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On S-n-Semiprimary Ideals

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Abstract In this paper, we introduce and explore the concept of S-n-semiprimary ideals, which generalizes the notion of n-semiprimary ideals in commutative ring theory. We provide a comprehensive analysis of their fundamental properties and investigate their relationships with other classes of ideals, such as S-prime and n-semiprimary ideals. We establish characterizations of S-n-semiprimary ideals and examine their behavior under various constructions, including homomorphic images, localizations, and direct products. Additionally, we delve into the structure and properties of S-n-powerful semiprimary ideals within integral domains. The paper further investigates how S-n-semiprimary ideals are transferred through trivial ring extensions and amalgamated algebras, offering a broad perspective on the interplay between these ideals and ring extensions. Various examples and corollaries illustrate the complex nature and applications of S-n-semiprimary ideals in commutative ring theory.

1 Introduction

Throughout this paper, we assume that all rings are commutative with identity, and all modules are unital. Let R be such a ring, and let $U(R)$ denote the set of unit elements of R. Recall that a nonempty subset S of R is a multiplicative set if it satisfies the following conditions: (i) $0 \notin S$, (ii) $1 \in S$, and (iii) $ab \in S$ for all $a, b \in S$. The saturation of S is denoted by S^*.

An ideal of R is proper if $I \neq R$. For a proper ideal I of R, the radical of I is denoted by $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$. On the other hand, for an ideal I of R and an element $a \in R$, we denote $(I :_R a) = \{x \in R \mid xa \in I\}$ as the conductor of a in I.

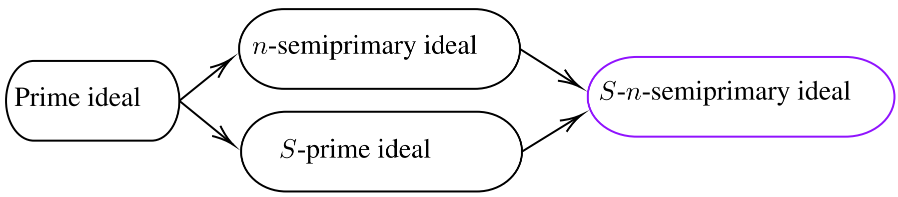
Let R be a ring and M an R-module. The trivial extension of R by M, also called the idealization of M, was introduced by Nagata. This ring, denoted by $R \ltimes M$, has the additive structure of the external direct sum $R \oplus M$ and a multiplication defined by $(a, m)(b, n) = (ab, bm + an)$. The canonical projection from $R \ltimes M$ to R, whose kernel $0 \ltimes M$ has zero square, induces an inclusion-preserving bijection between Spec$(R \ltimes M)$ and Spec(R). Background on trivial ring extensions is provided in [1, 7, 12].

Let (A, B) be a pair of rings, J an ideal of B, and $f : A \to B$ a ring homomorphism. In this setting, we consider the subring of $A \times B$:

$$A \bowtie^f B = \{(a, f(a) + j) \mid a \in A, j \in J\},$$

called the amalgamation of A and B along J with respect to f. This construction is a generalization of the amalgamated duplication of a ring along an ideal, introduced and studied by D'Anna and Fontana. Subsequent work by D'Anna, Finocchiaro, and Fontana has expanded the context of these amalgamations. They have explored these constructions within the framework of pullbacks, establishing numerous results on the transfer of various ideal and ring-theoretic properties between A and $f(A) + J$. The concept of amalgamation has received considerable attention over the decades, with motivations and additional applications discussed in detail in [4, 16].

The concept of prime ideals, a cornerstone of ideal theory in commutative rings, has been extensively studied by various authors. In 2020, Hamed and Malek defined S-prime ideals as a generalization of prime ideals ([10]). An ideal I of a commutative ring R, disjoint from a multiplicative subset S, is considered S-prime if for any $x, y \in R$ with $xy \in I$, there exists $s \in S$ such that either $sx \in I$ or $sy \in I$. In addition to S-prime ideals, this paper also explores n-semiprimary ideals and n-pseudo valuation domains, as discussed by Anderson and Badawi ([2]). An ideal I is called n-semiprimary if for any $x, y \in R$, $x^n y^n \in I$ implies that either $x^n \in I$ or $y^n \in I$. The survey paper by Anderson and Badawi, along with its references, details several properties of n-semiprimary ideals. We further extend these concepts to define S-n-semiprimary ideals of R: an ideal I is S-n-semiprimary if for all $x, y \in R$ with $x^n y^n \in I$, there exists $s \in S$ such that either $sx^n \in I$ or $sy^n \in I$. It is evident that any S-prime or n-semiprimary ideal, when disjoint from S, is an S-n-semiprimary ideal for any positive integer n. Our research shows that many properties typical of S-prime and n-semiprimary ideals are also applicable to S-n-semiprimary ideals, though certain properties do not transfer without complications. Below, we present a diagram summarizing these relationships.



