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Maps Preserving the ∂-Spectrum of Generalized Products of Operators

H. Benbouziane, A. Daoudi and M. Ech-Chérif El Kettani

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Corresponding Author: A. Daoudi

Abstract Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on an infnite-dimensional Banach space X. A map Δ, from $\mathcal{B}(X)$ into a closed subsets of $\mathbb{C}$, is said to be ∂-spectrum if, $\partial(\sigma(T)) \subseteq \Delta(T) \subseteq \sigma(T)$ *for all* $T \in \mathcal{B}(X)$, where $\sigma(T)$ is spectrum of T and $\partial(\sigma(T))$ is its the boundary.

For an integer $k \geq 2$, let $(i_1, ..., i_m)$ be a finite sequence with terms belong to $\{1, ..., k\}$, such that at least one of the terms in $(i_1, ..., i_m)$ appears exactly once. The generalized product of k operators $T_1, T_2, ..., T_k \in \mathcal{B}(X)$ is defined by $T_1 * T_2 * ... * T_k = T_{i_1}T_{i_2}...T_{i_m}$. The integer m is called the width of the generalized product.

Fixing a ∂-spectrum map Δ, in this paper we describe all maps ϕ from $\mathcal{B}(X)$ into itself for which the range contains all operators with rank at most two and satisfy

$$\Delta(T_1 * ... * T_k) = \Delta(\phi(T_1) * ... * \phi(T_k)) \text{ for all } T_1, ..., T_k \in \mathcal{B}(X).$$

1 Introduction and preliminaries

Let X be an infinite-dimensional complex Banach space, X^* the dual space of X and $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X. The identity operator on X (resp. X^*) will denote by I (resp. I_{X^*}).

In the sequel, for any operator $T \in \mathcal{B}(X)$, let $R(T)$, $N(T)$, T^*, $\sigma(T)$, $r(T)$, $\sigma_l(T)$, $\sigma_r(T)$, $\sigma_{sur}(T)$, $\sigma_{ap}(T)$, $\sigma_g(T)$, $\sigma_K(T)$ and $\partial(\sigma(T))$ respectively, be the range, the kernel, the adjoint, the spectrum, the spectral radius, the left spectrum, the right spectrum, the surjectivity spectrum, the approximate point spectrum, the genaralized spectrum, the kato (semi regular) spectrum and the boundary of spectrum of T.

For $x \in X \setminus \{0\}$ and $f \in X^* \setminus \{0\}$, let $x \otimes f$ stand for the operator of rank at most one defined by $(x \otimes f)y = f(y)x$ for all $y \in X$. Note that every operator on X of rank one can be written in this form. Recall that $\sigma(x \otimes f) = \{0, f(x)\}$ and $x \otimes f$ is a nilpotent operator if and only if $f(x) = 0$. Denote by $\mathcal{F}(X)$ the ideal of all finite rank operators in $\mathcal{B}(X)$.

A map $\Delta : \mathcal{B}(X) \to \{\text{closed subsets of } \mathbb{C}\}$ is said to be ∂-spectrum in $\mathcal{B}(X)$ if

$$\partial(\sigma(T)) \subseteq \Delta(T) \subseteq \sigma(T).$$

for all $T \in \mathcal{B}(X)$; see [13, Def 3.2]. Note that if Δ is ∂-spectrum, then for every $T \in \mathcal{B}(X)$, $\Delta(T)$ is non-empty and

$$\Delta(T) \textit{ is countable } \iff \sigma(T) \textit{ is countable},$$

and in this case we have $\Delta(T) = \sigma(T)$. In particular, $\Delta(x \otimes f) = \sigma(x \otimes f) = \{0, f(x)\}$ for all $x \in X$ and $f \in X^*$.

Consider the function $\Delta : \mathcal{B}(X) \to \{\text{closed subsets of } \mathbb{C}\}$ with $\Delta(.)$ is any one of the spectral functions $\sigma(.)$, $\sigma_l(.)$, $\sigma_r(.)$, $\sigma_{ap}(.)$, $\sigma_{sur}(.)$,$\sigma_g(.)$, $\sigma_K(.)$. It is known that these spectral functions satisfies

$$\partial(\sigma(T)) \subseteq \Delta(T) \subseteq \sigma(T) \; for \; all \; T \in \mathcal{B}(X).$$

Therefore, they are ∂-spectrum.

