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# Balanced Hermitian structures on twisted cartesian products

M. W. Mansouri and A. Oufkou

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**Abstract** We study Hermitian structures on the twisted cartesian product $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathrm{J}, \mathcal{K})$ of two Hermitian Lie algebras according to two representations $\rho_1$ and $\rho_2$. We give the conditions on $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathrm{J}, \mathcal{K})$ to be balanced and locally conformally balanced. As an application, we classify six-dimensional balanced Hermitian twisted cartesian products Lie algebras.

## 1 Introduction

The twisted product structure of Lie algebras defined by means of linear representations is a well known construction which can be regarded as the generalization of the semidirect product of algebras, see for instance [5]. By using this construction we can obtain examples of some special Hermitian metrics in Lie algebras.

In literature, a Hermitian structure on a $2n$-dimensional smooth manifold $M$ is a pair $(\mathrm{J}, \mathcal{K})$ where J is an integrable almost complex structure which is compatible with a Riemannian metric $\mathcal{K}$ on $M$, namely $\mathcal{K}(\mathrm{J}.\,,\mathrm{J}.) = \mathcal{K}(.\,,.)$. The fundamental form is given by $\omega(.\,,.) = \mathcal{K}(\mathrm{J}.\,,.)$ and the Lee form is defined by $\theta = \mathrm{J}\mathrm{d}^*\omega = -\mathrm{d}^*\omega \circ \mathrm{J}$. A fundamental class of Hermitian metrics is provided by the Kähler metrics by means $\mathrm{d}\omega = 0$. In literature, many generalizations of the Kähler condition have been introduced. Indeed, $(M, \mathrm{J}, \mathcal{K})$ is called:

- Balanced if $\theta = 0$ .
- Locally conformally balanced (shortly, LCB) if $\mathrm{d}\theta = 0$.
- Locally conformally Kähler (shortly, LCK) if $\mathrm{d}\omega = \theta \wedge \omega$ where the Lee form $\theta$ is a closed 1-form.

For general results about these generalized Kähler metrics, we refer the reader to [3], [2], [4].

The aim of this paper is to study Hermitian structures on the twisted cartesian product $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathrm{J}, \mathcal{K})$. We give the conditions on $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathrm{J}, \mathcal{K})$ to be balanced, locally conformally balanced and Kählerian.

The article is organized as follows: In section 2 we give the framework of the Hermitian Lie algebra $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathrm{J}, \mathcal{K})$. In particular, we provide a formula of the Lee form $\theta$ and we investigate it to know when the Hermitian twisted cartesian product structure becomes balanced, LCB and Kählerian. We also give a method to construct balanced Hermitian Lie algebras from two Hermitian Lie algebras and apply it in section 3.

**Notation.** Using the Salamon notation, we write structure equations for Lie algebras: e.g. $\mathfrak{rr}_{3,1} = (0, -12, -13, 0)$ fixing a coframe $(e^1, e^2, e^3, e^4)$ for $\mathfrak{rr}^*_{3,1}$ that means $\mathrm{d}e^1 = \mathrm{d}e^4 = 0$ and $\mathrm{d}e^2 = -e^1 \wedge e^2 = -e^{12}$, $\mathrm{d}e^3 = -e^1 \wedge e^3 = -e^{13}$.

## 2 Hermitian twisted cartesian products

Let $(\mathfrak{g}_1, \mathrm{J}_1, \mathrm{k}_1)$ and $(\mathfrak{g}_2, \mathrm{J}_2, \mathrm{k}_2)$ be two Hermitian Lie algebras for which there exist two linear Lie algebra representations

$$\rho_1 : \mathfrak{g}_1 \longrightarrow Der(\mathfrak{g}_2) \quad \text{and} \quad \rho_2 : \mathfrak{g}_2 \longrightarrow Der(\mathfrak{g}_1).$$

We will say that $(\rho_1, \rho_2)$ is a representation compatible couple if

$$\rho_1(\rho_2(a)x)b = \rho_1(\rho_2(b)x)a, \tag{2.1}$$

$$\rho_2(\rho_1(x)a)y = \rho_2(\rho_1(y)a)x, \tag{2.2}$$

for any $x$, $y \in \mathfrak{g}_1$ and $a$, $b \in \mathfrak{g}_2$.

The non-zero Lie brackets $[.,.]$ on $\mathfrak{g}_{(\rho_1,\rho_2)}$ are defined on the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ (direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$) by

$$\begin{aligned} [x,y] &= [x,y]_1, && x, y \in \mathfrak{g}_1 \\ [a,b] &= [a,b]_2, && a, b \in \mathfrak{g}_2 \\ [x,a] &= \rho_1(x)a - \rho_2(a)x, && x \in \mathfrak{g}_1,\ a \in \mathfrak{g}_2. \end{aligned}$$

A direct calculation shows that $(\mathfrak{g}_{(\rho_1,\rho_2)}, [\,.\,,\,.\,])$ is a Lie algebra if and only if $\rho_1$ and $\rho_2$ are compatible. The Lie algebra $(\mathfrak{g}_{(\rho_1,\rho_2)}, [\,.\,,\,.\,])$ is called twisted cartesian product of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ according to the representation compatible couple $(\rho_1, \rho_2)$ also noted $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ (see [5] and [6] for more details).

**Remark 2.1.** If $\rho_2 = 0$ then the twisted cartesian product becomes the semi-direct product of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ by $\rho_1$ and vice-versa.

We let $\mathcal{K}$ be the scalar product on $\mathfrak{g}_{(\rho_1,\rho_2)}$ which is defined by $\mathcal{K}(x+a, y+b) = \mathrm{k}_1(x,y) + \mathrm{k}_2(a,b)$, the almost-complex structure J on $\mathfrak{g}_{(\rho_1,\rho_2)}$ verify $\mathrm{J}(x+a) = \mathrm{J}_1(x) + \mathrm{J}_2(a)$ for $x, y \in \mathfrak{g}_1$ and $a, b \in \mathfrak{g}_2$. Clearly, J and $\mathcal{K}$ are compatible i.e., $\mathcal{K}(\mathrm{J}.,\mathrm{J}.) = \mathcal{K}(.,.)$. The following proposition gives a necessary and sufficient condition for J to be integrable.

**Proposition 2.2.** *Let* $(\mathfrak{g}_1, \mathrm{J}_1, \mathrm{k}_1)$ *and* $(\mathfrak{g}_2, \mathrm{J}_2, \mathrm{k}_2)$ *be two Hermitian Lie algebras and let* $(\rho_1, \rho_2)$ *a representation compatible couple. Then,* $(\mathfrak{g}_{(\rho_1,\rho_2)}, \omega, \mathrm{J})$ *is a* ***Hermitian*** *Lie algebra if and only if*

$$[\rho_1(\mathrm{J}_1(x)), \mathrm{J}_2] = \mathrm{J}_2 \circ \rho_1(x) \circ \mathrm{J}_2 + \rho_1(x), \qquad \forall x \in \mathfrak{g}_1 \tag{2.3}$$

$$[\rho_2(\mathrm{J}_2(a)), \mathrm{J}_1] = \mathrm{J}_1 \circ \rho_2(a) \circ \mathrm{J}_1 + \rho_2(a), \qquad \forall a \in \mathfrak{g}_2. \tag{2.4}$$

*Proof.* Using that $\mathrm{J}_1$ and $\mathrm{J}_2$ are integrable and the bilinearity of the Nijenhuis operator, the completeness of J reduces to $N_J(x,a) = 0$ for all $x \in \mathfrak{g}_1$ and $a \in \mathfrak{g}_2$.

$$\begin{aligned} N_{\mathrm{J}}(x,a) &= [\mathrm{J}_1 x, \mathrm{J}_2 a] - [x,a] - \mathrm{J}([\mathrm{J}_1 x, a] + [x, \mathrm{J}_2 a]) \\ &= \rho_1(\mathrm{J}_1 x)\mathrm{J}_2 a - \rho_2(\mathrm{J}_2 a)\mathrm{J}_1 x - \rho_1(x)a + \rho_2(a)x \\ &\quad - \mathrm{J}_2(\rho_1(\mathrm{J}_1 x)a) + \mathrm{J}_1(\rho_2(a)\mathrm{J}_1 x) - \mathrm{J}_2(\rho_1(x)\mathrm{J}_2 a) + \mathrm{J}_1(\rho_2(\mathrm{J}_2 a)x) \\ &= \Big([\mathrm{J}_1, \rho_2(\mathrm{J}_2 a)]x + \rho_2(a)x + \mathrm{J}_1(\rho_2(a)\mathrm{J}_1 x)\Big) + \Big([\rho_1(\mathrm{J}_1 x), \mathrm{J}_2]a \\ &\quad - \rho_1(x)a - \mathrm{J}_2(\rho_1(x)\mathrm{J}_2 a)\Big) \end{aligned}$$

and the result follows. □

**Remark 2.3.** The conditions (2.3) and (2.4) are satisfied if $\rho_1(x)$ and $\mathrm{J}_2$ (resp; $\rho_2(a)$ and $\mathrm{J}_1$) commute, for all $x \in \mathfrak{g}_1$ (resp; $a \in \mathfrak{g}_2$).

