamma_0\| \|R_{\delta,1,0} D\mathfrak{N}(a_n) R_{\delta,0,1} - D\mathfrak{N}(a_0)\|} \\ \leq & \frac{\xi}{1 - A\xi M\zeta}. \end{aligned}$$

Now, let a^{**} be the another singularity of $\mathfrak{N}$ in $V(a_0, M\zeta)$ and let $\vartheta : [0, 1] \to \mathbb{Z}$ be the minimizing geodesic such that $\vartheta(0) = a^*$ and $\vartheta(1) = a^{**}$. Then, we have

$$\|R_{\vartheta,t,0} D\mathfrak{N}(\vartheta(t)) R_{\vartheta,0,t} - D\mathfrak{N}(a^*)\| \leq A \int_0^t \|\vartheta'(0)\| ds$$

$$= Atd(a^*, a^{**}) \leq At\Big(d(a_0, a^*) + d(a_0, a^{**})\Big).$$

Hence

$$\begin{aligned}&\|D\mathfrak{N}(a^*)^{-1}\|\int_0^1\|R_{\vartheta,t,0}D\mathfrak{N}(\vartheta(t))R_{\vartheta,0,t}dt-D\mathfrak{N}(a^*)\|dt\\ \leq&\frac{\xi}{(1-A\xi M\zeta)}\int_0^1 At\Big(d(a_0,a^*)+d(a_0,a^{**})\Big)dt\\ \leq&\frac{\xi}{(1-A\xi M\zeta)}\frac{A}{2}(M\zeta+M\zeta)<1.\end{aligned}$$

By Banach's lemma, the operator $\int_0^1 R_{\vartheta,t,0}D\mathfrak{N}(\vartheta(t))R_{\vartheta,0,t}dt$ is invertible and we have

$$0=R_{\vartheta,1,0}\mathfrak{N}(a^{**})-\mathfrak{N}(a^*)=\int_0^1 R_{\vartheta,t,0}D\mathfrak{N}(\vartheta(t))R_{\vartheta,0,t}(\vartheta'(0))dt.$$

Therefore $\vartheta'(0)=0$. As $0=\|\vartheta'(0)\|=d(a^*,a^{**})$, we get $a^*=a^{**}$. Hence the proof is complete. □

4 The Order of convergence of the Newton-like method in Riemannian manifolds

In this section, we will study the order of convergence of the Newton-like method in Riemannian manifolds.

Definition 4.1. ([8]) Let $\mathbb{Z}$ be a complete Riemannian manifold and $\{a_n\}$ be a sequence in $\mathbb{Z}$ converges to a^*. If there is a chart (U,h) of a^* and constants $l>0$ and $\mathfrak{T}\geq 0$ such that

$$\|h^{-1}(a_{n+1})-h^{-1}(a^*)\|\leq\mathfrak{T}\|h^{-1}(a_n)-h^{-1}(a^*)\|^l \tag{4.1}$$

holds for all sufficiently large n, we say that $\{a_n\}$ converges to a^* with order at least l.

Remark 4.2. The above definition do not depend on the choice of the chart i.e. if (Y,z) is another chart of a^*, then (4.1) holds changing h by z and constant $\mathfrak{T}$ by $\mathfrak{K}$ [10]. So that, we can assume U is a normal neighborhood of each of its points, see Theorem 3.7 in [9]. Since in a totally normal neighborhood U of a^*,

$$d(a,b)=\|\exp_{a_n}^{-1}(a)-\exp_{a_n}^{-1}(b)\|, \tag{4.2}$$

for all $a,b\in U$ and for all sufficiently large n, we can rewrite (4.1) as

$$d(a_{n+1},a^*)\leq\mathfrak{T}d(a_n,a^*)^l.$$

Now, we will prove the order of convergence of the Newton-like method in Riemannian manifolds. Before that, we will prove the order of convergence of the iterative method defined by

$$\left.\begin{aligned}g_n&=-\left(\frac{D\mathfrak{N}(a_n)+R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1}}{2}\right)^{-1}\mathfrak{N}(a_n),\\ a_{n+1}&=\exp_{a_n}(g_n),\quad\text{for each } n=0,1,2,\dots\end{aligned}\right\} \tag{4.3}$$

.For this, we will prove a lemma.

