Palestine Journal of Mathematics

Vol 14(3)(2025) , 651–658 © Palestine Polytechnic University-PPU 2025

SOME IDENTITIES INVOLVING ODD HOMOGENEOUS MULTIPLE HARMONIC NUMBERS AND OTHER RELATED SUMS

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Communicated by Ayman Badawi

MSC 2020 Classifications: 11Bxx, 11B65, 11M32.

Keywords and phrases: Odd multiple harmonic sums, Multiple harmonic sums, Multiple zeta values, Binomial coefficient.

The author would like to thank the reviewers and the editor for their constructive comments and valuable suggestions, which have improved the quality of this paper.

Abstract. In this paper, we study a certain type of polynomials that yield new identities for (odd) homogeneous multiple harmonic numbers and other related sums.

1 Notation and preliminaries

The generalized harmonic number of order n of m is defined by

$$H_m(n) = \sum_{i=1}^{m} \frac{1}{i^n}.$$

For $(n_1, \dots, n_k) \in \mathbb{N}^k$ we define the multiple harmonic sum (number) and the multiple harmonic star sum (number), respectively, by

$$H_m(n_1, \dots, n_k) := \sum_{1 \le j_1 < j_2 < \cdots < j_k \le m} \frac{1}{j_1^{n_1} \dots j_k^{n_k}},$$

and

$$H_m^{\star}(n_1, \dots, n_k) := \sum_{1 \le j_1 \le j_2 \le \cdots \le j_k \le m} \frac{1}{j_1^{n_1} \dots j_k^{n_k}}.$$

When $n_1 = n_2 = \cdots = n_k = n$ we call these sums homogeneous multiple harmonic sums and homogeneous multiple harmonic star sums and denote them by

$$H_m(\{n\}^k) := \sum_{1 \le j_1 < j_2 < \cdots < j_k \le m} \frac{1}{j_1^{n} \cdots j_k^{n}}, \qquad H_m^{\star}(\{n\}^k) := \sum_{1 \le j_1 \le j_2 \le \cdots \le j_k \le m} \frac{1}{j_1^{n} \cdots j_k^{n}}.$$

When $k = 1$ we omit the parentheses and find the generalized harmonic numbers.

If a bar is placed over some n_r, this indicates the insertion of a factor $(-1)^{j_r}$ in the summand. For example,

$$H_m(\bar{n}_1, n_2, \dots, \bar{n}_k) := \sum_{1 \le j_1 < j_2 < \cdots < j_k \le m} \frac{(-1)^{j_1 + j_k}}{j_1^{n_1} \cdots j_k^{n_k}}.$$

The odd multiple harmonic sum and the odd multiple harmonic star sum are defined as follow (other convention are also being used, see for example [2]. Here we adopt a simpler description)

$$O_m(n_1, n_2, \dots, n_k) := \sum_{\substack{1 \le j_1 < j_2 < \cdots < j_k \le m \\ j_i \text{ odd}}} \frac{1}{j_1^{n_1} j_2^{n_2} \cdots j_k^{n_k}},$$

$$O_m^{\star}(n_1, n_2, \dots, n_k) := \sum_{\substack{1 \le j_1 \le j_2 \le \cdots \le j_k \le m \\ j_i \text{ odd}}} \frac{1}{j_1^{n_1} j_2^{n_2} \cdots j_k^{n_k}}.$$

In the homogeneous case we set

$$O_m(\{n\}^k) := \sum_{\substack{1 \le j_1 < j_2 < \cdots < j_k \le m \\ j_i \text{ odd}}} \frac{1}{j_1^{n} \cdots j_k^{n}}, \qquad O_m^{\star}(\{n\}^k) := \sum_{\substack{1 \le j_1 \le j_2 \le \cdots \le j_k \le m \\ j_i \text{ odd}}} \frac{1}{j_1^{n} \cdots j_k^{n}}.$$

For $n_1 \ge 2$, the limits

$$\zeta(n_1, \dots, n_k) := \lim_{m \to \infty} H_m(n_1, \dots, n_k), \qquad \zeta^{\star}(n_1, \dots, n_k) := \lim_{m \to \infty} H_m^{\star}(n_1, \dots, n_k)$$

exist. These are the well-known *multiple zeta values* (MZVs) and *multiple zeta star values* (MZSVs); see for instance [3, 4, 11] for further properties and results.