The Hermitian Lie algebra $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathrm{J}, \mathcal{K})$ is called Hermitian twisted cartesian product of two Hermitian Lie algebras according to two representations $\rho_1$ and $\rho_2$, or simply Hermitian twisted product if there is no confusion. Moreover, we consider in what follows that $\rho_1(x)$ and $\mathrm{J}_2$ (resp; $\rho_2(a)$ and $\mathrm{J}_1$) commute for all $x \in \mathfrak{g}_1$ (resp; $a \in \mathfrak{g}_2$).

**Lemma 2.4.** *The Levi-Civita product associated to* $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathcal{K})$ *is given by:*

$$\nabla_x a = \frac{1}{2}((\rho_1(x) - \rho_1^*(x))a - (\rho_2(a) + \rho_2^*(a))x),$$
$$\nabla_a x = \frac{1}{2}((\rho_2(a) - \rho_2^*(a))x - (\rho_1(x) + \rho_1^*(x))a),$$
$$\nabla_x y = \overset{1}{\nabla}_x y, \quad \nabla_a b = \overset{2}{\nabla}_a b,$$

*for any* $x, y \in \mathfrak{g}_1$ *and* $a, b \in \mathfrak{g}_2$*,*

*Proof.* The Koszul formula for the Levi-Civita connection in the invariant setting, is given by:

$$2\mathcal{K}(\nabla_{(x,a)}(y,b),(z,c)) = \mathcal{K}([(x,a),(y,b)],(z,c)) - \mathcal{K}([(y,b),(z,c)],(x,a)) - \mathcal{K}([(x,a),(z,c)],(y,b)).$$

For $(x,a) = (x,0)$ and $(y,b) = (y,0)$ and $(z,c) = (z,0)$, we have

$$\begin{aligned}2\mathcal{K}(\nabla_{(x,0)}(y,0),(z,0)) &= \mathcal{K}([(x,0),(y,0)],(z,0)) - \mathcal{K}([(y,0),(z,0)],(x,0)) \\ &\quad - \mathcal{K}([(x,0),(z,0)],(y,0)) \\ &= \mathrm{k}_1([x,y]_1,z) - \mathrm{k}_1([y,z]_1,x) - \mathrm{k}_1([x,z]_1,y) \\ 2\mathcal{K}(\nabla_{(x,0)}(y,0),(z,0)) &= 2\mathrm{k}_1(\overset{1}{\nabla}_x y, z)\end{aligned}$$

and therefore

$$\mathcal{K}(\nabla_x y, z) = \mathrm{k}_1(\overset{1}{\nabla}_x y, z) = \mathcal{K}(\overset{1}{\nabla}_x y, z).$$

Hence $\nabla_x y = \overset{1}{\nabla}_x y$. A similar calculation show that

$$2\mathcal{K}(\nabla_{(0,a)}(0,b),(0,c)) = 2\mathrm{k}_2(\overset{2}{\nabla}_a b, c)$$

and therefore

$$\mathcal{K}(\nabla_a b, c) = \mathrm{k}_2(\overset{2}{\nabla}_a b, c) = \mathcal{K}(\overset{2}{\nabla}_a b, c).$$

Hence $\nabla_a b = \overset{2}{\nabla}_a b$.

For $(x,a) = (x,0)$ and $(y,b) = (0,b)$, we have

$$\begin{aligned}2\mathcal{K}(\nabla_{(x,0)}(0,b),(z,c)) &= \mathcal{K}([(x,0),(0,b)],(z,c)) - \mathcal{K}([(0,b),(z,c)],(x,0)) \\ &\quad - \mathcal{K}([(x,0),(z,c)],(0,b)) \\ &= \mathcal{K}(\rho_1(x)b - \rho_2(b)x,(z,c)) - \mathrm{k}_1(\rho_2(b)z,x) \\ &\quad - \mathrm{k}_2(\rho_1(x)c,b) \\ &= \mathrm{k}_2(\rho_1(x)b,c) - \mathrm{k}_1(\rho_2(b)x,z) - \mathrm{k}_1(\rho_2(b)z,x) \\ &\quad - \mathrm{k}_2(\rho_1(x)c,b) \\ &= \mathrm{k}_2((\rho_1(x) - \rho_1^*(x))b,c) - \mathrm{k}_1((\rho_2(b) + \rho_2^*(b))x,z).\end{aligned}$$

Replacing $(z,c)$ by $(z,0)$, then by $(0,c)$ we find

$$2\nabla_x b = (\rho_1(x) - \rho_1^*(x))b - (\rho_2(b) + \rho_2^*(b))x.$$

A similar calculation show that

$$2\nabla_a x = (\rho_2(a) - \rho_2^*(a))x - (\rho_1(y) + \rho_1^*(x))a$$

and we get the desired result. □

**Lemma 2.5.** *The differential of the fundamental form $\omega$ associated to the twisted cartesian product $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathrm{J}, \mathcal{K})$ is given by:*

$$\mathrm{d}\omega(x,y,c) = -\omega_1\big((\rho_2^*(c)+\rho_2(c))x,y\big),$$
$$\mathrm{d}\omega(x,b,c) = -\omega_2\big((\rho_1^*(x)+\rho_1(x))b,c\big),$$
$$\mathrm{d}\omega(x,y,z) = \mathrm{d}\omega_1(x,y,z) \quad \textit{and} \quad \mathrm{d}\omega(a,b,c) = \mathrm{d}\omega_2(a,b,c),$$

*for any $x,y,z \in \mathfrak{g}_1$ and $a,b,c \in \mathfrak{g}_2$.*

*Proof.* Using the exterior derivative of the 2-form $\omega$, we have

$$\begin{aligned}
\mathrm{d}\omega(x,y,c) &= -\omega([x,y],c) + \omega([x,c],y) - \omega([y,c],x)\\
&= -\omega([x,y]_1,c) + \omega(\rho_1(x)c - \rho_2(c)x, y) - \omega(\rho_1(y)c - \rho_2(c)y, x)\\
&= \omega_1(\rho_2(c)y,x) - \omega_1(\rho_2(c)x,y)\\
&= \mathrm{k}_1\big((\mathrm{J}_1\circ\rho_2(c))y,x\big) - \mathrm{k}_1\big((\mathrm{J}_1\circ\rho_2(c))x,y\big)\\
&= \mathrm{k}_1\big((\rho_2(c)\circ\mathrm{J}_1)y,x\big) - \mathrm{k}_1\big((\mathrm{J}_1\circ\rho_2(c))x,y\big)\\
&= \mathrm{k}_1\big(J_1y,\rho_2^*(c)x\big) - \mathrm{k}_1\big((\mathrm{J}_1\circ\rho_2(c))x,y\big)\\
&= \mathrm{k}_1\big(\rho_2^*(c)x,J_1y\big) - \mathrm{k}_1\big((\mathrm{J}_1\circ\rho_2(c))x,y\big)\\
&= -\mathrm{k}_1\big((J_1\circ\rho_2^*(c))x,y\big) - \mathrm{k}_1\big((\mathrm{J}_1\circ\rho_2(c))x,y\big)\\
&= -\mathrm{k}_1\Big(\mathrm{J}_1\circ\big(\rho_2^*(c)+\rho_2(c)\big)x,y\Big)\\
\mathrm{d}\omega(x,y,c) &= -\omega_1\big((\rho_2^*(c)+\rho_2(c))x,y\big).
\end{aligned}$$