The problem of characterizing linear or nonlinear maps between algebras that leave a certain property, a particular relation, or even a subset invariant have been widely studied; see [2, 5, 6, 12, 16, 18] for more informations on this subject.

Over the past few years, there has been a lot of studies on describing maps which preserve these particular property, relation or subset, of certain products of operators, without assuming linearity or additivity. We refer the interested reader to [4, 7, 9, 10, 14, 19, 20, 21].

Motivated by the above results, in [3], the authors characterized nonlinear maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ preserving any one of spectral functions $\Delta(.)$, that satisfies $\partial(\sigma(.)) \subseteq \Delta(.) \subseteq \sigma(.)$, which are called ∂-spectrum, of the product or triple product of operators, and they proved that a such map must be a Jordan isomorphism multiplied by a square or cubic root of unity, respectively.

Throughout this paper, we fix a positive integer $k \geq 2$ and let $(i_1, ..., i_m)$ be a finite sequence with terms belong to $\{1, ..., k\}$ such that at least one of the terms in $(i_1, ..., i_m)$ appears exactly once. Recall that the generalized product of width m of k operators $T_1, T_2, ..., T_k \in \mathcal{B}(X)$ is defined by

$$T_1 * T_2 * ... * T_k = T_{i_1} T_{i_2} ... T_{i_m}.$$

Obviously, the usual product and the triple product are special cases of the generalized product.

In the last two decades, there has been a lot of activity on maps that preserve the generalized product of operators; see for example [9, 11, 20].

We continue this line of study. The purpose of this paper is to determine the structure of nonlinear map on $\mathcal{B}(X)$ (not necessarily surjective) preserving any one of spectral functions $\Delta(.)$, that satisfies $\partial(\sigma(.)) \subseteq \Delta(.) \subseteq \sigma(.)$ (∂-spectrum), of the generalized product of operators , which cover main result obtained in [9] and for certain spectral functions showed in [11], using a different approach and without assuming the condition of low rank operators.

Throughout this paper, let Δ be a ∂-spectrum. The aim of this work is to prove the following result.

Theorem 1.1. *Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a map with the range containing all operators of rank at most two, such that*

$$\Delta(T_1 * ... * T_k) = \Delta(\phi(T_1) * ... * \phi(T_k)) \; for \; all \; T_1, ..., T_k \in \mathcal{B}(X). \tag{1.1}$$

Then there exists a scalar $\alpha \in \mathbb{C}$ with $\alpha^m = 1$ and either

(i) there is a bounded invertible operator $A : X \to X$ such that $\phi(T) = \alpha A T A^{-1}$ for all $T \in \mathcal{B}(X)$, or

(ii) there is a bounded invertible operator $C : X^* \to X$ such that $\phi(T) = \alpha C T^* C^{-1}$ for all $T \in \mathcal{B}(X)$. In the last case, the space X is reflexive.

2 Proof of the main result

Let $T, S \in \mathcal{B}(X)$. Set $T_{i_p} = S$ and $T_{i_j} = T$ for $j \neq p$ where i_p is the term which appears exactly once in $(i_1, ..., i_m)$. Thus there exist nonnegative integers r, s with $r + s = m - 1 \geq 1$ such that $T_1 * ... * T_k = T^r S T^s$. Then, the Theorem 1.1 is a consequence of the following one.

Theorem 2.1. *Let r and s be two nonnegative integers with $r + s \geq 1$. Assume that $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ is a map with the range containing all operators of rank at most two satisfying*

$$\Delta(T^r S T^s) = \Delta(\phi(T)^r \phi(S) \phi(T)^s) \; for \; all \; T, S \in \mathcal{B}(X). \tag{2.1}$$

Then there exists a scalar $\alpha \in \mathbb{C}$ with $\alpha^{r+s+1} = 1$, and either *(i) there is a bounded invertible operator* $A : X \to X$ *such that*

$$\phi(T) = \alpha ATA^{-1} \; for \; all \; T \in \mathcal{B}(X),$$

or

(ii) there is a bounded invertible operator $C : X^* \to X$ *such that*

$$\phi(T) = \alpha CT^*C^{-1} \; for \; all \; T \in \mathcal{B}(X).$$

In this case, the space X *is reflexive.*

In order to prove the main theorem, we need some auxiliary results. For an operator $T \in \mathcal{B}(X)$, we use this notation introduced in [3]:

$$\Delta^*(T) = \begin{cases} \Delta(T) \setminus \{0\} & \text{if } \Delta(T) \neq \{0\} \\ \{0\} & \text{if } \Delta(T) = \{0\} \end{cases} \tag{2.2}$$

where Δ is ∂-spectrum. In particular, we have $\Delta^*(x \otimes f) = \{f(x)\}$ for all $x \in X$ and $f \in X^*$, and for all $\lambda \in \mathbb{C} \setminus \{0\}$ we have

$$\Delta^*(\lambda x \otimes f) = \{f(\lambda x)\} = \lambda\{f(x)\} = \lambda\Delta^*(x \otimes f). \tag{2.3}$$

Thus, for every rank one operator $R \in \mathcal{B}(X)$, and for all $\lambda \in \mathbb{C} \setminus \{0\}$, we get

$$\Delta^*(\lambda R) = \lambda\Delta^*(R). \tag{2.4}$$

The first lemma characterizes in term of ∂-spectrum when two operators are equal, which will be used to prove our main result.

Lemma 2.2. *Let* $T,\ S \ \in \mathcal{B}(X)$*,* r *and* s *be two nonnegative integers such that* $r + s \geq 1$*. Then, the following statements are equivalent.*

(i) $T = S$.

(ii) $\Delta(R^rTR^s) = \Delta(R^rSR^s)$ *for all rank one operators* $R \in \mathcal{B}(X)$.

(iii) $\Delta^*(R^rTR^s) = \Delta^*(R^rSR^s)$ *for all rank one operators* $R \in \mathcal{B}(X)$.

Proof. The implications $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are obvious. So we only need to show that the implication $(3) \implies (1)$ holds. Indeed, assume that $\Delta^*(R^rTR^s) = \Delta^*(R^rSR^s)$ for all rank one operators $R \in \mathcal{B}(X)$.
Let $R = x \otimes f$, where $x \in X \setminus \{0\}$ and $f \in X^*$ such that $f(x) \neq 0$. Note that $R^rTR^s = f(x)^{r+s-2}f(Tx)x \otimes f$ and $R^rSR^s = f(x)^{r+s-2}f(Sx)x \otimes f$. Then

$$\begin{aligned} \Delta^*(R^rTR^s) = \Delta^*(R^rSR^s) \quad &\Rightarrow \quad \Delta^*(f(x)^{r+s-2}f(Tx)x \otimes f) = \Delta^*(f(x)^{r+s-2}f(Sx)x \otimes f). \\ &\Rightarrow \quad \{f(x)^{r+s-1}f(Tx)\} = \{f(x)^{r+s-1}f(Sx))\} \\ &\Rightarrow \quad f(Tx) = f(Sx) \; for \; all \; x \in X, \; f \in X^*, \; with \; f(x) \neq 0. \end{aligned}$$

Next, consider the case when $f(x) = 0$ and take a linear functional $g \in X^*$ such that $g(x) \neq 0$. Then from what has been shown we have $(f+g)(Tx) = (f+g)(Sx)$ and $g(Tx) = g(Sx)$. Then

$$\begin{aligned} f(Tx) + g(Sx) &= f(Tx) + g(Tx) \\ &= (f+g)(Tx) \\ &= (f+g)(Sx) \\ &= f(Sx) + g(Sx). \end{aligned}$$

Which establishes that $f(Tx) = f(Sx)$ in this case too. So, by Hahn-Banach theorem, $Tx = Sx$ for all $x \in X$, which proves that $T = S$. □

For any $T \in \mathcal{B}(X)$, let $r(T) := \{sup|\lambda| : \lambda \in \sigma(T)\}$ be the spectral radius of T. The peripheral spectrum of T is defind by

$$\sigma_\pi(T) = \{\lambda \in \sigma(T) : |\lambda| = r(T)\},$$

and is nothing but $\sigma_\pi(T) = \{\lambda \in \partial\sigma(T) : |\lambda| = r(T)\}$. In particular,

$$\sigma_\pi(T) \subseteq \partial\sigma(T) \subseteq \Delta(T) \tag{2.5}$$

for all operators $T \in \mathcal{B}(X)$.

In [20], Zhang and Hou proved the following lemma.