Likewise, we define the odd multiple zeta values and odd multiple zeta star values by

$$O(n_1,\ldots,n_k) := \lim_{m\to\infty} O_m(n_1,\ldots,n_k), \qquad O^\star(n_1,\ldots,n_k) := \lim_{m\to\infty} O^\star_m(n_1,\ldots,n_k).$$

The identities for multiple harmonic sums established in this paper can be converted into identities for multiple zeta values by taking limits. For instance, applying Theorem 2.4, Eq(2.8), we obtain

$$O(\{n\}^k) = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{2^{n(k-j)}}\,\zeta(\{n\}^j)\,\zeta^\star(\{n\}^{k-j}), \tag{1.1}$$

and from Eq(3.6) we deduce

$$\begin{aligned}\zeta(3n,3n) &= 3\Big[\zeta(\{n\}^3)^2 + \zeta(\{n\}^6) + \zeta(n)^2\zeta(\{n\}^4)\Big] + \zeta(\{n\}^2)^3\\ &\quad - 3\Big[\zeta(n)\zeta(\{n\}^2)\zeta(\{n\}^3) + \zeta(\{n\}^2)\zeta(\{n\}^4) + \zeta(n)\zeta(\{n\}^5)\Big].\end{aligned} \tag{1.2}$$

Let $X_1, X_2, \ldots, X_m$ be m variables. The k-th elementary symmetric polynomial and the complete homogeneous symmetric polynomial are defined by

$$e_k(X_1,\ldots,X_m) = \sum_{1\le j_1<j_2<\cdots<j_k\le m} X_{j_1}\cdots X_{j_k},$$

$$h_k(X_1,\ldots,X_m) = \sum_{1\le j_1\le j_2\le\cdots\le j_k\le m} X_{j_1}\cdots X_{j_k}.$$

These polynomials are characterized by the generating function identities

$$\prod_{i=1}^{m}(1-X_i t) = \sum_{k=0}^{m}(-1)^k e_k(X_1,\ldots,X_m)\,t^k, \qquad \prod_{i=1}^{m}\frac{1}{1-X_i t} = \sum_{k\ge 0} h_k(X_1,\ldots,X_m)\,t^k.$$

We also define the k-th power sum symmetric polynomial by

$$s_k(X_1,\ldots,X_m) = \sum_{i=1}^{m} X_i^k.$$

For a partition $\lambda = (\lambda_1,\ldots,\lambda_\ell)$, the monomial symmetric function is defined as

$$m_\lambda(X_1,\ldots,X_m) = \sum_{\alpha} X_1^{\alpha_1}\cdots X_m^{\alpha_m},$$

where the sum runs over all distinct rearrangements $\alpha = (\alpha_1,\ldots,\alpha_m)$ of the sequence $(\lambda_1,\ldots,\lambda_\ell,0,\ldots,0)$ of length m (see [1] for further details).

When m is a non-negative integer, it follows directly from the above definitions that

$$e_k\left(1,\frac{1}{2^n},\ldots,\frac{1}{m^n}\right) = \sum_{1\le j_1<j_2<\cdots<j_k\le m}\frac{1}{j_1^n\cdots j_k^n} = H_m(\{n\}^k),$$

$$s_k\left(1,\frac{1}{2^n},\ldots,\frac{1}{m^n}\right) = \sum_{i=1}^{m}\frac{1}{i^{nk}} = H_m(nk),$$

$$e_k\left(1,\frac{1}{3^n},\ldots,\frac{1}{(m')^n}\right) = \sum_{\substack{1\le j_1<j_2<\cdots<j_k\le m\\ j_i \text{ odd}}}\frac{1}{j_1^n\cdots j_k^n} = O_m(\{n\}^k),$$

where $m' = m$ if m is odd, and $m' = m-1$ if m is even.

For the star analogues we have

$$h_k\left(1,\frac{1}{2^n},\ldots,\frac{1}{m^n}\right) = \sum_{1\le j_1\le j_2\le\cdots\le j_k\le m}\frac{1}{j_1^n\cdots j_k^n} = H^\star_m(\{n\}^k),$$

$$h_k\left(1,\frac{1}{3^n},\ldots,\frac{1}{(m')^n}\right) = \sum_{\substack{1\le j_1\le j_2\le\cdots\le j_k\le m\\ j_i \text{ odd}}}\frac{1}{j_1^n\cdots j_k^n} = O^\star_m(\{n\}^k),$$

where m' is equal to m or $m-1$ depending on the parity of m.

2 Identities for (odd) homogeneous multiple harmonic sums

The identities in this paper are expressed in terms of a particular product of binomial coefficients defined over a field $\mathbb{K}$. For any $x \in \mathbb{K}$ and integer k, we set

$$\binom{x}{k} = \begin{cases}\dfrac{x(x-1)\cdots(x-k+1)}{k!}, & k\ge 0,\\[2mm] 0, & k<0.\end{cases}$$

For convenience, we define the even and odd parts of the binomial coefficient $\binom{x-1}{m}$ by

$$\binom{x-1}{m}^e = \prod_{\substack{1\le j\le m\\ j \text{ even}}} \frac{x-j}{j}, \qquad \binom{x-1}{m}^o = \prod_{\substack{1\le j\le m\\ j \text{ odd}}} \frac{x-j}{j}.$$

It follows immediately that

$$\binom{x-1}{m} = \binom{x-1}{m}^e \binom{x-1}{m}^o.$$

Lemma 2.1. *For all integers $m \ge 0$, We have the following decomposition:*

$$\binom{x-1}{m} = \binom{x-1}{m}^e \binom{x-1}{m}^o = \binom{\frac{x}{2}-1}{\lfloor \frac{m}{2} \rfloor} \binom{x-1}{m}^o.$$

Proof. It suffices to show that

$$\binom{x-1}{m}^e = \binom{\frac{x}{2}-1}{\lfloor \frac{m}{2} \rfloor}.$$

We consider two cases.