A similar calculation show that $\mathrm{d}\omega(x,b,c) = -\omega_2\big((\rho_1^*(x)+\rho_1(x))b,c\big)$. □

**Corollary 2.6.** *The Lie algebra $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathrm{J}, \mathcal{K})$ is* ***Kählerian*** *if and only if $(\mathfrak{g}_i, \mathrm{J}_i, \mathrm{k}_i)_{i=1.2}$ is Kählerian and $\rho_i = -\rho_i^*$ for $i \in \{1,2\}$.*

*Proof.* By Lemma théorème 2.5 we have, $\mathrm{d}\omega(x,y,z) = \mathrm{d}\omega_1(x,y,z)$ and $\mathrm{d}\omega(a,b,c) = \mathrm{d}\omega_2(a,b,c)$ and since $\mathrm{J}_1$ commute with $\rho_2(c)$ we get :

$$\begin{aligned}
\mathrm{d}\omega(x,y,c) &= -\omega_1(\rho_2(c)x,y) + \omega_1(\rho_2(c)y,x)\\
&= \omega_1(y,\rho_2(c)x) + \omega_1(\rho_2(c)y,x)\\
&= \mathrm{k}_1(\mathrm{J}_1y,\rho_2(c)x) + \mathrm{k}_1(\mathrm{J}_1\circ\rho_2(c)y,x)\\
&= \mathrm{k}_1(\rho_2^*(c)\circ\mathrm{J}_1y,x) + \mathrm{k}_1(\mathrm{J}_1\circ\rho_2(c)y,x)\\
&= \mathrm{k}_1\Big(\big(\rho_2^*(c)\circ\mathrm{J}_1 + \mathrm{J}_1\circ\rho_2(c)\big)y,x\Big)\\
&= \mathrm{k}_1\Big(\big(\rho_2^*(c)\circ\mathrm{J}_1 + \rho_2(c)\circ\mathrm{J}_1\big)y,x\Big)\\
\mathrm{d}\omega(x,y,c) &= \mathrm{k}_1\Big(\big((\rho_2^*(c)+\rho_2(c))\circ\mathrm{J}_1\big)y,x\Big)
\end{aligned}$$

and similarly since $\mathrm{J}_2$ commute with $\rho_1(x)$ we get

$$\mathrm{d}\omega(x,b,c) = \mathrm{k}_2\Big(\big((\rho_1^*(x)+\rho_1(x))\circ\mathrm{J}_2\big)c,b\Big).$$

If we suppose that $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathrm{J}, \mathcal{K})$ is Kählerian then $\mathrm{d}\omega_1 = 0 = \mathrm{d}\omega_2$, which means that $\omega_1$ and $\omega_2$ are both Kählerian. In addition to that

$$\mathrm{d}\omega(x,y,c) = \mathrm{k}_1\Big(\big((\rho_2^*(c)+\rho_2(c))\circ\mathrm{J}_1\big)y,x\Big) = 0,$$
$$\mathrm{d}\omega(x,b,c) = \mathrm{k}_2\Big(\big((\rho_1^*(x)+\rho_1(x))\circ\mathrm{J}_2\big)c,b\Big) = 0,$$

by non-degeneracy of $k_1$ and $k_2$ and the fact that $J_1^2 = -Id_{\mathfrak{g}_1}$ and $J_2^2 = -Id_{\mathfrak{g}_2}$ thus

$$\rho_1 = -\rho_1^*, \quad \text{and} \quad \rho_2 = -\rho_2^*.$$

For the inverse it can be check easily. □

We note that $\mathcal{X}_{\rho_1}$ (resp. $\mathcal{X}_{\rho_2}$) is the character of the representation $\rho_1$ (resp. $\rho_2$) defined by $\mathcal{X}_{\rho_1}(x) = tr_2(\rho_1(x))$ and $\mathcal{X}_{\rho_2}(a) = tr_1(\rho_2(a))$. Our main result is the following theorem:

**Theorem 2.7.** *Let $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathrm{J}, \mathcal{K})$ be a Hermitian twisted product. Then, its associated Lee form $\theta$ is given by*

$$\theta(x) = \theta_1(x) - \mathcal{X}_{\rho_1}(x), \qquad x \in \mathfrak{g}_1 \tag{2.5}$$

$$\theta(a) = \theta_2(a) - \mathcal{X}_{\rho_2}(a), \qquad a \in \mathfrak{g}_2. \tag{2.6}$$

*Moreover, the Hermitian Lie algebra $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathrm{J}, \mathcal{K})$ is*

*i. **Balanced** if and only if $\theta_1(x) = \mathcal{X}_{\rho_1}(x)$ and $\theta_2(a) = \mathcal{X}_{\rho_2}(a)$ for all $x \in \mathfrak{g}_1$ and $a \in \mathfrak{g}_2$.*

*ii. **LCB** if and only if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are both LCB and*

$$\theta_1\big(\rho_2(a)x\big) - \theta_2\big(\rho_1(x)a\big) = \mathcal{X}_{\rho_1}\big(\rho_2(a)x\big) - \mathcal{X}_{\rho_2}\big(\rho_1(x)a\big)$$

*for all $x \in \mathfrak{g}_1$ and $a \in \mathfrak{g}_2$.*

*Proof.* i. Let $\mathcal{B}_1 = \{e_1, \ldots, e_{2n_1}\}$ and $\mathcal{B}_2 = \{f_1, \ldots, f_{2n_2}\}$ an orthonormal basis of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Recall the definition of $\mathrm{d}^*\omega$ :

$$\mathrm{d}^*\omega(X) = -\sum_{i=1}^{2n_1}(\nabla_{e_i}\omega)(e_i, X) - \sum_{j=1}^{2n_2}(\nabla_{f_j}\omega)(f_j, X)$$

where $X \in \mathfrak{g}_{(\rho_1,\rho_2)}$ and $\nabla$ is the Levi-Civita connection associated to $(\mathfrak{g}_{(\rho_1,\rho_2)}, \mathcal{K})$. Taking into consideration the Koszul formula for the Levi-Civita connection, we get

$$\begin{aligned}(\nabla_{e_i}\omega)(e_i, X) &= -\omega(\nabla_{e_i}e_i, X) - \omega(e_i, \nabla_{e_i}X) \\ &= \mathcal{K}(\nabla_{e_i}e_i, \mathrm{J}X) - \mathcal{K}(\mathrm{J}e_i, \nabla_{e_i}X) \\ &= \frac{1}{2}\Big(2\mathcal{K}([\mathrm{J}X, e_i], e_i) + \mathcal{K}([X, e_i], \mathrm{J}e_i) + \mathcal{K}([X, \mathrm{J}e_i], e_i) \\ &\quad - \mathcal{K}([\mathrm{J}e_i, e_i], X)\Big)\end{aligned}$$

Therefore

$$\begin{aligned}-\sum_{i=1}^{2n_1}(\nabla_{e_i}\omega)(e_i, X) &= -\mathrm{tr}_1(\mathrm{ad}_{\mathrm{J}X}) - \frac{1}{2}\Big( - \mathrm{tr}_1(\mathrm{J}\mathrm{ad}_X) + \mathrm{tr}_1(\mathrm{ad}_X J) \\ &\qquad - \sum_{i=1}^{2n_1}\mathcal{K}([\mathrm{J}_1e_i, e_i]_1, X)\Big) \\ &= -\mathrm{tr}_1(\mathrm{ad}_{\mathrm{J}X}) + \frac{1}{2}\sum_{i=1}^{2n_1}\mathcal{K}([\mathrm{J}_1e_i, e_i]_1, X).\end{aligned}$$

