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e-Zumkeller Numbers and e-Unitary Zumkeller Numbers

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Abstract The aim of this paper is to define e-Zumkeller numbers and e-unitary Zumkeller numbers generalizing the concepts of e-perfect numbers and e-unitary perfect numbers, respectively. Different characteristics of these numbers are investigated and illustrative examples are provided in support of these numbers.

1 Introduction

A positive integer n is called a *perfect number*, if the sum of proper positive divisors of n is equal to n. It is a well known result from the work of Euclid and Euler that the even perfect number is of the form $n = 2^k p$, where $p = 2^{k+1} - 1$ is a Mersenne prime and $k \geq 1$. But no odd perfect number has been found till now and it is an open problem for the mathematicians whether there exist odd perfect numbers. This open problem has motivated many researchers to generalize the concept of perfect numbers and to investigate their characteristics. Researchers have studied various aspects of arithmetical functions and associated concepts like Zumkeller numbers, harmonic divisor numbers, Zumkeller graph labeling([9, 5, 1, 13, 10]).

The concept of Zumkeller number is one of the generalizations of perfect numbers, which is introduced by R. H. Zumkeller, in 2003. A positive integer n is called *Zumkeller number*, if the set of all positive divisors of n can be partitioned into two disjoint subsets of equal sum. It is a point to be noted that every perfect number is a Zumkeller number. In 2013, Peng and Rao [8] proved several results of Zumkeller numbers. There exist odd Zumkeller numbers and 945 is the first odd Zumkeller number. In latter Mahanta, Saikia and Yaqubi [7] characterized all Zumkeller numbers with two distinct prime factors and proved several results connecting Zumkeller numbers with harmonic mean numbers. In [4] Kalita and Saikia introduced the concept of s-Zumkeller numbers and derived some properties of these numbers.

Let $n > 1$ be a positive integer and $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be the prime factorization. A positive divisor d of n is called an *exponential divisor(e-divisor)* of n , if $d = p_1^{b_1} p_2^{b_2} \cdots p_m^{b_m}$ with $b_i | a_i$, for all $i = 1, 2, \cdots, m$. For example e-divisors of $36(= 2^2 \times 3^2)$ are $6(= 2 \times 3)$, $12(= 2^2 \times 3)$, $(18 = 2 \times 3^2)$ and $36(= 2^2 \times 3^2)$. Collaborating the concept of perfect numbers and e-divisors of a positive integer, Straus and Subbarao[11] introduced exponentially perfect or e-perfect numbers. A positive integer n is called an *exponentially perfect or e-perfect number* if the sum of proper e-divisors of n is equal to n, or equivalently if $\sigma^{(e)}(n) = 2n$, where $\sigma^{(e)}(n)$ denotes the sum of e-divisors of n. Some examples of e-perfect numbers are $2^2 \times 3^2, 2^2 \times 3^3 \times 5^2, 2^4 \times 3^2 \times 11^2, 2^4 \times 3^3 \times 5^2 \times 11^2$ etc. Straus and Subbarao proved that there are no odd e-perfect numbers and proved that for each k, the number of e-perfect numbers with k prime factors is finite. Fabrykowski and Subbarao[2] proved that any e-perfect number not divisible by 3 must be divisible by 2^{117}, greater than 10^{664}, and have at least 118 distinct prime factors.

A positive divisor d of n is called *unitary exponential divisor(unitary e-divisor)* of n, if

$d = p_1^{b_1} p_2^{b_2} \cdots p_m^{b_m}$ with $b_i|_* a_i$, for all $i = 1, 2, \cdots, m$, where $b_i|_* a_i$ denotes that b_i is a unitary divisors of a_i. For example, the unitary e-divisors of $n = p^{12}$, with p prime, are $d = p, p^3, p^4, p^{12}$, while its e-divisors are $d = p, p^2, p^3, p^4, p^6, p^{12}$. Tóth and Minculete[12] studied the properties of related arithmetical functions of e-unitary Zumkeller numbers and introduced the notion of exponential unitary perfect(e-unitary perfect) numbers. A positive integer n is called an *e-unitary perfect number* if $\sigma^{(e)*}(n) = 2n$, where $\sigma^{(e)*}(n)$ denotes the sum of the e-unitary divisors of n.