Case 1: m even. If $m \ge 0$ is even, then

$$\binom{x-1}{m}^e = \frac{(x-2)(x-4)\cdots(x-m)}{2\cdot 4\cdots m} = \frac{2^{\frac{m}{2}}\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)\cdots\left(\frac{x}{2}-\frac{m}{2}\right)}{2^{\frac{m}{2}}\left(1\cdot 2\cdots \frac{m}{2}\right)} = \binom{\frac{x}{2}-1}{\frac{m}{2}}.$$

Since $\lfloor \frac{m}{2} \rfloor = \frac{m}{2}$, the claim follows.

Case 2: m odd. If $m \ge 1$ is odd, then

$$\binom{x-1}{m}^e = \frac{(x-2)(x-4)\cdots(x-(m-1))}{2\cdot 4\cdots (m-1)} = \frac{2^{\frac{m-1}{2}}\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)\cdots\left(\frac{x}{2}-\frac{m-1}{2}\right)}{2^{\frac{m-1}{2}}\left(1\cdot 2\cdots \frac{m-1}{2}\right)} = \binom{\frac{x}{2}-1}{\frac{m-1}{2}}.$$

Since $\lfloor \frac{m}{2} \rfloor = \frac{m-1}{2}$, the claim follows. $\square$

Theorem 2.2. *Let $\mathbb{K}$ be any extension of $\mathbb{Q}$ that contains the n-th roots of unity. Then, for all $x \in \mathbb{K}$, we have*

$$\prod_{\omega^n=1} \binom{\omega x-1}{m} =(-1)^{mn} \sum_{k=0}^{m} (-1)^k H_m(\{n\}^k) x^{kn}, \tag{2.1}$$

$$\frac{1}{\prod_{\omega^n=1}\binom{\omega x-1}{m}} =(-1)^{mn} \sum_{k\ge 0} H_m^\star(\{n\}^k) x^{kn}. \tag{2.2}$$

Proof. This is a slight modification of the proof given in [5, Theorem 2]. Replacing the polynomial $F_n(x)$ used there by

$$F_{n,m}(x) = \frac{(x^n-1^n)(x^n-2^n)\cdots(x^n-m^n)}{1^n 2^n \cdots m^n},$$

we can establish that

$$F_{n,m}(x) =(-1)^{m(n-1)} \prod_{\omega^n=1} \binom{\omega x-1}{m} = (-1)^m \sum_{k=0}^{m} (-1)^k H_m(\{n\}^k) x^{kn}, \tag{2.3}$$

$$\frac{1}{F_{n,m}(x)} =(-1)^{m(n-1)} \frac{1}{\prod_{\omega^n=1}\binom{\omega x-1}{m}} = (-1)^m \sum_{k\ge 0} H_m^\star(\{n\}^k) x^{kn}. \tag{2.4}$$

$\square$

Theorem 2.3. *Let m be a non-negative integer, then*

$$\prod_{\omega^n=1} \binom{\omega x-1}{m}^o =(-1)^{n\lfloor \frac{m+1}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} (-1)^k O_m(\{n\}^k) x^{kn}, \tag{2.5}$$

$$\frac{1}{\prod_{\omega^n=1}\binom{\omega x-1}{m}^o} =(-1)^{n\lfloor \frac{m+1}{2} \rfloor} \sum_{k\ge 0} O_m^\star(\{n\}^k) x^{kn}. \tag{2.6}$$

Proof. We first consider the case when m is even. Define

$$F^o_{n,m}(x) = \frac{(x^n-1^n)(x^n-3^n)\cdots(x^n-(m-1)^n)}{1^n 3^n \cdots (m-1)^n}.$$

Let $\{1, \omega_2, \dots, \omega_n\}$ denote the n-th roots of unity. Then the polynomial $F_n(x)$ can be written as

$$\begin{aligned} F^o_{n,m}(x) &= \prod_{\substack{j=1\\ j \text{ odd}}}^{m} \frac{x-j}{m!!} \prod_{\substack{j=1\\ j \text{ odd}}}^{m} \frac{x-j\omega_2}{m!!} \cdots \prod_{\substack{j=1\\ j \text{ odd}}}^{m} \frac{x-j\omega_n}{m!!} \\ &= \left(1^{\frac{m}{2}} \omega_2^{\frac{m}{2}} \cdots \omega_n^{\frac{m}{2}}\right) \prod_{\substack{j=1\\ j \text{ odd}}}^{m} \frac{x-j}{m!!} \prod_{\substack{j=1\\ j \text{ odd}}}^{m} \frac{\omega_2^{-1}x-j}{m!!} \cdots \prod_{\substack{j=1\\ j \text{ odd}}}^{m} \frac{\omega_n^{-1}x-j}{m!!}, \end{aligned}$$

where $m!!$ denotes the double factorial

$$m!! = \begin{cases} m(m-2)(m-4)\cdots 1 \text{ when } m \text{ is odd,} \\ m(m-2)(m-4)\cdots 2 \text{ when } m \text{ is even.} \end{cases}$$