Similarly, we get

$$-\sum_{j=1}^{2n_2}(\nabla_{f_j}\omega)(f_j, X) = -\mathrm{tr}_2(\mathrm{ad}_{\mathrm{J}X}) + \frac{1}{2}\sum_{j=1}^{2n_2}\mathcal{K}([\mathrm{J}_2f_j, f_j]_2, X).$$

So

$$\mathrm{d}^*\omega(X) = -\mathrm{tr}(\mathrm{ad}_{\mathrm{J}X}) + \frac{1}{2}\Big(\sum_{i=1}^{2n_1}\mathcal{K}([\mathrm{J}_1 e_i, e_i]_1, X) + \sum_{j=1}^{2n_2}\mathcal{K}([\mathrm{J}_2 f_j, f_j]_2, X)\Big).$$

We have for $X = e_k$, $k = 1\dots 2n_1$

$$\begin{aligned}
\mathrm{d}^*\omega(e_k) &= -\,\mathrm{tr}\Big(\mathrm{ad}_{\mathrm{J}_1 e_k}\Big) + \frac{1}{2}\sum_{i=1}^{2n_1}\mathrm{k}_1\Big([\mathrm{J}_1 e_i, e_i]_1, e_k\Big)\\
&= -\sum_{i=1}^{2n_1} e^i\Big([\mathrm{J}_1 e_k, e_i]_1\Big) - \sum_{i=1}^{2n_2} f^i\Big([\mathrm{J}_1 e_k, f_i]\Big)\\
&\quad + \frac{1}{2}\sum_{i=1}^{2n_1}\mathrm{k}_1\Big([\mathrm{J}_1 e_i, e_i]_1, e_k\Big)\\
&= \mathrm{d}^*\omega_1(e_k) - \sum_{i=1}^{2n_2} f^i\Big(\rho_1(\mathrm{J}_1 e_k) f_i\Big) + \sum_{i=1}^{2n_2} f^i\Big(\rho_2(f_i)\mathrm{J}_1 e_k\Big)\\
&= -\theta_1 \circ \mathrm{J}_1^{-1}(e_k) - \mathrm{tr}_2\Big(\rho_1(\mathrm{J}_1 e_k)\Big).
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{d}^*\omega(\mathrm{J}_1 e_k) &= -\theta_1(e_k) - \mathrm{tr}_2\big(\rho_1(-e_k)\big)\\
&= -\theta_1(e_k) + \mathrm{tr}_2\Big(\rho_1(e_k)\Big)\\
&= -\theta_1(e_k) + \mathcal{X}_{\rho_1}(e_k).
\end{aligned}$$

Since $\theta = -\mathrm{d}^*\omega \circ \mathrm{J}$, we get

$$\begin{aligned}
\theta(e_k) &= -\mathrm{d}^*\omega \circ \mathrm{J}(e_k)\\
&= -\mathrm{d}^*\omega(\mathrm{J}_1 e_k)\\
\theta(e_k) &= \theta_1(e_k) - \mathcal{X}_{\rho_1}(e_k).
\end{aligned}$$

The same is true for the second identity $\theta(f_r) = \theta_2(f_r) - \mathcal{X}_{\rho_2}(f_r)$ that we get for $X = f_r$, $r = 1,\dots 2n_2$. Taking into consideration the balanced condition, we get the result.

ii. By the assertion $(i)$, we know that

$$\begin{aligned}
\theta(x) &= \theta_1(x) - \mathcal{X}_{\rho_1}(x),\\
\theta(a) &= \theta_2(a) - \mathcal{X}_{\rho_2}(a).
\end{aligned}$$

for all $x \in \mathfrak{g}_1$ and $a \in \mathfrak{g}_2$. So the differential of $\theta$ is defined by

$$\begin{aligned}
\mathrm{d}\theta(x,y) &= \mathrm{d}\theta_1(x,y), \quad \forall x, y \in \mathfrak{g}_1\\
\mathrm{d}\theta(a,b) &= \mathrm{d}\theta_2(a,b), \quad \forall a, b \in \mathfrak{g}_2\\
\mathrm{d}\theta(x,a) &= \theta_1\big(\rho_2(a)x\big) - \theta_2\big(\rho_1(x)a\big) - \Big(\mathcal{X}_{\rho_1}\big(\rho_2(a)x\big) - \mathcal{X}_{\rho_2}\big(\rho_1(x)a\big)\Big),
\end{aligned}$$

where $(x,a) \in \mathfrak{g}_1 \times \mathfrak{g}_2$.
Then $\mathrm{d}\theta = 0$ if and only if $\mathrm{d}\theta_1 = \mathrm{d}\theta_2 = 0$ and

$$\theta_1\big(\rho_2(a)x\big) - \theta_2\big(\rho_1(x)a\big) = \mathcal{X}_{\rho_1}\big(\rho_2(a)x\big) - \mathcal{X}_{\rho_2}\big(\rho_1(x)a\big)$$

and the result follows.

□

**Remark 2.8.** If $(\mathfrak{g}_1,\omega_1,\mathrm{J}_1)$ and $(\mathfrak{g}_2,\omega_2,\mathrm{J}_2)$ are balanced then the Hermitian twisted cartesian product is balanced if and only if $\mathrm{tr}_2\big(\rho_1(x)\big)=0$ and $\mathrm{tr}_1\big(\rho_2(a)\big)=0$, for all $x\in\mathfrak{g}_1$ and $a\in\mathfrak{g}_2$.

A simple case that generates several examples is when $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are abelians.

**Corollary 2.9.** *Let* $(\mathfrak{g}_1,\mathrm{J}_1,\mathrm{k}_1)$ *and* $(\mathfrak{g}_2,\mathrm{J}_2,\mathrm{k}_2)$ *be two abelian Hermitian Lie algebras and* $(\mathfrak{g}_{(\rho_1,\rho_2)},\mathrm{J},\mathcal{K})$ *their Hermitian twisted cartesian product. The associated Lee form* $\theta$ *is given by*

$$\theta(x)=-\mathcal{X}_{\rho_1}(x)\quad \textit{and}\quad \theta(a)=-\mathcal{X}_{\rho_2}(a).$$

*Moreover, the Lie algebra* $(\mathfrak{g}_{(\rho_1,\rho_2)},\omega,\mathrm{J})$ *is* ***balanced*** *if and only if* $\mathcal{X}_{\rho_1}(x)=\mathcal{X}_{\rho_2}(a)=0$ *and LCB if and only if* $tr_1\big(\rho_2(\rho_1(x)a)\big)=tr_2\big(\rho_1(\rho_2(a)x)\big)$.

**Example 2.10.** A general result of the corollary above is the balanced Hermitian twisted cartesian product $\big(\mathbb{R}^{2p}\bowtie\mathbb{R}^{2q},\mathrm{J},\mathcal{K}\big)$ such that
$\big(\mathbb{R}^{2p}=span\{e_1,\ldots,e_{2p}\},\mathrm{J}_1,\mathrm{k}_1\big)$ and $\big(\mathbb{R}^{2q}=span\{f_1,\ldots,f_{2q}\},\mathrm{J}_2,\mathrm{k}_2\big)$ are the two abelian Hermitian Lie algebras associated respectively to the representations
$\rho_1:\mathbb{R}^{2p}\to\mathrm{End}(\mathbb{R}^{2q})$ and $\rho_2:\mathbb{R}^{2q}\to\mathrm{End}(\mathbb{R}^{2p})$ defined by:
$\rho_1(e_i)=\left(\begin{array}{c|c}A_i & B_i\\ \hline -B_i & A_i\end{array}\right)$ and $\rho_2(f_j)=\left(\begin{array}{c|c}C_j & D_j\\ \hline -D_j & C_j\end{array}\right)$
where $(A_i,B_i)\in\mathrm{Diag}(a_1,\ldots,a_q)\times\mathrm{Diag}(b_1,\ldots,b_q)$ and
$(C_j,D_j)\in\mathrm{Diag}(c_1,\ldots,c_p)\times\mathrm{Diag}(d_1,\ldots,d_p)$ and $\mathrm{tr}\,A_i=0=\mathrm{tr}\,C_j$ for the standard complex structures $\mathrm{J}_1=\begin{pmatrix}0 & -I_p\\ I_p & 0\end{pmatrix}$ and $\mathrm{J}_2=\begin{pmatrix}0 & -I_q\\ I_q & 0\end{pmatrix}$.