Lemma 2.3. *Let R be a nonzero operator on $\mathcal{B}(X)$ where X is of dimension at least two. Suppose r and s are two nonnegative integers such that $r + s \geq 1$. The following statements are equivalent.*

(i) *R has rank one.*

(ii) *$\sigma_\pi(T^r RT^s)$ is a singleton for all operators $T \in \mathcal{B}(X)$.*

(iii) *$\sigma_\pi(T^r RT^s)$ is a singleton for all rank two operators $T \in \mathcal{B}(X)$.*

The following lemma gives a characterization of rank one operators in terms of ∂-spectrum of the generalized products.

Lemma 2.4. *Let R be a nonzero operator on $\mathcal{B}(X)$, r and s are two nonnegative integers. The following statements are equivalent.*

(i) *R has rank one.*

(ii) *$\Delta^*(T^r RT^s)$ is a singleton for all operators $T \in \mathcal{B}(X)$.*

(iii) *$\Delta^*(T^r RT^s)$ is a singleton for all rank two operators $T \in \mathcal{B}(X)$.*

Proof. We only need to prove that the implication $(iii) \Rightarrow (i)$.
Suppose that $\Delta^*(T^r RT^s)$ is a singleton for all rank two operators $T \in \mathcal{B}(X)$. In this case, we have $\Delta^*(T^r RT^s) = \sigma_\pi(T^r RT^s)$. Lemma 2.3 ensurs that R is of rank one. □

We end this section with the following technical lemma which will be used to show that a map ϕ satisfying (2.1) is additive.

Lemma 2.5. *Let $T, S \in \mathcal{B}(X)$. Then for all rank one operators $R \in \mathcal{B}(X)$ we have*

$$\Delta^*(R^r(T + S)R^s) = \Delta^*(R^r TR^s) + \Delta^*(R^r SR^s).$$

Proof. Let $R = x \otimes f$ be a rank one operator where $x \in X \setminus \{0\}$ and $f \in X^*$. We have

$$R^r(T + S)R^s = f(x)^{r+s-2} f((T + S)x)x \otimes f.$$

Then

$$\begin{aligned}
\Delta^*(R^r(T + S)R^s) &= \{f(x)^{r+s-1} f((T + S)x)\} \\
&= \{f(x)^{r+s-1} f(Tx) + f(x)^{r+s-1} f(Sx)\} \\
&= \{f(x)^{r+s-1} f(Tx)\} + \{f(x)^{r+s-1} f(Sx)\} \\
&= \Delta^*(R^r TR^s) + \Delta^*(R^r SR^s),
\end{aligned}$$

proving the result. □

Proof of Theorem 2. 1. Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a map with the range contains all operators of rank at most two. Suppose that ϕ satisfies

$$\Delta(T^r ST^s) = \Delta(\phi(T)^r \phi(S)\phi(T)^s) \ for\ all\ T, S \in \mathcal{B}(X).$$

We proceed in several steps.

Step 1. ϕ is injective and $\phi(0) = 0$.
Assume that $\phi(A) = \phi(B)$ for some operators $A, B \in \mathcal{B}(X)$. For every rank one operator $R \in \mathcal{B}(X)$, we have

$$\Delta(R^r AR^s) = \Delta(\phi(R)^r \phi(A)\phi(R)^s) = \Delta(\phi(R)^r \phi(B)\phi(R)^s) = \Delta(R^r BR^s).$$

This proves, by Lemma 2.2, that $A = B$. Thus ϕ is injective.
On the other hand, set R a rank one operator, by hypothesis, there exists $T \in \mathcal{B}(X)$ such that $\phi(T) = R$. Then we have

$$\begin{aligned}\Delta^*(R^r 0R^s) &= \Delta^*(\phi(T)^r 0\phi(T)^s)\\ &= \{0\}\\ &= \Delta^*(T^r 0T^s)\\ &= \Delta^*(\phi(T)^r \phi(0)\phi(T)^s)\\ &= \Delta^*(R^r \phi(0)R^s).\end{aligned}$$

By lemma 2.2, we see that $0 = \phi(0)$.

Step 2. ϕ preserves rank one operator and rank one non-nilpotent operator in both directions.
Let R be a rank one operator. So, $\phi(R) \neq 0$. By hypothesis, for every operator of rank at most two $T \in \mathcal{B}(X)$, there exists an operator $S \in \mathcal{B}(X)$ such that $\phi(S) = T$ and $\Delta^*(T^r \phi(R)T^s) = \Delta^*(\phi(S)^r \phi(R)\phi(S)^s) = \Delta^*(S^r RS^s)$ is a singleton. Its follows, by Lemma 2.4, that $\phi(R)$ is rank one operator.