As the product of roots $1\cdot\omega_2\cdots\omega_n = (-1)^{n-1}$, it follows

$$\begin{aligned} F^o_{n,m}(x) &= (-1)^{\frac{m(n-1)}{2}} \prod_{\substack{j=1\\ j\ odd}}^{m} \frac{x-j}{m!!} \prod_{\substack{j=1\\ j\ odd}}^{m} \frac{\omega_2^{-1}x-j}{m!!} \cdots \prod_{\substack{j=1\\ j\ odd}}^{m} \frac{\omega_n^{-1}x-j}{m!!} \\ &= (-1)^{\frac{m(n-1)}{2}} \binom{x-1}{m}^o \binom{\omega_2^{-1}x-1}{m}^o \cdots \binom{\omega_n^{-1}x-1}{m}^o \\ &= (-1)^{\frac{m(n-1)}{2}} \prod_{\omega^n=1} \binom{\omega^{-1}x-1}{m}^o \\ &= (-1)^{\frac{m(n-1)}{2}} \prod_{\omega^n=1} \binom{\omega x-1}{m}^o. \end{aligned}$$

On the other hand,

$$\begin{aligned} F^o_{n,m}(x) &= \frac{(x^n-1)(x^n-3^n)\cdots(x^n-(m-1)^n)}{(m-1)!!^n} \\ &= \left(\frac{x^n-1}{1^n}\right)\left(\frac{x^n-3^n}{3^n}\right)\cdots\left(\frac{x^n-(m-1)^n}{(m-1)^n}\right) \\ &= (-1)^{\frac{m}{2}}(1-x^n)\left(1-\frac{x^n}{3^n}\right)\cdots\left(1-\frac{x^n}{(m-1)^n}\right) \\ &= (-1)^{\frac{m}{2}} \sum_{k=0}^{\frac{m}{2}} (-1)^k e_k(1,\frac{1}{3^n},\ldots,\frac{1}{(m-1)^n})x^{nk} \\ &= (-1)^{\frac{m}{2}} \sum_{k=0}^{\frac{m}{2}} (-1)^k O_m(\{n\}^k)x^{kn}. \end{aligned}$$

For odd m the proof is analogous. Combining both cases, we conclude that

$$F^a_{n,m}(x) = (-1)^{(n-1)\lfloor\frac{m+1}{2}\rfloor} \prod_{\omega^n=1} \binom{\omega x-1}{m}^o = (-1)^{\lfloor\frac{m+1}{2}\rfloor} \sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor} (-1)^k O_m(\{n\}^k)x^{kn}.$$

□

Theorem 2.4. *Let m be a non-negative integer. Then, for all $k\geq 0$ we have*

$$H_m(\{n\}^k) = \sum_{j=0}^{k} \frac{O_m(\{n\}^j)H_{\lfloor\frac{m}{2}\rfloor}(\{n\}^{k-j})}{2^{n(k-j)}}. \tag{2.7}$$

$$O_m(\{n\}^k) = \sum_{j=0}^{k} \frac{(-1)^{k-j}H_m(\{n\}^j)H^\star_{\lfloor\frac{m}{2}\rfloor}(\{n\}^{k-j})}{2^{n(k-j)}}. \tag{2.8}$$

$$\frac{H_{\lfloor\frac{m}{2}\rfloor}(\{n\}^k)}{2^{kn}} = \sum_{j=0}^{k}(-1)^{k-j}H_m(\{n\}^j)O^\star_m(\{n\}^{k-j}). \tag{2.9}$$

Proof. We use the simple observation $\left\lfloor\frac{m+1}{2}\right\rfloor + \left\lfloor\frac{m}{2}\right\rfloor = m$.

(i) For the full product, we factor into odd and even parts:

$$\begin{aligned} \prod_{\omega^n=1}\binom{\omega x-1}{m} &= \prod_{\omega^n=1}\binom{\omega x-1}{m}^o\binom{\omega x-1}{m}^e \\ &= \left[(-1)^{n\lfloor\frac{m+1}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{m+1}{2}\rfloor}(-1)^j O_m(\{n\}^j)x^{jn}\right] \\ &\times \left[(-1)^{n\lfloor\frac{m}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^j H_{\lfloor\frac{m}{2}\rfloor}(\{n\}^j)\left(\frac{x}{2}\right)^{jn}\right] \\ &= (-1)^{mn}\sum_{k=0}^{m}(-1)^k\left(\sum_{j=0}^{k}\frac{O_m(\{n\}^j)\,H_{\lfloor\frac{m}{2}\rfloor}(\{n\}^{k-j})}{2^{\,n(k-j)}}\right)x^{kn}. \end{aligned}$$

Comparing with the expansion in Eq(2.3) yields the first identity.