## 3 Applications in dimension six

In this section we look for six-dimensional balanced Hermitian twisted cartesian products. In order to do that only two cases are presented, the first one is $\mathbb{R}^2\bowtie\mathfrak{g}$ and the second one is $\mathrm{aff}(\mathbb{R})\bowtie\mathfrak{g}$, where $\dim(\mathfrak{g})=4$.

### 3.1 $\mathbb{R}^2\bowtie\mathfrak{g}$ with $\dim(\mathfrak{g})=4$

Let $\mathbb{R}^2=span\{e_1,e_2\}$ with $[e_1,e_2]=0$, $\omega_1=e^{12}$ and $\mathrm{J}_1(e_1)=e_2$ and $(\mathfrak{g},\omega_2,\mathrm{J}_2)$ be a four-dimensional Hermitian Lie algebra and its associated Lee form $\theta_2$, in order to applicate our Theorem théorème 2.7 $\mathfrak{g}$ is necessary Lck (because $d\theta_2=d\,\mathrm{tr}(\rho_2)=0$ and $d\omega_2=\theta_2\wedge\omega_2$). In the other hand, the article [1] gives a classification of Lck structures on four dimensional Lie algebras up to linear equivalence. In fact basing on results of Table 2 see [1], we twist $\mathbb{R}^2$ with each one of the four-dilensional Lie algebras listed in this table. We obtain the following theorem.

**Theorem 3.1.** *Balanced Hermitian twisted cartesian products Lie algebras of type* $(\mathbb{R}^2\bowtie\mathfrak{g},\omega,\mathrm{J})$ *where* $\dim(\mathfrak{g})=4$ *are described as follows:*

- $\mathbb{R}^2\rtimes\mathfrak{rr}_{3,1}$ :

  $(-13-x23-t26,x13+t16-23,0,-34,-35,0)$
  $\mathrm{J}(e_1)=e_2$, $\mathrm{J}(e_3)=e_6$, $\mathrm{J}(e_4)=e_5$.
  $\omega=e^{12}+\sigma e^{36}+e^{45}$ *with* $\sigma>0$.

- $\mathbb{R}^2\rtimes\mathfrak{rr}_{3,0}$ :

  $\big(\frac{\delta}{2}13+\frac{\sigma}{2\delta}15-x23-z25-t26,x13+z15+t16+\frac{\delta}{2}23+\frac{\sigma}{2\delta}25,0,-34,0,0\big)$
  $\mathrm{J}(e_1)=e_2$, $\mathrm{J}(e_3)=e_4$, $\mathrm{J}(e_5)=e_6$.
  $\omega=e^{12}+\frac{\delta(\delta+1)}{\sigma}e^{34}+e^{36}-e^{45}+\frac{\sigma}{\delta^2}e^{56}$ *with* $\delta>0$, $\sigma>0$.

- $\mathbb{R}^2 \rtimes \mathfrak{rh}_3$ :

  $(-\frac{1}{2}16 - x23 - y24 - t26, x13 + y14 + t16 - \frac{1}{2}26, 0, 0, -34, 0)$
  $\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = e_4,\ \mathrm{J}(e_5) = e_6.$
  $\omega = e^{12} + \sigma\left(e^{34} + e^{56}\right)$ *with* $\sigma > 0.$

- $\mathbb{R}^2 \rtimes \mathfrak{rr}'_{3,\gamma} \quad \gamma > 0$ :

  $(-\gamma 13 - x23 - t26, x13 + t16 - \gamma 23, 0, -\gamma 34 - 35, 34 - \gamma 35, 0)$
  $\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = e_6,\ \mathrm{J}(e_4) = \pm e_5.$
  $\omega = e^{12} + \sigma e^{36} + e^{45}$ *with* $\sigma > 0.$

- $\mathbb{R}^2 \rtimes_1 \mathfrak{r}_2\mathfrak{r}_2$ :

  $\left(-\frac{1}{2}13 - x23 - z25, x13 - \frac{1}{2}23 + z15, 0, -34, 0, -56\right)$
  $\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = e_4,\ \mathrm{J}(e_5) = e_6$
  $\omega = e^{12} + \omega_{34}e^{34} + \omega_{56}(-e^{36} + e^{45} + e^{56})$ *with* $\omega_{34} > \omega_{56} > 0.$

- $\mathbb{R}^2 \rtimes_2 \mathfrak{r}_2\mathfrak{r}_2$ :

  $\left(-\frac{1}{2}15 - x23 - z25, x13 + z15 - \frac{1}{2}25, 0, -34, 0, -56\right)$
  $\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = e_4,\ \mathrm{J}(e_5) = e_6.$
  $\omega = e^{12} + \omega_{34}(e^{34} - e^{36} + e^{45}) + \omega_{56}e^{56}$ *with* $\omega_{56} > \omega_{34} > 0.$

- $\mathbb{R}^2 \rtimes_3 \mathfrak{r}_2\mathfrak{r}_2$ :

  $(\frac{\sigma}{2}13 + \frac{\tau}{2}15 - x23 - z25, x13 + z15 + \frac{\sigma}{2}23 + \frac{\tau}{2}25, 0, -34, 0, -56)$
  $\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = e_4,\ \mathrm{J}(e_5) = e_6.$
  $\omega = e^{12} + \mu\left(\frac{1+\sigma}{\tau}e^{34} + e^{36} - e^{45} + \frac{\tau+1}{\sigma}e^{56}\right)$ *with* $\sigma\tau \neq 0,\ \sigma + \tau \neq -1,\ \mu \neq 0,$ $\frac{\mu(1+\sigma)}{\tau} > 0,\ \frac{\mu(1+\tau)}{\sigma} > 0,\ \frac{\sigma+\tau+1}{\sigma\tau} > 0.$

- $\mathbb{R}^2 \rtimes_1 \mathfrak{r}'_2$ :

  $(-13 - x23 - y24, x13 + y14 - 23, 0, 0, -35 + 46, -36 - 45)$
  $\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = e_5,\ \mathrm{J}(e_4) = e_6.$
  $\omega = e^{12} + \omega_{34}(e^{34} + e^{56}) + \omega_{35}(e^{35} + e^{46})$ *with* $\omega_{35} > 0,\ \omega_{35}^2 - \omega_{34}^2 > 0,\ \omega_{34} > 0.$

- $\mathbb{R}^2 \rtimes_2 \mathfrak{r}'_2$ :

  $\left(\frac{\alpha}{2}13 - x23 - y24, x13 + y14 + \frac{\alpha}{2}23, 0, 0, -35 + 46, -36 - 45\right)$
  $\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = e_5,\ \mathrm{J}(e_4) = e_6.$
  $\omega = \omega_{46}(-(\alpha + 1)e^{35} + e^{46})$ *with* $\omega_{46} > 0,\ \alpha + 1 < 0.$

- $\mathbb{R}^2 \rtimes_3 \mathfrak{r}'_2$ :

  $(-13 - x23 - y24, x13 + y14 - 23, 0, 0, -35 + 46, -36 - 45)$
  $\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = -ae_3 + \frac{a^2+1}{b}e_4,\ \mathrm{J}(e_5) = e_6$ *with* $(a, b) \neq (0, 1).$
  $\omega = e^{12} + \omega_{34}e^{34} + e^{56}$ *with* $b\,\omega_{34} > 0.$

- $\mathbb{R}^2 \ltimes_4 \mathfrak{r}'_2$ :

  $(-13 - 24, 14 - 23, 0, 0, -a(13 + 24) + b(23 - 14) - 35 + 46, a(23 - 14) + b(13 + 24) - 36 - 45)$
  $\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = e_4,\ \mathrm{J}(e_5) = e_6\ .$
  $\omega = e^{12} + \sigma e^{34} + \omega_{56}e^{56}$ *with* $\sigma > 0.$

- $\mathbb{R}^2 \rtimes_5 \mathfrak{r}_2'$ :

  $(-13 - x23 - y24, x13 + y14 - 23, 0, 0, -35 + 46, -36 - 45)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = e_4$, $\mathrm{J}(e_5) = e_6$.
  $\omega = e^{12} + \sigma e^{34} + \omega_{56} e^{56}$ *with* $\sigma > 0$.