Using the concepts of e-divisors, e-perfect numbers and Zumkeller numbers, we introduce the concept of e-Zumkeller numbers in Section 2 of this paper. We present some examples and results of e-Zumkeller numbers .In Section 3, we introduce the notion of e-unitary Zumkeller numbers and study their characteristics.

2 e-Zumkeller numbers

We first define some arithmetic functions and preliminary results of these functions that are needed in the sequel.

The arithmetic functions $\sigma^{(e)}(n)$ and $\tau^{(e)}(n)$ denote the sum of all positive e-divisors of n and the number of positive e-divisors of n, respectively.

Lemma 2.1. *[11] Let $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be a positive integer. Then*

$$\tau^{(e)}(n) = \tau(a_1)\tau(a_2)\cdots\tau(a_m).$$

Also,

$$\sigma^{(e)}(n) = \prod_{i=1}^{m} \sigma^{(e)}(p_i^{a_i}) = \prod_{j=1}^{m} (\sum_{b_j|a_j} p_i^{b_j}).$$

Now we define e-Zumkeller numbers as follows:

Definition 2.2. A positive integer n is called an *exponential Zumkeller number (or e-Zumkeller number)* if the set of exponential divisors (e-divisors) of n can be partitioned into two subsets of equal sum.

Example 2.3. Some e-Zumkeller numbers are 36, 180, 252, 396, 468, 612, 684, 828, 900, 1044, 1116, 1260, 1332 etc.

Proposition 2.4. *If n is an e-Zumkeller number, then $\sigma^{(e)}(n)$ is even and $\sigma^{(e)}(n) \geq 2n$.*

Proposition 2.5. *A positive integer n is an e-Zumkeller number if and only if $\frac{\sigma^{(e)}(n)}{2} - n$ is either zero or the sum of distinct proper e-divisors of n.*

Proof. Let n be an e-Zumkeller number with e-Zumkeller partition $\{A, B\}$. Without loss of generality, we may assume that $n \in A$.

Case1 If A does not contain any element other than n, then $\frac{\sigma^{(e)}(n)}{2} = n$ and hence $\frac{\sigma^{(e)}(n)}{2} - n$ is zero.

Case2 If A contains elements other than n, then the sum of remaining elements of A is $\frac{\sigma^{(e)}(n)}{2} - n$. Hence $\frac{\sigma^{(e)}(n)}{2} - n$ is the sum of distinct proper e-divisors of n.

Conversely, Suppose $\frac{\sigma^{(e)}(n)}{2} - n$ is zero. Then obviously n is an e-Zumkeller number with partition $\{A, B\}$, where $n \in A$ and B contains other e-divisors of n.

Again suppose $\frac{\sigma^{(e)}(n)}{2} - n$ is the sum of distinct proper e-divisors of n. If we augment this set of distinct e-divisors with n, then we get a set of e-divisors of n summing to $\frac{\sigma^{(e)}(n)}{2}$. The complementary set of these e-divisors of n sums to the same value, and so these two sets form an e-Zumkeller partition of n. Hence n is an e-Zumkeller number. □

Proposition 2.6. *Every e-perfect number is an e-Zumkeller number.*

Proof. Let n be an e-perfect number. Then $\sigma^{(e)}(n) = 2n$, which implies $\frac{\sigma^{(e)}(n)}{2} - n = 0$. Then by Proposition 2.5, n is an e-Zumkeller number. □

Remark 2.7. Converse of the above Proposition does not hold. There exist e-Zumkeller numbers which are not e-perfect numbers. For example, 900 is an e-Zumkeller number, but not e-perfect.

For an integer n and a set A of integers, let $nA = \{na : a \in A\}$.

Proposition 2.8. *If n is an e-Zumkeller number and p is a prime with $(n, p) = 1$, then for any positive integer l, np^l is an e-Zumkeller number.*

Proof. Let n be an e-Zumkeller number with e-Zumkeller partition $\{A, B\}$. Let $\tau(l) = k$ and the positive divisors of l are $b_1(= 1), b_2, b_3, \cdots, b_k(= l)$. Therefore $\{(pA) \cup (p^{b_2}A) \cup \cdots \cup (p^{b_k}A), (pB) \cup (p^{b_2}B) \cup \cdots \cup (p^{b_k}B)\}$ be the e-Zumkeller partition of np^l. Hence, np^l is an e-Zumkeller number. □

As a consequence of the above Proposition, we get the following corollary.