(ii) For the odd part, we compute

$$\prod_{\omega^n=1}\binom{\omega x-1}{m}^o = \frac{\prod_{\omega^n=1}\binom{\omega x-1}{m}}{\prod_{\omega^n=1}\binom{\frac{\omega x}{2}-1}{\lfloor\frac{m}{2}\rfloor}}$$
$$= \left[(-1)^{mn}\sum_{j=0}^{m}(-1)^j H_m(\{n\}^j)x^{jn}\right]\left[(-1)^{n\lfloor\frac{m}{2}\rfloor}\sum_{j\geq 0}H^\star_{\lfloor\frac{m}{2}\rfloor}(\{n\}^j)\left(\frac{x}{2}\right)^{jn}\right]$$
$$= (-1)^{n\left(m-\lfloor\frac{m}{2}\rfloor\right)}\sum_{k\geq 0}(-1)^k\left(\sum_{j=0}^{k}\frac{(-1)^{k+j}H_m(\{n\}^j)H^\star_{\lfloor\frac{m}{2}\rfloor}(\{n\}^{k-j})}{2^{(k-j)n}}\right)x^{kn}$$
$$= (-1)^{n\lfloor\frac{m+1}{2}\rfloor}\sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor}(-1)^k O_m(\{n\}^k)x^{kn}.$$

This gives the second identity.

(iii) For the even part, we similarly have

$$\prod_{\omega^n=1}\binom{\omega x-1}{m}^e = \frac{\prod_{\omega^n=1}\binom{\omega x-1}{m}}{\prod_{\omega^n=1}\binom{\omega x-1}{m}^o}$$
$$= \left[(-1)^{mn}\sum_{j=0}^{m}(-1)^j H_m(\{n\}^j)x^{jn}\right]\left[(-1)^{n\lfloor\frac{m+1}{2}\rfloor}\sum_{j\geq 0}O^\star_m(\{n\}^j)x^{jn}\right]$$
$$= (-1)^{n\lfloor\frac{m}{2}\rfloor}\sum_{k\geq 0}(-1)^k\left(\sum_{j=0}^{k}(-1)^{k+j}H_m(\{n\}^j)O^\star_m(\{n\}^{k-j})\right)x^{kn}$$
$$= (-1)^{n\lfloor\frac{m}{2}\rfloor}\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^k H_{\lfloor\frac{m}{2}\rfloor}(\{n\}^k)\left(\frac{x}{2}\right)^{kn}.$$

This proves the third identity.

□

Theorem 2.5. *Let m be a non-negative integer. Then, for all $k \geq 0$ we have*

$$H^\star_m(\{n\}^k) = \sum_{j=0}^{k}\frac{O^\star_m(\{n\}^j)H^\star_{\lfloor\frac{m}{2}\rfloor}(\{n\}^{k-j})}{2^{n(k-j)}}. \tag{2.10}$$

$$O^\star_m(\{n\}^k) = \sum_{j=0}^{k}\frac{(-1)^j H_{\lfloor\frac{m}{2}\rfloor}(\{n\}^j)H^\star_m(\{n\}^{k-j})}{2^{nj}}. \tag{2.11}$$

$$\frac{H^\star_{\lfloor\frac{m}{2}\rfloor}(\{n\}^k)}{2^{kn}} = \sum_{j=0}^{k}(-1)^j O_m(\{n\}^j)H^\star_m(\{n\}^{k-j}). \tag{2.12}$$

Proof. The result follows by inverting the relations established in the previous proof. □

Theorem 2.6. *Let m be a non-negative integer and let*

$$\overline{N}_k = (\overline{2n}, 2n, \ldots, 2n, \overline{2n}) \quad or \quad (\overline{2n}, 2n, \ldots, \overline{2n}, 2n),$$

and similarly

$$N_k = (2n, \overline{2n}, \ldots, 2n, \overline{2n}) \quad or \quad (2n, \overline{2n}, \ldots, \overline{2n}, 2n).$$