- $\mathbb{R}^2 \ltimes_6 \mathfrak{r}_2'$ :

  $(-13 + 24, -14 - 23, 0, 0, a(24 - 13) - b(14 + 23) - 35 + 46, -a(14 + 23) + b(13 - 24) - 36 - 45)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = e_4$, $\mathrm{J}(e_5) = e_6$.
  $\omega = e^{12} + \sigma e^{34} + \omega_{56} e^{56}$ *with* $\sigma > 0$.

- $\mathbb{R}^2 \rtimes_7 \mathfrak{r}_2'$ :

  $(\frac{\alpha}{2}13 + \frac{\beta}{2}14 - x23 - y24, x13 + y14 + \frac{\alpha}{2}23 + \frac{\beta}{2}24, 0, 0, -35 + 46, -36 - 45)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = e_5$, $\mathrm{J}(e_4) = e_6$.
  $\omega = e^{12} + \omega_{35}\left(e^{35} + \frac{\beta}{\alpha}(e^{36} + e^{45}) + \frac{\beta^2 - \alpha}{\alpha(\alpha+1)} e^{46}\right)$ *with* $\omega_{35} > 0$, $\alpha \neq -1$, $\alpha \neq 0$, $\beta > 0$, $\frac{\beta^2 - \alpha}{\alpha(\alpha+1)} > \frac{\beta^2}{\alpha^2}$.

- $\mathbb{R}^2 \rtimes \mathfrak{r}_{4,\alpha,1}$ , $\alpha \notin \{0; 1\}$ :

  $(-16 - t25, t16 - 26, 36, \alpha 46, 56, 0)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = e_5$, $\mathrm{J}(e_4) = -e_6$.
  $\omega = e^{12} + e^{35} + \sigma e^{46}$ *with* $\sigma < 0$.

- $\mathbb{R}^2 \rtimes \mathfrak{r}_{4,\alpha,\alpha}$ , $\alpha \notin \{0; 1\}$ :

  $(-\alpha 16 - t26, t16 - \alpha 26, 36, \alpha 46, \alpha 56, 0)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = -e_6$, $\mathrm{J}(e_5) = -e_4$.
  $\omega = e^{12} + \sigma e^{36} + e^{45}$ *with* $\sigma < 0$.

- $\mathbb{R}^2 \rtimes \mathfrak{r}'_{4,\gamma,\delta}$ , $\delta > 0$, $\gamma \neq 0$ :

  $(-\gamma 16 - t26, t16 - \gamma 26, 36, \gamma 46 + \delta 56, -\delta 46 + \gamma 56, 0)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = -e_6$, $\mathrm{J}(e_5) = \pm e_4$
  $\omega = e^{12} + \sigma e^{36} \pm e^{45}$ *with* $\sigma < 0$.

- $\mathbb{R}^2 \rtimes \mathfrak{d}_4$ :

  $\left(-\frac{1}{2}16 - t26, t16 - \frac{1}{2}26, 36, -46, -34, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = -e_5$, $\mathrm{J}(e_4) = -e_6$.
  $\omega = e^{12} - e^{35} + \sigma e^{46}$ *with* $\sigma < 0$.

- $\mathbb{R}^2 \rtimes_1 \mathfrak{d}_{4,1}$ :

  $\left(-\frac{1}{2}16 - y24 - t26, y14 + t16 - \frac{1}{2}26, 36, 0, -34 + 56, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = e_6$, $\mathrm{J}(e_4) = e_5$.
  $\omega = e^{12} + \sigma e^{36} + e^{45}$ *with* $\sigma > 0$.

- $\mathbb{R}^2 \rtimes_2 \mathfrak{d}_{4,1}$ :

  $\left(\frac{1}{2}14 + \frac{\alpha}{2}16 - y24 - t26, y14 + t16 + \frac{1}{2}24 + \frac{\alpha}{2}26, 36, 0, -34 + 56, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = e_6$, $\mathrm{J}(e_4) = e_5$.
  $\omega = e^{12} + \left(\frac{\omega_{45}(\alpha+1)}{\beta^2}\right) e^{34} + \left(\frac{(\omega_{34}\beta-\omega_{45})(\alpha+1)}{\beta^2}\right) e^{36} + \frac{\omega_{45}}{\beta^2} e^{45} + \left(-\frac{\omega_{45}(\alpha+1)}{\beta^2}\right) e^{56}$ *with* $\beta \neq 0$, $\omega_{45} > 0$, $(\omega_{34}\beta - \omega_{45})(\alpha+1) > 0$, $(\omega_{34}\beta - \omega_{45} - \omega_{45}(\alpha+1))\omega_{45}(\alpha+1) > 0$.

- $\mathbb{R}^2 \rtimes_1 \mathfrak{d}_{4,\frac{1}{2}}$ :

  $\left(\frac{\alpha}{2}16 - t26, t16 + \frac{\alpha}{2}26, \frac{1}{2}36, \frac{1}{2}46, -34 + 56, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = e_4$, $\mathrm{J}(e_5) = -e_6$.
  $\omega = e^{12} + \tau(e^{34} - (\sigma+1)e^{56})$ *with* $\tau > 0$, $\sigma + 1 > 0$.

- $\mathbb{R}^2 \rtimes_2 \mathfrak{d}_{4,\frac{1}{2}}$ :

  $\left(-\frac{3}{4}16 - t26, t16 - \frac{3}{4}26, \frac{1}{2}36, \frac{1}{2}46, -34 + 56, 0\right)$.
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = -e_4$, $\mathrm{J}(e_5) = -e_6$.
  $\omega = e^{12} + \sigma(e^{34} + \frac{1}{2}e^{56})$ *with* $\sigma < 0$.

- $\mathbb{R}^2 \rtimes_2 \mathfrak{d}_{4,\frac{1}{2}}$ :

  $\left(-\frac{3}{4}16 - t26, t16 - \frac{3}{4}26, \frac{1}{2}36, \frac{1}{2}46, -34 + 56, 0\right)$.
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = 2e_6$, $\mathrm{J}(e_4) = e_5$.
  $\omega = e^{12} + \omega_{22}e^{45} + \frac{\sigma}{2\omega_{22}}e^{36}$ *with* $\omega_{22} > 0$, $\sigma > 0$.

- $\mathbb{R}^2 \rtimes_1 \mathfrak{d}'_{4,0}$ :

  $\left(\frac{\mu}{2}16 - t26, t16 + \frac{\mu}{2}26, 46, -36, -34, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = e_4$, $\mathrm{J}(e_5) = e_6$.
  $\omega = e^{12} + \sigma(e^{34} - \mu e^{56})$ *with* $\sigma > 0$, $\mu < 0$.

- $\mathbb{R}^2 \rtimes_2 \mathfrak{d}'_{4,0}$ :

  $\left(\frac{\mu}{2}16 - t26, t16 + \frac{\mu}{2}26, 46, -36, -34, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = e_4$, $\mathrm{J}(e_5) = -e_6$.
  $\omega = e^{12} + \sigma(e^{34} - \mu e^{56})$ *with* $\sigma > 0$, $\mu > 0$.