Conversely, let R be an operator in $\mathcal{B}(X)$ such that $\phi(R)$ is of rank one. Then, for any $T \in \mathcal{B}(X)$, $\Delta^*(T^r RT^s) = \Delta^*(\phi(T)^r \phi(R)\phi(T)^s)$ is a singleton. Applying Lemma 2.4 once again , we conclude that R is of rank one.

Next, let $R = x \otimes f$ be a non-nilpotent rank one operator where $x \in X$ and $f \in X^*$ such that $f(x) \neq 0$. Since ϕ preserves rank one operators, $\phi(R)$ is a rank one operator. Then there existe $y \in X$ and $g \in X^*$ such that $\phi(R) = y \otimes g$. We have

$$\begin{aligned}\{f(x)^{r+s+1}\} &= \Delta^*(f(x)^{r+s} x \otimes f)\\ &= \Delta^*(R^r RR^s)\\ &= \Delta^*(\phi(R)^r \phi(R)\phi(R)^s)\\ &= \Delta^*(g(y)^{r+s} y \otimes g)\\ &= \{g(y)^{r+s+1}\}.\end{aligned}$$

This implies that $g(y) \neq 0$. Thus $\phi(R)$ is a non-nilpotent rank one operator. The converse holds in a similar way and thus ϕ preserves the rank one operators in both directions.

Step 3. ϕ is a linear map and preserves $\mathcal{F}(X)$ in both directions.

Let us first show that ϕ is additive. Let T, S be operators in $\mathcal{B}(X)$ and let $R \in \mathcal{B}(X)$ be a rank one operator, by (2.1) and Lemma 2.5, we have

$$\begin{aligned}\Delta^*(\phi(R)^r \phi(T+S)\phi(R)^s) &= \Delta^*(R^r (T+S)R^s)\\ &= \Delta^*(R^r TR^s) + \Delta^*(R^r SR^s)\\ &= \Delta^*(\phi(R)^r \phi(T)\phi(R)^s) + \Delta^*(\phi(R)^r \phi(S)\phi(R)^s)\\ &= \Delta^*(\phi(R)^r (\phi(T) + \phi(S))\phi(R)^s).\end{aligned}$$

Since R is an arbitrary rank one operator, by Lemma 2.2 and step 2, we get $\phi(T + S) = \phi(T) + \phi(S)$ and ϕ is additive.

Next, we show that ϕ is homogenous. Let $T \in \mathcal{B}(X)$ and λ be a nonzero scalar in $\mathbb{C}$. For every rank one operator $R \in \mathcal{B}(X)$, by (2.4), we have

$$\begin{aligned}\Delta^*(\lambda R^r TR^s) &= \lambda\Delta^*(R^r TR^s)\\ &= \lambda\Delta^*(\phi(R)^r \phi(T)\phi(R)^s)\\ &= \Delta^*(\phi(R)^r \lambda\phi(T)\phi(R)^s).\end{aligned}$$

On the other hand

$$\Delta^*(\lambda R^r T R^s) = \Delta^*(R^r \lambda T R^s) \quad = \quad \Delta^*(\phi(R)^r \phi(\lambda T) \phi(R)^s).$$

This implies that

$$\Delta^*(\phi(R)^r \phi(\lambda T) \phi(R)^s) = \Delta^*(\phi(R)^r \lambda \phi(T) \phi(R)^s).$$

Using Lemma 2.2 and step 2 to conclude that $\phi(\lambda T) = \lambda \phi(T)$. Since $\phi(0) = 0$ then ϕ is homogenous.

For the second assertion, by linearity and the fact that ϕ preserves rank one operators in both directions, ϕ preserves $\mathcal{F}(X)$ in both directions.

Step 4. $\phi(I) = \alpha I$ where $\alpha^{r+s+1} = 1$.

Suppose, by the way of contradiction, that $\phi(I)$ and I are linearly independent. Then there exists a nonzero vector $x \in X$ such that $\phi(I)x$ and x are linearly independent. Let f be a linear functional on X such that $f(x) = 1$ and $f(\phi(I)x) = 0$.