Then, for any non-negative integer m, the following relations hold

$$H_m(\{\overline{N}_k\}) = \sum_{j=0}^{k}(-1)^j\frac{O_m(\{2n\}^j)H_{\lfloor\frac{m}{2}\rfloor}(\{2n\}^{k-j})}{2^{2n(k-j)}}. \tag{2.13}$$

$$H_m(\{N_k\}) = \sum_{j=0}^{k}(-1)^{k-j}\frac{O_m(\{2n\}^j)H_{\lfloor\frac{m}{2}\rfloor}(\{2n\}^{k-j})}{2^{2n(k-j)}}. \tag{2.14}$$

$$H^\star_m(\{\overline{N}_k\}) = \sum_{j=0}^{k}(-1)^j\frac{O^\star_m(\{2n\}^j)H^\star_{\lfloor\frac{m}{2}\rfloor}(\{2n\}^{k-j})}{2^{2n(k-j)}}. \tag{2.15}$$

$$H^\star_m(\{N_k\}) = \sum_{j=0}^{k}(-1)^{k-j}\frac{O^\star_m(\{2n\}^j)H^\star_{\lfloor\frac{m}{2}\rfloor}(\{2n\}^{k-j})}{2^{2n(k-j)}}. \tag{2.16}$$

Proof.

(i) First assume that m is even. Consider the polynomial
$$F^a_{2n,m}(x) = \frac{(x^{2n}+1^{2n})(x^{2n}-2^{2n})\cdots(x^{2n}+m^{2n})}{1^{2n}2^{2n}\cdots m^{2n}}.$$
Let ζ be a primitive $4n$-th root of unity, so that $\zeta^{2n}=-1$. Then
$$\begin{aligned}
F^a_{2n,m}(x) &= \frac{(x^{2n}+1^{2n})(x^{2n}+3^{2n})\cdots(x^{2n}+(m-1)^{2n})}{1^{2n}3^{2n}\cdots(m-1)^{2n}} \\
&\quad\times \frac{(x^{2n}-2^{2n})(x^{2n}-4^{2n})\cdots(x^{2n}-m^{2n})}{2^{2n}4^{2n}\cdots m^{2n}} \\
&= \frac{(x^{2n}-(\zeta)^{2n})(x^{2n}-(3\zeta)^{2n})\cdots(x^{2n}-((m-1)\zeta)^{2n})}{1^{2n}3^{2n}\cdots(m-1)^{2n}} \\
&\quad\times \frac{\big((\frac{x}{2})^{2n}-1^{2n}\big)\big((\frac{x}{2})^{2n}-2^{2n}\big)\cdots\big((\frac{x}{2})^{2n}-(\frac{m}{2})^{2n}\big)}{1^{2n}2^{2n}\cdots(\frac{m}{2})^{2n}} \\
&= (\zeta^{2n})^{\frac{m}{2}}\,\frac{\big((\frac{x}{\zeta})^{2n}-1^{2n}\big)\big((\frac{x}{\zeta})^{2n}-3^{2n}\big)\cdots\big((\frac{x}{\zeta})^{2n}-(m-1)^{2n}\big)}{1^{2n}3^{2n}\cdots(m-1)^{2n}}\,F_{2n,\frac{m}{2}}\left(\frac{x}{2}\right) \\
&= (-1)^{\frac{m}{2}}F^o_{2n,m}\left(\frac{x}{\zeta}\right)F_{2n,\frac{m}{2}}\left(\frac{x}{2}\right).
\end{aligned}$$
If m is odd, a similar argument yields
$$F^a_{2n,m}(x) = (-1)^{\frac{m+1}{2}}F^o_{2n,m}\left(\frac{x}{\zeta}\right)F_{2n,\frac{m-1}{2}}\left(\frac{x}{2}\right).$$
Thus, in general,
$$F^a_{2n,m}(x) = (-1)^{\lfloor\frac{m+1}{2}\rfloor}F^o_{2n,m}\left(\frac{x}{\zeta}\right)F_{2n,\lfloor\frac{m}{2}\rfloor}\left(\frac{x}{2}\right).$$
Expanding via the product identities for $O_m(\{n\}^k)$ and $H_{\lfloor\frac{m}{2}\rfloor}(\{n\}^k)$ gives
$$F^a_{2n,m}(x) = (-1)^{\lfloor\frac{m}{2}\rfloor}\left[\sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor}O_m(\{2n\}^k)x^{2nk}\right]\left[\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^kH_{\lfloor\frac{m}{2}\rfloor}(\{2n\}^k)\left(\frac{x}{2}\right)^{2nk}\right].$$
On the other hand, by direct expansion one has
$$F^a_{2n,m}(x) = (-1)^{\lfloor\frac{m}{2}\rfloor}\sum_{k=0}^{m}(-1)^kH_m(\overline{N}_k)\,x^{2nk}.$$
Comparing coefficients yields the first identity.

(ii) For the case of N_k, consider instead
$$F^{a'}_{2n,m}(x) = \frac{(x^{2n}-1^{2n})(x^{2n}+2^{2n})\cdots(x^{2n}\pm m^{2n})}{1^{2n}2^{2n}\cdots m^{2n}},$$
and follow the same procedure.