- $\mathbb{R}^2 \rtimes_1 \mathfrak{d}_{4,\lambda}$, $\lambda > \frac{1}{2}, \lambda \neq 1$ :

  $\left(\left(\frac{\lambda}{2} - 1\right) 16 - t26, t16 + \left(\frac{\lambda}{2} - 1\right) 26, \lambda 36, (1-\lambda)46, -34 + 56, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = -e_5$, $\mathrm{J}(e_4) = -\frac{1}{\lambda-1}e_6$.
  $\omega = e^{12} - e^{35} - \sqrt{\sigma + \omega_{22}}(\lambda - 1)\, e^{36} - \sqrt{\sigma + \omega_{22}}\, e^{45} + (1-\lambda)\omega_{22}e^{46}$ *with* $\sigma < 0$, $\omega_{22} \geq -\sigma$.

- $\mathbb{R}^2 \rtimes_2 \mathfrak{d}_{4,\lambda}$ , $\lambda > \frac{1}{2}, \lambda \neq 1$ :

  $\left(\left(\frac{\lambda}{2}-1\right)16 - t26, t16 + \left(\frac{\lambda}{2}-1\right)26, \lambda 36, (1-\lambda)46, -34+56, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = -e_5$, $\mathrm{J}(e_4) = -\frac{1}{\lambda-1}e_6$.
  $\omega_2 = e^{12} - e^{35} + \sqrt{\sigma+\omega_{22}}(\lambda-1)\,e^{36} + \sqrt{\sigma+\omega_{22}}\,e^{45} + (1-\lambda)\omega_{22}e^{46}$ *with* $\sigma < 0$, $\omega_{22} \geq -\sigma$.

- $\mathbb{R}^2 \rtimes_3 \mathfrak{d}_{4,\lambda}$ , $\lambda > \frac{1}{2}, \lambda \neq 1$ :

  $\left(-\frac{\lambda+1}{2}16 - t26, t16 - \frac{\lambda+1}{2}26, \lambda 36, (1-\lambda)46, -34+56, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = -e_5$, $\mathrm{J}(e_4) = -\frac{1}{\lambda-1}e_6$.
  $\omega = e^{12} - \omega_{11}e^{35} - (\lambda-1)\omega_{22}e^{46}$ *with* $\omega_{11} > 0$, $\omega_{22} > 0$.

- $\mathbb{R}^2 \rtimes_4 \mathfrak{d}_{4,\lambda}$ , $\lambda > \frac{1}{2}, \lambda \neq 1$ :

  $\left(\left(\frac{\lambda}{2}-1\right)16 - t26, t16 + \left(\frac{\lambda}{2}-1\right)26, \lambda 36, (1-\lambda)46, -34+56, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = \frac{1}{\lambda}e_6$, $\mathrm{J}(e_4) = e_5$
  $\omega = e^{12} + \omega_{11}\lambda e^{36} + \omega_{22}e^{45}$ *with* $\omega_{11} > 0$, $\omega_{22} > 0$.

- $\mathbb{R}^2 \rtimes_1 \mathfrak{gl}_2$ :

  $\left(-\frac{\mu}{2}16, -\frac{\mu}{2}26, -45, -2\times 34, 2\times 35, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = -(e_4+e_5)$, $\mathrm{J}(e_4) = \frac{1}{2}e_3 - \frac{1}{\mu}e_6$ *with* $\mu \in \mathbb{R}\setminus\{0\}$.
  $\omega = e^{12} + \omega_{34}e^{34} + \omega_{35}e^{35} + \omega_{45}\mu e^{36} + \omega_{45}e^{45} + \frac{1}{2}\omega_{34}\mu e^{46} - \frac{1}{2}\omega_{35}\mu e^{56}$ *with* $\omega_{34} \geq \omega_{35} > 0$, $\omega_{45} \geq 0$, $\omega_{34}\omega_{35} - \omega_{45}^2 > 0$.

- $\mathbb{R}^2 \rtimes_2 \mathfrak{gl}_2$ :

  $\left(\frac{\alpha}{2}16, \frac{\alpha}{2}26, -45, -2\times 34, 2\times 35, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = -(e_4+e_5)$, $\mathrm{J}(e_4) = \frac{1}{2}e_3 - \frac{1}{\mu}e_6$ *with* $\mu \in \mathbb{R}\setminus\{0\}$.
  $\omega = e^{12} + \omega_{34}e^{34} + \omega_{34}e^{35} - \frac{1}{2}\omega_{34}\alpha e^{46} + \frac{1}{2}\omega_{34}\alpha e^{56}$ *with* $\alpha \neq -\mu$, $\omega_{34} > 0$, $\frac{\alpha}{\mu} < 0$.

- $\mathbb{R}^2 \rtimes_1 \mathfrak{u}_2$ :

  $\left(\frac{\alpha}{2}16 - t26, t16 + \frac{\alpha}{2}26, 45, -35, 34, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = ae_3 - be_6$, $\mathrm{J}(e_4) = -e_5$ *with* $a \neq 0$, $b \neq 0$.
  $\omega = e^{12} + \omega_{45}\alpha e^{36} + \omega_{45}e^{45}$ *with* $\alpha \neq 0$, $\omega_{45} < 0$, $\frac{\alpha}{b} > 0$.

- $\mathbb{R}^2 \rtimes_2 \mathfrak{u}_2$ :

  $\left(-\frac{\alpha}{2}16 + t26, -t16 - \frac{\alpha}{2}26, 45, -35, 34, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = -be_6$, $\mathrm{J}(e_4) = -e_5$ *with* $b \neq 0$.
  $\omega = e^{12} + \omega_{45}\alpha e^{36} + \omega_{45}e^{45}$ *with* $\alpha \notin \{0, -\frac{1}{b}\}$, $\omega_{45} < 0$, $\frac{\alpha}{b} > 0$.

- $\mathbb{R}^2 \rtimes_1 \mathfrak{d}'_{4,\delta}$ :

  $\left(\frac{\mu}{2}16 - t26, t16 + \frac{\mu}{2}26, \frac{\delta}{2}36 + 46, -36 + \frac{\delta}{2}46, -34 + \delta 56, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = -e_4$, $\mathrm{J}(e_5) = -e_6$.
  $\omega = e^{12} + \sigma(e^{34} - (\delta+\mu)e^{56})$ *with* $\sigma < 0$, $\delta + \mu < 0$, $\mu \neq 0$.

- $\mathbb{R}^2 \rtimes_2 \mathfrak{d}'_{4,\delta}$ :

  $\left(\frac{\mu}{2}16 - t26, t16 + \frac{\mu}{2}26, \frac{\delta}{2}36 + 46, -36 + \frac{\delta}{2}46, -34 + \delta 56, 0\right)$
  $\mathrm{J}(e_1) = e_2$, $\mathrm{J}(e_3) = -e_4$, $\mathrm{J}(e_5) = e_6$.
  $\omega = e^{12} + \sigma(e^{34} - (\delta+\mu)e^{56})$ *with* $\sigma < 0$, $\delta + \mu > 0$, $\mu \neq 0$.

- $\mathbb{R}^2 \rtimes_3 \mathfrak{d}'_{4,\delta}$ :

$$\left(\tfrac{\mu}{2}16 - t26, t16 + \tfrac{\mu}{2}26, \tfrac{\delta}{2}36 + 46, -36 + \tfrac{\delta}{2}46, -34 + \delta 56, 0\right)$$
$$\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = e_4,\ \mathrm{J}(e_5) = e_6.$$
$$\omega = e^{12} + \sigma(e^{34} - (\delta+\mu)e^{56}) \textit{ with } \sigma > 0,\ \delta+\mu < 0,\ \mu \neq 0.$$

- $\mathbb{R}^2 \rtimes_4 \mathfrak{d}'_{4,\delta}$ :

$$\left(\tfrac{\mu}{2}16 - t26, t16 + \tfrac{\mu}{2}26, \tfrac{\delta}{2}36 + 46, -36 + \tfrac{\delta}{2}46, -34 + \delta 56, 0\right)$$
$$\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = e_4,\ \mathrm{J}(e_5) = -e_6.$$
$$\omega = e^{12} + \sigma(e^{34} - (\delta+\mu)e^{56}) \textit{ with } \sigma > 0,\ \delta+\mu > 0,\ \mu \neq 0.$$