Corollary 2.9. *If n is an e-Zumkeller number and m is co-prime to n, then nm is e-Zumkeller number.*

Proposition 2.10. *Let n be a non e-Zumkeller number and p be a prime with $(n, p) = 1$. If np^l is an e-Zumkeller number, for any positive integer l, then $p \leq \sigma^{(e)}(n)$.*

Proof. Since np^l is an e-Zumkeller number, the set of e-divisors of np^l can be partitioned into two disjoint subsets of equal sum. Let $\tau(l) = k$ and $b_1 = 1, b_2, \cdots, b_k = l$ be the positive divisors of l. It is obvious that every e-divisor of np^l can be expressed as dp^{b_i}, for all $i = 1, 2, \cdots, k$, where d is an e-divisor of n. Since np^l is an e-Zumkeller number, these divisors can be partitioned into two equal summed subsets. Therefore, there exist s_i, t_i, each of which is a sum of some e-divisors of n, such that

$$\sum_{i=1}^{k} p^{b_i}(s_i - t_i) = 0,$$

and

$$s_i + t_i = \sigma^{(e)}(n).$$

So, $p|(s_1 - t_1)$. Additionally, $s_1 + t_1 = \sigma^{(e)}(n)$ and n is not an e-Zumkeller number. Therefore $s_1 - t_1 \neq 0$. This implies $|s_1 - t_1| \geq p$. Hence,

$$\sigma^{(e)}(n) = s_1 + t_1 \geq |s_1 - t_1| \geq p.$$

□

Proposition 2.11. *Let the positive e-divisors of n be $a_1 < a_2 < ... < a_k = n$. If $a_{i+1} \leq 2a_i$, for all $1 \leq i < k$ and $\frac{\sigma^{(e)}(n)}{a_1}$ is even, then n is an e-Zumkeller number.*

Proof. Let $b_i = \frac{a_i}{a_1}$, or $-\frac{a_i}{a_1}$, for each i. We explain how we choose the sign of b_i precisely. Then we show that $\sum_{j=1}^{k} b_j = 0$, which implies that $\sum_{j=1}^{k} a_1 b_j = 0$. Therefore it will imply that $\sigma^{(e)}(n)$ can be partitioned into two equal summed subsets. Let us illustrate the process precisely.

Let $b_k = \frac{a_k}{a_1}$, which is positive. So we assign negative sign to b_{k-1}. i.e. $b_{k-1} = -\frac{a_{k-1}}{a_1}$. Since $a_k \leq 2a_{k-1}$, $\frac{a_k}{a_1} \leq \frac{2a_{k-1}}{a_1}$. Therefore $0 \leq b_k + b_{k-1} = \frac{a_k}{a_1} - \frac{a_{k-1}}{a_1} \leq \frac{2a_{k-1}}{a_1} - \frac{a_{k-1}}{a_1} = \frac{a_{k-1}}{a_1}$, i.e $0 \leq b_k + b_{k-1} \leq \frac{a_{k-1}}{a_1}$. Since the current sum $b_k + b_{k-1}$ is positive, we assign the negative sign to b_{k-2}, i.e. $b_{k-2} = -\frac{a_{k-2}}{a_1}$. Then $b_{k-2} \leq b_k + b_{k-1} + b_{k-2} \leq \frac{a_{k-2}}{a_1}$. If $b_k + b_{k-1} + b_{k-2} \geq 0$, then we assign the negative sign to b_{k-3}. Otherwise we assign the positive sign to b_{k-3}.

Let $s_i = \sum_{j=i}^{k} b_j$. In general, the sign assign to b_i is the opposite of the sign of s_{i+1}. Now we show that $|s_i| \leq \frac{a_i}{a_1}$, for $1 \leq i \leq k$.