(iii) To obtain the last two identities, we consider the reciprocal polynomials $\frac{1}{F^a_{2n,m}(x)}$ and $\frac{1}{F^{a'}_{2n,m}(x)}$, respectively.

□

3 A shuffle like relation for homogeneous multiple harmonic sums indexed by integer partitions

In this section we establish a shuffle like relation (see for instance [3], [11]) between homogeneous multiple harmonic sums that involve the symmetric monomial polynomials evaluated at roots of unity.

Lemma 3.1. *For $a \geq 1$, we have*
$$F_{an,m}(x) = (-1)^{m(a-1)}F_{n,m}(x)F_{n,m}(\omega x)\cdots F_{n,m}(\omega^{a-1}x), \tag{3.1}$$
where ω is a primitive an-th root of unity.

Proof. Let ω be a primitive an-th root of unity. Set $\{\omega_0 = 1, \omega_1, \ldots, \omega_{a-1}\}$ to be the set of all a-th roots of unity. We can then rearrange terms so that $\omega^{in} = \omega_i$ for $0 \leq i \leq a-1$. We have the following
$$\begin{aligned}
F_{an,m}(x) &= \frac{(x^{an}-1)(x^{an}-2^{an})\cdots(x^{an}-m^{an})}{m!^{an}} \\
&= \prod_{i=0}^{a-1}\prod_{j=1}^{m}\frac{x^n-\omega_i j^n}{m!^{an}} = \prod_{i=0}^{a-1}\prod_{j=1}^{m}\frac{\frac{x^n}{\omega_i}-j^n}{m!^{an}} \\
&= \omega_0^m\omega_1^m\cdots\omega_{a-1}^m\prod_{i=0}^{a-1}\prod_{j=1}^{m}\frac{(\frac{x}{\omega^i})^n-j^n}{m!^{an}} \\
&= (-1)^{m(a-1)}F_{n,m}(x)F_{n,m}(\omega x)\cdots F_{n,m}(\omega^{a-1}x).
\end{aligned}$$
□

Theorem 3.2. *Let $a \geq 1$ and $k \geq 0$, we have*

(i) For homogeneous multiple harmonic sums, we have

$$
\begin{aligned}
(-1)^{(a-1)k} H_m(\{an\}^k) &= \sum_{l_0+\cdots+l_{a-1}=ak} \omega_0^{l_0}\omega_1^{l_1}\cdots\omega_{a-1}^{l_{a-1}} H_m(\{n\}^{l_0})H_m(\{n\}^{l_1})\cdots H_m(\{n\}^{l_{a-1}}),\\
&= \sum_{\substack{\lambda=(\lambda_1,\ldots,\lambda_a)\vdash ak\\ l(\lambda)\leq a}} m_\lambda(\omega_0,\ldots,\omega_{a-1}) H_m(\{n\}^{\lambda_1})H_m(\{n\}^{\lambda_2})\cdots H_m(\{n\}^{\lambda_a}).
\end{aligned}
\tag{3.2}
$$

(ii) For homogeneous multiple harmonic star sums, we have

$$
\begin{aligned}
H_m^\star(\{an\}^k) &= \sum_{l_0+\cdots+l_{a-1}=ak} \omega_0^{l_0}\omega_1^{l_1}\cdots\omega_{l_{a-1}}^{l_{a-1}} H_m^\star(\{n\}^{l_0})H_m^\star(\{n\}^{l_1})\cdots H_m^\star(\{n\}^{l_{a-1}}),\\
&= \sum_{\substack{\lambda=(\lambda_1,\ldots,\lambda_a)\vdash ak\\ l(\lambda)\leq a}} m_\lambda(\omega_0,\ldots,\omega_{a-1}) H_m^\star(\{n\}^{\lambda_1})H_m^\star(\{n\}^{\lambda_2})\cdots H_m^\star(\{n\}^{\lambda_a}).
\end{aligned}
\tag{3.3}
$$

Proof. The result follows by expanding the product $(-1)^{m(a-1)}F_{n,m}(x)F_{n,m}(\omega x)\cdots F_{n,m}(\omega^{a-1}x)$ and comparing coefficients with those of $F_{an,m}(x)$. □

Corollary 3.3. *When $a=2$ and $k\geq 0$, we find*

$$
\begin{aligned}
(-1)^k H_m(\{2n\}^k) = \sum_{l=0}^{2k}(-1)^l H_m(\{n\}^l)H_m(\{n\}^{2k-l}) = \\
2\sum_{l=0}^{\lfloor\frac{k+1}{2}\rfloor} H_m(\{n\}^{2l})H_m(\{n\}^{2k-2l}) + (-1)^k H_m(\{n\}^k)^2 - 2\sum_{l=0}^{\lfloor\frac{k}{2}\rfloor} H_m(\{n\}^{2l+1})H_m(\{n\}^{2k-2l-1}).
\end{aligned}
\tag{3.4}
$$