Set $R = x \otimes f$ which is a non-nilpotent rank one operator, by step 2 and since the range of Φ contains all operators of rank at most two, there is a non-nilpotent rank one operator T such that $\phi(T) = R$. Let $T = y \otimes g$ where $y \in$ X and $g \in X^*$ with $g(y) \neq 0$. We have

$$R^r \phi(I) R^s = (x \otimes f)^r \phi(I)(x \otimes f)^s = f(x)^{r+s-2} f(\phi(I)x) x \otimes f = 0.$$

Then

$$\Delta^*(\phi(T)^r \phi(I) \phi(T)^s) = \Delta^*(R^r \phi(I) R^s) = \{0\}.$$

On the other hand,

$$\Delta^*(T^r I T^s) = \Delta^*((y \otimes g)^{r+s}) = \Delta^*(g^{r+s-1}(y) y \otimes g) = \{g^{r+s}(y)\}.$$

Therefore, $g(y) = 0$ which is a contradiction. We conclude that $\phi(I) = \alpha I$ where α is a nonzero scalar in $\mathbb{C}$.

Moreover, by using (2.1) for $T = S = I$, we get

$$\{1\} = \Delta(I) = \Delta^*(\phi(I)^r \phi(I) \phi(I)^s) = \Delta(\alpha^{r+s+1} I) = \{\alpha^{r+s+1}\}.$$

Finally, $\alpha^{r+s+1} = 1$, as desired.

Step 5. ϕ takes the desired form.

Since ϕ is a injective and $\phi(\mathcal{F}(X)) = \mathcal{F}(X)$, then the restriction

$$\phi_{|\mathcal{F}(X)} : \mathcal{F}(X) \to \mathcal{F}(X)$$

is a linear bijective mapping that preserves rank one operators in both directions. It follows, form [16, Theorem 3.3], that one of the assertions holds.

(i) There exist bijective linear operators $A : X \to X$ and $B : X^* \to X^*$ such that

$$\phi(x \otimes f) = Ax \otimes Bf \quad (x \in X, f \in X^*).$$

(ii) There exist bijective linear operators $C : X^* \to X$ and $D : X \to X^*$ such that

$$\phi(x \otimes f) = Cf \otimes Dx \quad (x \in X, f \in X^*).$$

Suppose that (1) holds. For $x \in X$ and $f \in X^*$ we have

$$\begin{aligned} \{f(x)\} &= \Delta^*(x \otimes f) \\ &= \Delta^*(I^r (x \otimes f) I^s) \\ &= \Delta^*(\phi(I)^r \phi(x \otimes f) \phi(I)^s)) \\ &= \Delta^*(\alpha^{r+s} \phi(x \otimes f)) \\ &= \{\alpha^{r+s} Bf(Ax)\}. \end{aligned}$$

Which proves that $f(x) = \alpha^{r+s} Bf(Ax)$. Hence

$$Bf(Ax) = \alpha f(x). \tag{2.6}$$

Next, let us show that A is a bounded operator. To do that, let $(x_n)_n$ be a sequence in X converging to a vector $x \in X$ such that the sequence $(Ax_n)_n$ converge to a vector $y \in X$. For every linear functional f on X we have

$$Bf(y) = Bf(\lim_{n\to\infty} Ax_n) = \lim_{n\to\infty} Bf(Ax_n) = \lim_{n\to\infty} \alpha f(x_n) = \alpha f(x) = Bf(Ax).$$

Since B is bijective, then $f(Ax) = f(y)$ for all $x \in X$ and $f \in X^*$. By Hahn-Banach theorem, we conclude that $Ax = y$. Thus, A is bounded.

Moreover, by the equation (2.6), we see that $\alpha f(x) = A^* Bf(x)$ for all $x \in X$ and $f \in X^*$, which implies that $A^* B = \alpha I_{X^*}$. Therefore, $B = \alpha(A^*)^{-1} = \alpha(A^{-1})^*$. Thus, for every rank one operator $R \in \mathcal{B}(X)$, say $R = x \otimes f$ where $x \in X$ and $f \in X^*$, we get

$$\phi(R) = Ax \otimes Bf = Ax \otimes (\alpha(A^{-1})^*)f = \alpha A(x \otimes f)A^{-1} = \alpha ARA^{-1}.$$

Now, let $T \in \mathcal{B}(X)$ be an arbitrary operator. For every rank one operator $R \in \mathcal{B}(X)$, we have

$$\begin{aligned} \Delta(R^r T R^s) &= \Delta(\phi(R)^r \phi(T) \phi(R)^s) \\ &= \Delta((\alpha ARA^{-1})^r \phi(T) (\alpha ARA^{-1})^s) \\ &= \Delta(\alpha^{r+s} AR^r A^{-1} \phi(T) AR^s A^{-1}) \\ &= \Delta(\alpha^{r+s} R^r A^{-1} \phi(T) AR^s). \end{aligned}$$

We conclude, by Lemma 2.2, that $T = \alpha^{r+s} A^{-1} \phi(T) A$. Therefore $\phi(T) = \alpha ATA^{-1}$ with $\alpha^{r+s+1} = 1$ for all $T \in \mathcal{B}(X)$, as desired.