It is true for $i = k$. We assume that $|s_{i+1}| \leq \frac{a_{i+1}}{a_1}$. Since the sign of b_i is opposite of the sign of s_{i+1}, $|s_i| = ||s_{i+1}| - \frac{a_i}{a_1}|$. Note that $-\frac{a_i}{a_1} \leq |s_{i+1}| - \frac{a_i}{a_1} \leq \frac{a_{i+1}}{a_1} - \frac{a_i}{a_1} \leq \frac{a_i}{a_1}$, since $a_{i+1} \leq a_i$. Therefore, $|s_i| \leq \frac{a_i}{a_1}$. So, for $i = 1$, $|s_1| \leq \frac{a_1}{a_1} = 1$. Here s_1 is obtained by assigning positive or negative sign to each of the terms in $\frac{\sigma^{(e)}(n)}{a_1}$. Again $\frac{\sigma^{(e)}(n)}{a_1}$ is even. Therefore $s_1 = \sum_{j=1}^{k} b_j$ is even. Hence $s_1 = \sum_{j=1}^{k} b_j = 0$. This implies that $\sum_{j=1}^{k} a_1 b_j = 0$. Hence the set of the positive e-divisors of n can be partitioned into two subsets of equal sum. Therefore, n is an e-Zumkeller number. □

Proposition 2.12. *If n is an e-Zumkeller number, then $\tau^{(e)}(n) > 3$.*

Proof. Let n be an e-Zumkeller number. Clearly, $\tau^{(e)}(n) > 2$.

Suppose $\tau^{(e)}(n) = 3$. Let the prime factorization of n be $\prod_{i=1}^{m} p_i^{k_i}$. By Lemma 2.1, $\tau^{(e)}(n) = \tau^{(e)}(\prod_{i=1}^{m} p_i^{k_i}) = \prod_{i=1}^{m} \tau(k_i)$. Therefore, $\tau^{(e)}(n) = 3$ implies $\prod_{i=1}^{m} \tau(k_i) = 3$. In this case, $\tau(k_j) = 3$, for some j and for all other i, $\tau(k_i) = 1$ and say $n = p_1p_2 \dots p_j^{k_j} \dots p_m$ such that $\tau(k_j) = 3$. Let the divisors of k_j be $1, d$ and k_j. Therefore, the e-divisors of n are $p_1p_2 \dots p_j \dots p_m$, $p_1p_2 \dots p_j^{d} \cdots p_m$ and $p_1p_2 \dots p_j^{k_j} \dots p_m$. Since n is an e-Zumkeller number, $\sigma^{(e)}(n) \geq 2n$, this implies that $1 + p_j^{d-1} + p_j^{k_j-1} \geq 2p_j^{k_j-1}$, which is a contradiction. Therefore $\tau^{(e)}(n) > 3$. □

Proposition 2.13. *There is no e-Zumkeller number of the form $p_1^{\alpha}p_2^{\beta}$ except* $36(= 2^2 \times 3^2)$*, where p_1 and p_2 are distinct odd primes and α and β are primes.*

Proof. Let $n = p_1^{\alpha}p_2^{\beta}$ be an e-Zumkeller number, where p_1 and p_2 are distinct odd primes and α and β are primes. Then $\sigma^{(e)}(p_1^{\alpha}p_2^{\beta}) \geq 2p_1^{\alpha}p_2^{\beta}$. Since α and β are primes, $\sigma^{(e)}(p_1^{\alpha}p_2^{\beta}) = (p_1 + p_1^{\alpha})(p_2 + p_2^{\beta})$. This implies that $(p_1 + p_1^{\alpha})(p_2 + p_2^{\beta}) \geq 2p_1^{\alpha}p_2^{\beta}$ and simplifying this we have $1 + p_1^{\alpha-1} + p_2^{\beta-1} \geq p_1^{\alpha-1}p_2^{\beta-1}$, which is possible if and only if $p_1 = 2$, $p_2 = 3$, $\alpha = \beta = 2$. Again, $2^2 \times 3^2 = 36$ is an e-Zumkeller number. Hence there is no e-Zumkeller number of the form $p_1^{\alpha}p_2^{\beta}$, except 36. □

Proposition 2.14. *There is no e-Zumkeller number of the form $2^{\alpha} \prod_{i=1}^{m} p_i$, where p_i's are odd primes and α is a positive integer.*