We also have

$$
\begin{aligned}
H_m^\star(\{2n\}^k) = \sum_{l=0}^{2k}(-1)^l H_m^\star(\{n\}^l)H_m^\star(\{n\}^{2k-l}) = \\
2\sum_{l=0}^{\lfloor\frac{k+1}{2}\rfloor} H_m^\star(\{n\}^{2l})H_m^\star(\{n\}^{2k-2l}) + (-1)^k H_m^\star(\{n\}^k)^2 - 2\sum_{l=0}^{\lfloor\frac{k}{2}\rfloor} H_m^\star(\{n\}^{2l+1})H_m^\star(\{n\}^{2k-2l-1}).
\end{aligned}
\tag{3.5}
$$

Proof. We regroup similar terms. □

Examples 1. (i) When $k=3$, we find

$$H_m(2n,2n,2n) = 2\left[-H_m(\{n\}^6) + H(n)H_m(\{n\}^5) - H_m(\{n\}^2)H_m(\{n\}^4)\right] + H_m(\{n\}^3)^2.$$

(ii) If we set α and β the two 3-roots of unity then

$$
\begin{aligned}
H_m(3n,3n) =&(\alpha^2\beta^2)H_m(\{n\}^2)^3 + (\alpha^3+\beta^3+\alpha^3\beta^3)H_m(\{n\}^3)^2 + (1+\alpha^6+\beta^6)H_m(\{n\}^6)\\
&+ (\alpha^2\beta+\alpha\beta^2+\alpha^3\beta+\alpha\beta^3+\alpha^3\beta^2+\alpha^2\beta^3)H_m(n)H_m(\{n\}^2)H_m(\{n\}^3)\\
&+ (\alpha\beta+\alpha^4\beta+\alpha\beta^4)H_m(n)^2H_m(\{n\}^4)\\
&+ (\alpha^2+\alpha^4+\beta^2+\beta^4+\alpha^4\beta^2+\alpha^2\beta^4)H_m(\{n\}^2)H_m(\{n\}^4)\\
&+ (\alpha+\alpha^5+\beta+\beta^5+\alpha^5\beta+\alpha\beta^5)H_m(n)H_m(\{n\}^5).
\end{aligned}
$$

Now, using relations $\alpha\beta=1$, $\alpha^3=\beta^3=1$ and $\alpha+\beta=-1$ the above equation may be simplified to

$$
\begin{aligned}
H_m(3n,3n) = H_m(\{n\}^2)^3 + 3\left[H_m(\{n\}^3)^2 + H_m(\{n\}^6) + H_m(n)^2H_m(\{n\}^4)\right]\\
-3\left[H_m(n)H_m(\{n\}^2)H_m(\{n\}^3) + H_m(\{n\}^2)H_m(\{n\}^4) + H_m(n)H_m(\{n\}^5)\right].
\end{aligned}
\tag{3.6}
$$

A similar argument can be made to prove the following result on odd homogeneous multiple harmonic (star) sums.

Theorem 3.4. *Let $a\geq 1$ and $k\geq 0$, we have*

(i) For odd homogeneous multiple harmonic sums, we have

$$
\begin{aligned}
(-1)^{(a-1)k} O_m(\{an\}^k) &= \sum_{l_0+\cdots+l_{a-1}=ak} \omega_0^{l_0}\omega_1^{l_1}\cdots\omega_{l_{a-1}}^{l_{a-1}} O_m(\{n\}^{l_0})O_m(\{n\}^{l_1})\cdots O_m(\{n\}^{l_{a-1}}),\\
&= \sum_{\substack{\lambda=(\lambda_1,\cdots,\lambda_a)\vdash ak\\ l(\lambda)\leq a}} m_\lambda(\omega_0,\ldots,\omega_{a-1}) O_m(\{n\}^{\lambda_1})O_m(\{n\}^{\lambda_2})\cdots O_m(\{n\}^{\lambda_a}).
\end{aligned}
\tag{3.7}
$$

(ii) For odd homogeneous multiple harmonic star sums, we have

$$
\begin{aligned}
O_m^\star(\{an\}^k) &= \sum_{l_0+\cdots+l_{a-1}=ak} \omega_0^{l_0}\omega_1^{l_1}\cdots\omega_{l_{a-1}}^{l_{a-1}} O_m^\star(\{n\}^{l_0})O_m^\star(\{n\}^{l_1})\cdots O_m^\star(\{n\}^{l_{a-1}}),\\
&= \sum_{\substack{\lambda=(\lambda_1,\cdots,\lambda_a)\vdash ak\\ l(\lambda)\leq a}} m_\lambda(\omega_0,\ldots,\omega_{a-1}) O_m^\star(\{n\}^{\lambda_1})O_m^\star(\{n\}^{\lambda_2})\cdots O_m^\star(\{n\}^{\lambda_a}).
\end{aligned}
\tag{3.8}
$$

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Received: 2024-09-28
Accepted: 2025-09-19