*Proof.* We will give the proof in the case $\mathbb{R}^2 \bowtie \mathfrak{rr}_{3,1}$ since all cases should be handled in a similar way. Let $(\mathfrak{rr}_{3,1} = span\{e_3, e_4, e_5, e_6\}, \omega_2, \mathrm{J}_2)$ be the four-dimensional Hermitian Lie algebra with $[e_3,e_4] = e_4$, $[e_3,e_5] = e_5$, $\omega_2 = \sigma e^{36} + e^{45}$ $(\sigma > 0)$ and $\mathrm{J}_2(e_3) = e_6$, $\mathrm{J}_2(e_4) = e_5$ and $\theta_2 = -2e^3$. Using the definitions above, we know that $\rho_1 : \mathbb{R}^2 \longrightarrow Der(\mathfrak{rr}_{3,1})$ and $\rho_2 : \mathfrak{rr}_{3,1} \longrightarrow Der(\mathbb{R}^2)$. Lets look now for the derivations $\rho_1(e_1)$, $\rho_1(e_2)$ of $\mathfrak{rr}_{3,1}$ which commute with $\mathrm{J}_2$ and $\rho_2(e_3)$, $\rho_2(e_4)$, $\rho_2(e_5)$, $\rho_2(e_6)$ of $\mathbb{R}^2$ which commute with $\mathrm{J}_1$. We get

$$\rho_2(e_3) = \begin{pmatrix} x_2 & -x_1 \\ x_1 & x_2 \end{pmatrix},\ \rho_2(e_4) = \begin{pmatrix} y_2 & -y_1 \\ y_1 & y_2 \end{pmatrix},\ \rho_2(e_5) = \begin{pmatrix} z_2 & -z_1 \\ z_1 & z_2 \end{pmatrix},\ \rho_2(e_6) = \begin{pmatrix} t_2 & -t_1 \\ t_1 & t_2 \end{pmatrix}$$

and

$$\rho_1(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_1 & -a_2 & 0 \\ 0 & a_2 & a_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \rho_1(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & -b_2 & 0 \\ 0 & b_2 & b_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $x_1, x_2, y_1, y_2, z_1, z_2, t_1, t_2, a_1, a_2, b_1, b_2 \in \mathbb{R}$.
Taking into consideration that $\rho_1$ and $\rho_2$ are representations, we have:

$$\rho_2(e_3) = \begin{pmatrix} x_2 & -x_1 \\ x_1 & x_2 \end{pmatrix},\ \rho_2(e_4) = 0,\ \rho_2(e_5) = 0,\ \rho_2(e_6) = \begin{pmatrix} t_2 & -t_1 \\ t_1 & t_2 \end{pmatrix}.$$

Since $\theta_1 = 0$ and $\theta_2 = -2e^3$, we have : $0 = \theta_1(e_1) = tr_2(\rho_1(e_1)) = 2a_1$ and $0 = \theta_1(e_2) = tr_2(\rho_1(e_2)) = 2b_1$ and $-2 = \theta_2(e_3) = tr_1(\rho_2(e_3)) = 2x_2$ and $0 = \theta_2(e_6) = tr_1(\rho_2(e_6)) = 2t_2$.
So $a_1 = b_1 = 0$ and $x_2 = -1$ and $t_2 = 0$.
As a consequence, we get :

$$\rho_2(e_3) = \begin{pmatrix} -1 & -x_1 \\ x_1 & -1 \end{pmatrix},\ \rho_2(e_4) = 0,\ \rho_2(e_5) = 0,\ \rho_2(e_6) = \begin{pmatrix} 0 & -t_1 \\ t_1 & 0 \end{pmatrix}$$

and

$$\rho_1(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \rho_1(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -b_2 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A calculation show that the non-zero brackets on the balanced Hermitian Lie algebra $\mathbb{R}^2 \bowtie \mathfrak{rr}_{3,1}$ are defined by :

$$[e_1,e_3] = e_1 - x_1 e_2,\quad [e_1,e_6] = -t_1 e_2,\quad [e_2,e_3] = x_1 e_1 + e_2,\quad [e_2,e_6] = t_1 e_1,$$
$$[e_3,e_4] = e_4,\quad [e_3,e_5] = e_5,$$

and result follows. □

### 3.2 $\mathrm{aff}(\mathbb{R}) \bowtie \mathfrak{g}_2$ with $\dim(\mathfrak{g}) = 4$

Let $\mathrm{aff}(\mathbb{R}) = span\{e_1, e_2\}$ with $[e_1, e_2] = e_1$, $\omega_1 = e^{12}$ and $\mathrm{J}_1(e_1) = e_2$ and $(\mathfrak{g}, \omega_2, \mathrm{J}_2)$ be a four-dimensional Hermitian Lie algebra, The Lie algebra $\mathfrak{g}$ is of dimension four, so its associated Lee form $\theta_2$ satisfy $d\omega_2 = \theta_2 \wedge \omega_2$. In addition to that $\theta_2 = \mathrm{tr}_1(\rho_2(.))$ is a closed 1-form. As a consequence $(\mathfrak{g}, \mathrm{J}_2, \mathcal{K}_2)$ is LCK. On the other hand, LCK structures on four-dimensional Lie algebras are classified in [1]. We investigate this classification by twisting $\mathrm{aff}(\mathbb{R})$ with each of its Lie algebras.

**Theorem 3.2.** *Balanced Hermitian twisted cartesian products Lie algebras of type* $(\mathrm{aff}(\mathbb{R}) \bowtie \mathfrak{g}, \omega, \mathrm{J})$ *where* $\dim(\mathfrak{g}) = 4$ *are described as follows:*

- $\mathrm{aff}(\mathbb{R}) \rtimes_1 \mathfrak{d}_{4,2}$ :

$$(-12, 0, 2\times 36, -46, 56-34, 0)$$
$$\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = -e_5,\ \mathrm{J}(e_4) = -e_6.$$
$$\omega = e^{12} - e^{35} - \sqrt{\sigma + \omega_{22}}\, e^{36} - \sqrt{\sigma + \omega_{22}}\, e^{45} - \omega_{22} e^{46} \textit{ with } \sigma < 0,\ \omega_{22} \geq -\sigma.$$

- $\mathrm{aff}(\mathbb{R}) \rtimes_2 \mathfrak{d}_{4,2}$ :

$$(-12, 0, 2\times 36, -46, 56-34, 0)$$
$$\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = -e_5,\ \mathrm{J}(e_4) = -e_6.$$
$$\omega = e^{12} - e^{35} + \sqrt{\sigma + \omega_{22}}\, e^{36} + \sqrt{\sigma + \omega_{22}}\, e^{45} - \omega_{22} e^{46} \textit{ with } \sigma < 0,\ \omega_{22} \geq -\sigma.$$

- $\mathrm{aff}(\mathbb{R}) \rtimes_3 \mathfrak{d}_{4,2}$ :

$$(-12, 0, 2\times 36, -46, 56-34, 0)$$
$$\mathrm{J}(e_1) = e_2,\ \mathrm{J}(e_3) = \tfrac{1}{2} e_6,\ \mathrm{J}(e_4) = e_5.$$
$$\omega = e^{12} + 2\omega_{11} e^{36} + \omega_{22} e^{45} \textit{ with } \omega_{11} > 0,\ \omega_{22} > 0.$$

*Proof.* The proof is similar to Theorem théorème 3.1. □

## 4 conclusion

By Theorem théorème 3.1 we conclude that all the Hermitian twisted cartesian products $\mathbb{R}^2 \bowtie \mathfrak{g}$ carries a balanced structure, contrary to Theorem théorème 3.2 the $\mathrm{aff} \bowtie \mathfrak{d}_{4,2}$ is the only one.

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### Author information


M. W. Mansouri, Department of Mathematics, Ibn Tofail university, Morocco.
E-mail: mansourimohammed.wadia@uit.ac.ma

A. Oufkou, Department of Mathematics, Ibn Tofail university, Morocco.
E-mail: ahmed.oufkou@uit.ac.ma



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