Suppose, now, that (2) occurs. For every $x \in X$ and $f \in X^*$ we have

$$\begin{aligned} \{f(x)\} &= \Delta^*(x \otimes f) \\ &= \Delta^*(I^r (x \otimes f) I^s) \\ &= \Delta^*(\phi(I)^r \phi(x \otimes f) \phi(I)^s) \\ &= \Delta^*(\alpha^{r+s} \phi(x \otimes f)) \\ &= \Delta^*(\alpha^{r+s} Cf \otimes Dx) \\ &= \{\alpha^{r+s} Dx(Cf)\}. \end{aligned}$$

This implies that $f(x) = \alpha^{r+s} Dx(Cf)$. Thus

$$Dx(Cf) = \alpha f(x). \tag{2.7}$$

By using the closed graph theorem, separately for each of C and D, we conclude that these operators are bounded. Therefore, both $C^* : X^* \to X^{**}$ and $D^* : X^{**} \to X^*$ are invertible.

We will now prove that X is reflexive. For this, if we denote by $\jmath$ the canonical embedding of X to X^{**}. Then $D^* \circ \jmath \circ C = \alpha I_{X^*}$. This implies that $\jmath \circ C = \alpha(D^*)^{-1}$. Since C and $(D^*)^{-1}$ are surjective, we conclude that $\jmath$ is surjective and X is reflexive. By identifying X with X^{**}, we get that $D^* \circ C = \alpha I_{X^*}$ and $C^* \circ D = \alpha I$. This implies that

$$D = \alpha(C^*)^{-1} = \alpha(C^{-1})^*.$$

With the same method as before, we get

$$\phi(R) = \alpha CR^* C^{-1} \text{ for every rank one operator } R \in \mathcal{B}(X).$$

Finally, let T be an arbitrary operator in $\mathcal{B}(X)$. The fact that $\Delta(S) = \Delta(S^*)$ for every finite rank operator $S \in \mathcal{B}(X)$ implies that, for every rank one operator $R \in \mathcal{B}(X)$, we have

$$\begin{aligned}
\Delta(\phi(R)^r\phi(T)\phi(R)^s) &= \Delta(R^rTR^s)\\
&= \Delta((R^*)^sT^*(R^*)^r)\\
&= \Delta(\alpha^{r+s+1}(R^*)^sT^*(R^*)^r)\\
&= \Delta(\alpha^{r+s+1}C(R^*)^sT^*(R^*)^rC^{-1})\\
&= \Delta(\alpha CR^*C^{-1}\alpha CR^*C^{-1}...\alpha CT^*C^{-1}...\alpha CR^*C^{-1})\\
&= \Delta(\phi(R)^r\alpha CT^*C^{-1}\phi(R)^s).
\end{aligned}$$

Since ϕ preserves rank one operators, by Lemma 2.2, we conclude that $\phi(T) = \alpha CT^*C^{-1}$ for all $T \in \mathcal{B}(X)$ with $\alpha^{r+s+1} = 1$. The proof is complete.

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Author information

H. Benbouziane, Department of Mathematics, Faculty of Sciences DharMahraz Fes, University Sidi Mohammed Ben Abdellah, Fes, Morocco.
E-mail: hassane.benbouziane@usmba.ac.ma

A. Daoudi, Department of Mathematics, Faculty of Sciences DharMahraz Fes, University Sidi Mohammed Ben Abdellah, Fes, Morocco.
E-mail: aukacha.daoudi@usmba.ac.ma

M. Ech-Chérif El Kettani, Department of Mathematics, Faculty of Sciences DharMahraz Fes, University Sidi Mohammed Ben Abdellah, Fes, Morocco.
E-mail: mostapha.echcherifelkettani@usmba.ac.ma

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