Lemma 4.3. *Let $\mathbb{Z}$ be a complete Riemannian manifold, $\Omega\subseteq\mathbb{Z}$ be an open convex set, $\mathfrak{N}\in\chi(\mathbb{Z})$ and $\alpha(t)=\exp_{a_n}(tv)$.*
Then,

$$R_{\alpha,t,0}\mathfrak{N}(\alpha(t))=\mathfrak{N}(a_n)+t\left(\frac{R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1}+D\mathfrak{N}(a_n)}{2}\right)v+R(t)$$

with

$$\|R(t)\|\leq\frac{A}{2}\big(t\|v\|+d(a_n,b_n)\big)t\|v\|.$$

Proof. From (2.8), we have

$$R_{\alpha,t,0}\mathfrak{N}(\alpha(t)) = \mathfrak{N}(a_n) + \int_0^t R_{\alpha,s,0}D\mathfrak{N}(\alpha(s))R_{\alpha,0,s}(v)ds.$$

Thus

$$R_{\alpha,t,0}\mathfrak{N}(\alpha(t)) - \mathfrak{N}(a_n) - t\left(\frac{R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)}{2}\right)v$$
$$= \int_0^t \left(R_{\alpha,s,0}D\mathfrak{N}(\alpha(s))R_{\alpha,0,s}(v) - \left(\frac{R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)}{2}\right)v\right)ds,$$

letting

$$R(t) = \int_0^t \left(R_{\alpha,s,0}D\mathfrak{N}(\alpha(s))R_{\alpha,0,s}(v) - \left(\frac{R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)}{2}\right)v\right)ds.$$

We obtain

$$\|R(t)\| \le \frac{1}{2}\int_0^t \|2R_{\alpha,s,0}D\mathfrak{N}(\alpha(s))R_{\alpha,0,s} - D\mathfrak{N}(a_n) - R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1}\|\|v\|ds$$
$$=\frac{1}{2}\int_0^t \|2R_{\alpha,s,0}D\mathfrak{N}(\alpha(s))R_{\alpha,0,s} - 2D\mathfrak{N}(a_n) + D\mathfrak{N}(a_n) - R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1}\|\|v\|ds$$
$$\le\frac{A}{2}\big(t^2\|v\| + td(a_n,b_n)\big)\|v\|.$$

Therefore

$$\|R(t)\| \le \frac{A}{2}\big(t\|v\| + d(a_n,b_n)\big)t\|v\|.$$

□

Theorem 4.4. *(Order of convergence) The iterative method given in* (4.3) *is of order one.*

Proof. By Lemma 4.3, if α is a minimizing geodesic joining a_n to a^* defined by

$$\alpha(t) = \exp_{a_n}(tv_n),$$

where $v_n \in T_{a_n}\mathbb{Z}$ and $d(a_n, a^*) = \|v_n\|$. Then,

$$R_{\alpha,t,0}\mathfrak{N}(a^*) = \mathfrak{N}(a_n) + \left(\frac{R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)}{2}\right)v_n + R(1) \tag{4.4}$$

with

$$\|R(1)\| \le \frac{A}{2}\big(\|v_n\| + d(a_n,b_n)\big)\|v_n\|.$$

Therefore, from (4.4), we have

$$\left(\frac{R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)}{2}\right)^{-1}\mathfrak{N}(a_n) + v_n + \left(\frac{R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)}{2}\right)^{-1}R(1)$$
$$= 0.$$

Since

$$-\left(\frac{R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)}{2}\right)^{-1}\mathfrak{N}(a_n) = \exp_{a_n}^{-1}(a_{n+1}) \text{ and } v_n = \exp_{a_n}^{-1}(a^*).$$

We have

$$\exp_{a_n}^{-1}(a_{n+1}) - \exp_{a_n}^{-1}(a^*) = \left(\frac{R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)}{2}\right)^{-1} R(1),$$

we conclude that

$$\begin{aligned}
d(a_{n+1}, a^*) &= \| \exp_{a_n}^{-1}(a_{n+1}) - \exp_{a_n}^{-1}(a^*)\| \\
&= \left\| \left(\frac{R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)}{2}\right)^{-1} R(1)\right\| \\
&\leq \left\| \left(\frac{R_{\Psi_n,1,0}D\mathfrak{N}(b_n)R_{\Psi_n,0,1} + D\mathfrak{N}(a_n)}{2}\right)^{-1}\right\| \|R(1)\| \\
&\leq \frac{2A}{4-2a_n}\big(\|v_n\| + d(a_n, b_n)\big)\|v_n\| \\
&\leq \frac{2A}{4-2a_n}\|\Gamma_n\|\big(d(a_n, a^*) + d(a_n, b_n)\big)d(a_n, a^*) \\
&= \frac{2A}{4-2a_n}\|\Gamma_n\|\left(\frac{d(a_n, a^*)}{d(a_n, b_n)} + 1\right)d(a_n, b_n)d(a_n, a^*).
\end{aligned}$$