Proof. Suppose, $n = 2^{\alpha} \prod_{i=1}^{m} p_i$ is an e-Zumkeller number, where α is a positive integer. Let $\tau(\alpha) = k$ and divisors of α be $d_1 = 1, d_2, d_3, \dots, d_k = \alpha$. Then $\sigma^{(e)}(2^{\alpha} \prod_{i=1}^{m} p_i) = (2 + 2^{d_2} + 2^{d_3} + \cdots + 2^{\alpha}) \prod_{i=1}^{m} p_i$. Since n is an e-Zumkeller number, $\sigma^{(e)}(2^{\alpha} \prod_{i=1}^{m} p_i) \geq 2(2^{\alpha} \prod_{i=1}^{m} p_i)$. This implies that $(2 + 2^{d_2} + 2^{d_3} + \cdots + 2^{\alpha}) \prod_{i=1}^{m} p_i \geq 2(2^{\alpha} \prod_{i=1}^{m} p_i)$ and simplifying this we have $1 + 2^{d_2-1} + 2^{d_3-1} + \cdots + 2^{\alpha-1} \geq 2^{\alpha}$, which is impossible. Hence there does not exist e-Zumkeller number of the form $2^{\alpha} \prod_{i=1}^{m} p_i$. □

Lemma 2.15. *For any prime number p,*

$$1 + \frac{1}{p^2} + \frac{1}{p^3} < 1 + \frac{1}{p}.$$

In [11], the authors have proved that there is no odd e-perfect numbers. Now we investigate whether there exists odd e-Zumkeller numbers.

Proposition 2.16. *If $n = p_1^{k_1}p_2^{k_2} \dots p_m^{k_m}$ is an odd e-Zumkeller number, then*

$$2 \leq \prod_{i=1}^{m} \frac{p_i + 1}{p_i}.$$

Proof. Let $n = p_1^{k_1}p_2^{k_2} \dots p_m^{k_m}$ be an odd number , where p_i's are odd primes. Then $\sigma^{(e)}(n) = \sigma^{(e)}(p_1^{k_1})\sigma^{(e)}(p_2^{k_2}) \dots \sigma^{(e)}(p_m^{k_m})$.

Let n be an e-Zumkeller number. Then $\sigma^{(e)}(n)$ is even. This implies that either some $\sigma^{(e)}(p_i^{k_i})$'s are odd and at least one $\sigma^{(e)}(p_i^{k_i})$'s is even or all $\sigma^{(e)}(p_i^{k_i})$'s are even.

For odd values of $\sigma^{(e)}(p_i^{k_i})$, we have k_i's are perfect square and then we have

$$\frac{\sigma^{(e)}(p_i^{k_i})}{p_i^{k_i}} \leq \frac{\sigma^{(e)}(p_i^{4})}{p_i^{4}} = 1 + \frac{1}{p_i^2} + \frac{1}{p_i^3} < \frac{p_i + 1}{p_i}, \text{(using lemma 2.15)}.$$

Again for even values of $\sigma^{(e)}(p_i^{k_i})$, we have

$$\frac{\sigma^{(e)}(p_i^{k_i})}{p_i^{k_i}} \leq \frac{\sigma^{(e)}(p_i^{2})}{p_i^{2}} = \frac{p_i + 1}{p_i}.$$

Thus, for both cases

$$\frac{\sigma^{(e)}(n)}{n} = \prod_{i=1}^{m} \frac{\sigma^{(e)}(p_i^{k_i})}{p_i^{k_i}} \le \prod_{i=1}^{m} \frac{p_i+1}{p_i}.$$

Again, if n is an e-Zumkeller numbers, then $2n \le \sigma^{(e)}(n)$. Hence, n is an odd e-Zumkeller number, which implies that

$$2 \le \prod_{i=1}^{m} \frac{p_i+1}{p_i}.$$

□

Corollary 2.17. *If n is an odd e-Zumkeller number, then n contains at least 5 distinct odd primes.*

Proof. Let $n = \prod_{i=1}^{m} p_i^{k_i}$ be an odd e-Zumkeller number. Then Proposition 2.16 implies that

$$2 \le \prod_{i=1}^{m} \frac{p_i+1}{p_i}. \tag{2.1}$$

If $m < 5$, then we have

$$\prod_{i=1}^{m} \frac{p_i+1}{p_i} \le \frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times \frac{12}{11} < 2, \text{ which contradicts (2.1).}$$

Therefore $m \ge 5$. Hence n contains at least 5 distinct primes. □

Corollary 2.18. *If $n = \prod_{i=1}^{5} p_i^{k_i}$ is an odd e-Zumkeller number, then $k_i \neq 1$ for all i.*

Proof. Let $n = \prod_{i=1}^{5} p_i^{k_i}$ be an odd e-Zumkeller number. Suppose for some j, $k_j = 1$. Without loss of generality, let $k_5 = 1$. Then $\frac{\sigma^{(e)}(p_5^{k_5})}{p_5^{k_5}} = 1$. Therefore

$$\frac{\sigma^{(e)}(n)}{n} = \prod_{i=1}^{4} \frac{\sigma^{(p_i^{k_i})}}{p_i^{k_i}} \le \prod_{i=1}^{4} \frac{p_i+1}{p_i} < 2, \text{ which is a contradiction.}$$

Therefore for all i, $k_i \neq 1$ □

Corollary 2.19. *The smallest odd e-Zumkeller number is* $3^2 \times 5^2 \times 7^2 \times 11^2 \times 13^2$.

Proof. Let n be an odd e-Zumkeller number. Then by Corollary 2.17, n contains at least 5 distinct odd primes. Again by Corollary 2.18, power of each distinct prime is greater than 1. Hence possible smallest odd e-Zumkeller number is $3^2 \times 5^2 \times 7^2 \times 11^2 \times 13^2$.

Again $3^2 \times 5^2 \times 7^2 \times 11^2 \times 13^2$ is an e-Zumkeller number with the e-Zumkeller partition $\{A, B\}$, where

$A = \{3\times5\times7\times11\times13^2, 3\times5^2\times7\times11^2\times13^2, 3\times5^2\times7^2\times11^2\times13, 3^2\times5^2\times7^2\times11^2\times13^2\}$

and

$B = \{3 \times 5 \times 7 \times 11 \times 13, 3 \times 5 \times 7 \times 11^2 \times 13, 3 \times 5 \times 7 \times 11^2 \times 13^2, 3 \times 5 \times 7^2 \times 11 \times 13, 3 \times 5 \times 7^2 \times 11 \times 13^2, 3 \times 5 \times 7^2 \times 11^2 \times 13, 3 \times 5 \times 7^2 \times 11^2 \times 13^2, 3 \times 5^2 \times 7 \times 11 \times 13, 3 \times 5^2 \times 7 \times 11 \times 13^2, 3 \times 5^2 \times 7 \times 11^2 \times 13, 3 \times 5^2 \times 7^2 \times 11 \times 13, 3 \times 5^2 \times 7^2 \times 11 \times 13^2, 3 \times 5^2 \times 7^2 \times 11^2 \times 13^2, 3^2 \times 5 \times 7 \times 11 \times 13, 3^2 \times 5 \times 7 \times 11 \times 13^2, 3^2 \times 5 \times 7 \times 11^2 \times 13, 3^2 \times 5 \times 7 \times 11^2 \times 13^2, 3^2 \times 5 \times 7^2 \times 11 \times 13, 3^2 \times 5 \times 7^2 \times 11 \times 13^2, 3^2 \times 5 \times 7^2 \times 11^2 \times 13, 3^2 \times 5 \times 7^2 \times 11^2 \times 13^2, 3^2 \times 5^2 \times 7 \times 11 \times 13, 3^2 \times 5^2 \times 7 \times 11 \times 13^2, 3^2 \times 5^2 \times 7 \times 11^2 \times 13, 3^2 \times 5^2 \times 7 \times 11^2 \times 13^2, 3^2 \times 5^2 \times 7^2 \times 11 \times 13, 3^2 \times 5^2 \times 7^2 \times 11 \times 13^2, 3^2 \times 5^2 \times 7^2 \times 11^2 \times 13\}$,

and the sum of elements of each subset is 242161920.