If n is sufficiently large, then $d(a_n, a^*) \leq d(a_n, b_n)$, and therefore

$$\left(\frac{d(a_n, a^*)}{d(a_n, b_n)} + 1\right) \leq 2$$

and then, for b_n sufficiently close to a^*,

$$d(a_{n+1}, a^*) \leq N_0 d(a_n, b_n)d(a_n, a^*),$$

with $N_0 \leq \frac{4A}{4-2a_n}\|\Gamma_n\|$. □

Remark 4.5. If we fix a_j sufficiently close to a^*, then, the calculations made in the Theorem 4.4 become in

$$d(a_{n+1}, a^*) \leq N_j d(a_j, a^*)d(a_n, a^*),$$

with $N_j \leq \frac{4A}{4-2a_n}\|\Gamma_n\|$. Thus,

$$d(a_{n+1}, a^*) \leq N d(a_j, a^*)d(a_n, a^*), \tag{4.5}$$

with $N \leq \frac{4A}{4-2a_n}\|\Gamma_n\|$.

Now, we can prove the order of convergence of the Newton-like method in Riemannian manifolds.

Theorem 4.6. *Under the hypotheses of Theorem 3.6, the method described in* (3.1) *converges with order of convergence* 3.

Proof. Since

$$d(a_{n+1}, a_n) \leq d(a_{n+1}, b_n) + d(b_n, a_n).$$

Now, from Theorem 3.6 the sequence $\{a_n\}$ converges to a^*. Also, by using the Lemma 28 (i) (see [8]) and (4.5), we have

$$d(a_{n+1}, a^*) \leq N d(a_n, a^*)d(b_n, a^*) \leq N d(a_n, a^*)C d(a_n, a^*)^2 = NC d(a_n, a^*)^3.$$

□

5 Numerical examples

In this section, two numerical examples are given to show the effectiveness of our results.

Example 5.1. Consider the vector field $W : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$W\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_2 \\ a_1 - a_1 a_3^2 \\ a_1 a_2 a_3 \end{pmatrix}$$

with the Frobenius norm and let $\mathfrak{N} = W|_{\boldsymbol{S}^2}$ be a vector field on $\boldsymbol{S}^2 = \{a_1, a_2, a_3 \in \mathbb{R} : a_1^2 + a_2^2 + a_3^2 = 5\}$. Then it can be easily verified that

$$W|_{\boldsymbol{S}^2}(a) \in T_a\boldsymbol{S}^2 \quad \forall\, a \in \boldsymbol{S}^2.$$

Now, the derivative $D\mathfrak{N}$ using the method given in [6] of $\mathfrak{N}$ in the tangent plane of $\boldsymbol{S}^2$ is given by

$$\begin{pmatrix} -a_1a_2(a_1^2+1) & -1-a_1^2(a_2^2+a_3^2-1) \\ 1-a_2^2-a_1^2(-2+a_2^2)-a_3^2 & -a_1a_2(a_2^2+a_3^2-3) \end{pmatrix}.$$

Next, by using the method of Lagrange's multipliers, we get

$$A = \sup\{D\mathfrak{N}(a_1, a_2, a_3) : a_1^2 + a_2^2 + a_3^2 = 5\} = 11$$

is a Lipschitz constant of $D\mathfrak{N}$. Initially for $a_0 = (2, -0.0013091, 1)^T$, we get $\|\Gamma_0\| = 1.00779 = \xi$, $\|\Gamma_0\mathfrak{N}(a_0)\| = 0.0013192979 = \zeta$. Hence all the assumptions for convergence are satisfied. By the Newton-like method on $\mathbf{S}^2$, we get the solution. The results are in the Table 1, which shows that $\{a_i\}$ converges to the singularity $(2, 0, 1)^T$.