Hence, $3^2 \times 5^2 \times 7^2 \times 11^2 \times 13^2$ is the smallest odd e-Zumkeller number. □

Proposition 2.20. *Let $\alpha_0, \ldots, \alpha_k$ be non-negative integers. If α is an e-Zumkeller number, $a|\alpha_i$ for all $1 \le i \le k$, and $(\alpha, \alpha_0) = 1$, then*

$$P(n) = \alpha\alpha_0 + \alpha\alpha_1 n + \cdots + \alpha\alpha_k n^k$$

is an e-Zumkeller number, for all non-negative integers n.

Proof. Let α be an e-Zumkeller number, $a|\alpha_i$ for all $1 \leq i \leq k$, and $(\alpha, \alpha_0) = 1$. Then for all positive integers n,

$$(\alpha, \alpha_0 + \alpha_1 n + \alpha_2 n^2 + \cdots + \alpha_k n^k) = 1.$$

Therefore by Corollary 2.9, we have

$$\alpha(\alpha_0 + \alpha_1 n + \alpha_2 n^2 + \cdots + \alpha_k n^k) = \alpha\alpha_0 + \alpha\alpha_1 n + \cdots + \alpha\alpha_k n^k$$

is an e-Zumkeller number. □

3 e-Unitary Zumkeller numbers

The arithmetic function $\tau^{(e)*}(n)$ denotes the number of e-unitary divisors of n and $\sigma^{(e)*}(n)$ denotes the sum of the e-unitary divisors of n. The function $\omega(n)$ denotes the number of distinct prime divisors of n.

Lemma 3.1. *[12] If $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, where $p_1, p_2, \cdots, p_m$ are primes, then*

(i) $\tau^{(e)*}(n) = \tau^*(a_1)\tau^*(a_2)\cdots\tau^*(a_m) = 2^{\omega(a_1)+\omega(a_2)+\cdots+\omega(a_m)}$*;*

(ii) $\sigma^{(e)*}(n) = \left(\sum_{d_1|_* a_1} p_1^{d_1}\right)\left(\sum_{d_2|_* a_2} p_2^{d_2}\right)\cdots\left(\sum_{d_m|_* a_m} p_m^{d_m}\right)$.

We note that for any $n > 1$, the values of $\tau^{(e)*}(n)$ and $\sigma^{(e)*}(n)$ are even.

Definition 3.2. An integer n is called an *e-square free number* if all the exponents of the prime factorization of n are square free. By convention 1 is square free number.

Lemma 3.3. *[12] If n is an e-square free number, then a divisor d of n is a unitary e-divisor if and only if d is an e-divisor of n.*

Now we define e-unitary Zumkeller numbers:

Definition 3.4. A positive integer n is called an *e-unitary Zumkeller number* if the set of exponential unitary divisors(e-unitary divisors) of n can be partitioned into two subsets of equal sum.

Proposition 3.5. *An e-square free number n is e-unitary Zumkeller number if and only if n is e-Zumkeller number.*

Proposition 3.6. *A positive integer n is an e-unitary Zumkeller number if and only if $\frac{\sigma^{(e)*}(n)}{2} - n$ is either zero or the sum of distinct proper e-unitary divisors of n.*

Proof. The proof is similar as the proof of Proposition 2.5. □

Using Proposition 3.6 and following the proof of Proposition 2.6, we obtain the following Proposition.

Proposition 3.7. *Every e-unitary perfect number is an e-unitary Zumkeller number.*

Proposition 3.8. *If n is an e-unitary Zumkeller number and p is a prime with $(n, p) = 1$, then for any positive integer l, np^l is an e-unitary Zumkeller number.*

Proof. The proof is similar as the proof of Proposition 2.10. □

Corollary 3.9. *If n is an e-unitary Zumkeller number and m is co-prime to n, then nm is an e-unitary Zumkeller number.*

Proposition 3.10. *If n is an e-unitary Zumkeller number, then $\tau^{(e)*}(n) > 3$.*

Proof. The proof is similar as the proof of Proposition 2.12 □

Following the proof of Proposition 2.16, we obtain a similar result for e-unitary Zumkeller numbers as follows:

Proposition 3.11. *If $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ is an odd e-unitary Zumkeller number, then*

$$2 \leq \prod_{i=1}^{m} \frac{p_i + 1}{p_i}.$$

Remark 3.12. The smallest e-Zumkeller number , which is not e-unitary Zumkeller number, is $2^4 \times 3^2 \times 11^2 = 17424$.

Remark 3.13. By Corollary 2.19, the smallest odd e-Zumkeller number is $3^2 \times 5^2 \times 7^2 \times 11^2 \times 13^2$, which is an e-square free number. Therefore by Proposition 3.5, the smallest odd e-unitary Zumkeller number is $3^2 \times 5^2 \times 7^2 \times 11^2 \times 13^2$.

Remark 3.14. The smallest e-unitary Zumkeller number, which is not e-squarefree, is $2^4 \times 3^2 \times 5^2 \times 7^2 \times 11^2 = 21344400$.

Proposition 3.15. *Let $\alpha_0, \dots, \alpha_k$ be non-negative integers. If α is an e-unitary Zumkeller number, $a|\alpha_i$ for all $1 \leq i \leq k$, and $(\alpha, \alpha_0) = 1$, then*

$$P(n) = \alpha\alpha_0 + \alpha\alpha_1 n + \cdots + \alpha\alpha_k n^k$$

is an e-unitary Zumkeller number, for all non-negative integer n.

Proof. The proof is similar to the proof of Proposition 2.20. □

4 Conclusion

In this paper we have defined e-Zumkeller numbers and have studied certain properties. We have noticed that though there does not exist any odd e-perfect numbers but there exists odd e-Zumkeller numbers and $3^2 \times 5^2 \times 7^2 \times 11^2 \times 13^2$ is the smallest odd e-Zumkeller number. In future we can investigate more results of these numbers and can generalize e-Zumkeller numbers partitioning the set of e-divisors into k number of distinct subsets. We can also study graph labeling using the notion of e-Zumkeller numbers.

References

[1] B. J. Balamurugan, K. Thirusangu and D. G. Thomas, *Zumkeller labeling algorithms for complete bipartite graphs and wheel graphs*, Adv. Intell. Syst. Comput., **324**, 405–413, (2015).

[2] J. Fabrykowski and M.V. Subbarao, *On e-perfect numbers not divisible by 3*, Nieuw Arch. Wiskd., **4(4)**, 165–173, (1986).

[3] P. Hagis, Jr., *Some results concerning exponential divisors*, Int. J. Math.& Math. Sci., **11**, 343–349, (1988).

[4] J. Kalita and H. K. Saikia, *A Study on generalized Zumkeller numbers*, Integers, **24**, #A95, (2024).

[5] J. Kalita and H. K. Saikia,*A note on near harmonic divisor number and associated concepts*, Palest. J. Math., **13(4)**, 693-700, (2024).

[6] I. Kátai and M. V. Subbarao, *On the distribution of exponential divisors*, Annales Univ. Sci. Budapest., Sect. Comp., **22**, 161–180, (2003).

[7] P. J. Mahanta, M. P. Saikia and D. Yaqubi, *Some Properties of Zumkeller Numbers and k-Layered Numbers*, J. Number Theory, **217**, 218–236, (2020).

[8] Y. Peng and K. P. S. Bhaskara Rao, *On Zumkeller numbers*, J. of Number Theory, **133**, 1135–1155, (2013).

[9] H. Potodia and H. K. Saikia, *A note on m-Zumkeller cordial labbeling of graphs*, Proyecciones, **42(1)**, 65-84, (2023).

[10] R. Rajendra, R. Kemparaju and P. S. K. Reddy,*Arithmetic function graph of a finite group*, Palest. J. Math., **11(2)**, 488-495, (2022).

[11] E. G. Straus and M. V. Subbarao,*On exponential divisors*, Duke Math. J., **41** , 465-471, (1974).

[12] L. Tóth and N. Minculete, *Exponential unitary divisors*, Ann. Univ. Sci. Budapest. Sect. Comput., **35**, (2011).

[13] K. L. Verma ,*On an arithmetical functions involving general exponential*, Palest. J. Math., **11(2)**, 496-504, (2022).

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