Table 1. Results of the Newton-like method on $\mathbf{S}^2$:

Iterations	a_i	$\|\|\mathfrak{N}(a_i)\|\|$	$d(a_{i+1}, a_i)$
0	$\begin{pmatrix} 2 \\ -1.309100e-03 \\ 1 \end{pmatrix}$	2.927237e-03	0
1	$\begin{pmatrix} 1.999999e+00 \\ 6.356511e-09 \\ 9.999991e-01 \end{pmatrix}$	2.927237e-03	1.309107e-03
2	$\begin{pmatrix} 1.999999e+00 \\ -8.170206e-15 \\ 1.000000e+00 \end{pmatrix}$	3.427543e-06	9.580408e-07

Table 2. Results of the Newton's method on $\mathbf{S}^2$:

Iterations	a_i	$\|\|\mathfrak{N}(a_i)\|\|$	$d(a_{i+1}, a_i)$
0	$\begin{pmatrix} 2 \\ -1.309100e-03 \\ 1 \end{pmatrix}$	2.927237e-03	0
1	$\begin{pmatrix} 1.999999e+00 \\ 2.991289e-09 \\ 9.999991e-01 \end{pmatrix}$	2.927237e-03	1.309104e-03
2	$\begin{pmatrix} 1.999999e+00 \\ -7.689527e-15 \\ 1.000000e+00 \end{pmatrix}$	3.427503e-06	9.580201e-07

Example 5.2. Consider the vector field $\mathfrak{N} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathfrak{N}(a) = \mathfrak{N}(a_1, a_2)^T = \left(\frac{\cos a_1 + 20a_1}{20}, a_2\right)^T \tag{5.1}$$

with the max norm $\|.\| = \|.\|_\infty$. The first and second Fréchet derivatives of $\mathfrak{N}$ are respectively:

$$D\mathfrak{N}(a) = \begin{bmatrix} \frac{-\sin a_1 + 20}{20} & 0 \\ 0 & 1 \end{bmatrix},$$

$$D^2\mathfrak{N}(a) = \left[\begin{array}{cc|cc} \frac{-\cos a_1}{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Initially for $a_0 = (0,0)^T$, we have

$$\|D\mathfrak{N}(a_0)^{-1}\| = 1 = \xi, \quad \|D\mathfrak{N}(a_0)^{-1}\mathfrak{N}(a_0)\| = 0.05 = \zeta,$$

$$\|D^2\mathfrak{N}(a)\| = \max\left(\left|\frac{-\cos a_1}{20}\right|, 0\right) = \left|\frac{\cos a_1}{20}\right| \leq A = \frac{1}{20}.$$

Hence all the assumptions for convergence are satisfied. By the Newton-like method on $\mathbb{R}^2$, we get the solution. The results are in the Table 3, which shows that $\{a_i\}$ converges to the singularity $(-0.04994, 0)^T$.

Table 3. Results of the Newton-like method on $\mathbb{R}^2$:

Iterations	a_i	$\|\|\mathfrak{N}(a_i)\|\|$	$d(a_{i+1}, a_i)$
0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	5.000000e-02	0
1	$\begin{pmatrix} -4.993760e-02 \\ 0 \end{pmatrix}$	5.000000e-02	4.993760e-02
2	$\begin{pmatrix} -4.993767e-02 \\ 0 \end{pmatrix}$	6.484663e-08	6.468519e-08

Table 4. Results of the Newton's method on $\mathbb{R}^2$:

Iterations	a_i	$\|\|\mathfrak{N}(a_i)\|\|$	$d(a_{i+1}, a_i)$
0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	5.000000e-02	0
1	$\begin{pmatrix} -5.000000e-02 \\ 0 \end{pmatrix}$	5.000000e-02	5.000000e-02
2	$\begin{pmatrix} -4.993767e-02 \\ 0 \end{pmatrix}$	6.248698e-05	6.233122e-05
3	$\begin{pmatrix} -4.993767e-02 \\ 0 \end{pmatrix}$	9.700823e-11	9.676672e-11

6 Conclusion

In this paper, we have extended the Newton-like method from Banach space to Riemannian manifolds to find the singular point of a vector field and studied the convergence analysis of the Newton-like method under ω-continuity condition on the second order covariant derivative of vector field. Also, we have derived the order of convergence using normal coordinates. Finally, two numerical examples are provided.

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