In this paper, we introduce and investigate the concept of S-n-semiprimary ideals, which generalize n-semiprimary ideals. We explore numerous properties of these ideals and highlight their shared characteristics with n-semiprimary ideals. Our analysis includes examining the relationships between S-n-semiprimary ideals and other related classes of ideals. Additionally, we provide several characterizations of S-n-semiprimary ideals and examine their stability across various contexts in commutative ring theory, such as homomorphic images, localizations, and direct products.

Furthermore, we establish that if an ideal is S-n-semiprimary, then its radical is S-prime. We also delve into the concept of S-n-powerful semiprimary ideals in integral domains, discussing some of their notable properties. The concluding section of the paper focuses on investigating the structure of trivial ring extensions and amalgamation rings, with particular attention to the behavior of S-n-semiprimary ideals.

Any undefined notation and terminology can be found in [15].

2 Properties of S-n-Semiprimary Ideals

In this section, we define and explore fundamental properties of S-n-semiprimary ideals. Specifically, we investigate conditions under which well-known ideals are S-n-semiprimary ideals and examine the relationship between S-prime ideals and S-n-semiprimary ideals.

Definition 2.1. Let R be a commutative ring with identity, n be a positive integer, and S be a multiplicative subset of R. An ideal I of R, disjoint from S, is called an S-n-semiprimary ideal if there exists an $s \in S$ such that for any $x, y \in R$ with $x^n y^n \in I$, either $sx^n \in I$ or $sy^n \in I$.

Note that an S-1-semiprimary ideal is equivalent to an S-prime ideal, and every S-prime ideal is an S-n-semiprimary ideal for all $n \in \mathbb{N}$. However, it is noteworthy that while any n-semiprimary ideal becomes an S-n-semiprimary ideal for every multiplicative subset S of R, the converse is not generally true, as illustrated in the following example:

Example 2.2. Let $R = \mathbb{Z}_2[X, Y]$ be a ring and $S = \{X^k \mid k \in \mathbb{N}\}$ a multiplicative subset of R. For any integer $n \geq 2$, consider the ideal $I_n = (XY, Y^{n+1})$ of R. While I_n is an $(n+1)$-semiprimary ideal, it is not an n-semiprimary ideal of R. Nevertheless, I_n is an S-prime ideal of R.

Proof. Since $Y^n Y^n = Y^{2n} = Y^{n+1} Y^{n-1} \in I_n$ and $Y^n \notin I_n$, it follows that I_n is not an n-semiprimary ideal. Let $P, Q \in R$ such that $PQ \in I_n \subseteq (Y)$, and since (Y) is prime (as

R is a domain), either $P \in (Y)$ or $Q \in (Y)$. Consequently, either $XP \in (XY) \subseteq I_n$ or $XQ \in (XY) \subseteq I_n$. Thus, I_n is an S-prime ideal of R, as required. □

The following Proposition 2.3(4) generalizes [2, Theorem 2.2(a)] to the S-case.

Proposition 2.3. *Let R be a ring, S a multiplicative subset of R, and I an ideal of R disjoint from S. Then the following assertions hold:*

(1) *Assume $S_1 \subseteq S_2$ are multiplicative subsets. If I is an S_1-n-semiprimary ideal and $I \cap S_2 = \emptyset$, then I is an S_2-n-semiprimary ideal.*

(2) *I is an S-n-semiprimary ideal if and only if I is an S^*-n-semiprimary ideal.*

(3) *Let T be a commutative ring containing R, and let J be an ideal of T disjoint from S. If J is an S-n-semiprimary ideal of T, then $J \cap R$ is an S-n-semiprimary ideal of R.*

(4) *If I is an S-n-semiprimary ideal, then I is an S-mn-semiprimary ideal for every positive integer m.*

Proof. (1) This is clear.

(2) ($\Rightarrow$) First, we demonstrate that $I \cap S^* = \emptyset$. Assume the contrary; let $u \in I \cap S^*$. Then $\frac{u}{1}$ is a unit in $S^{-1}R$, implying that $\frac{u}{1}\frac{v}{t} = 1$ for some $v \in R$ and $t \in S$, hence leading to $uvs \in I \cap S$ for some $s \in S$, which is a contradiction since S^* contains S and no element of S is in I. Therefore, the conclusion is evident.

($\Leftarrow$) Let $s' \in S$, and consider $x, y \in R$ such that $x^n y^n \in I$. Then either $s'x^n \in I$ or $s'y^n \in I$. Furthermore, there exist $r \in R$ and $s \in S$ such that $\frac{s'}{1} \cdot \frac{r}{s} = \frac{1}{1}$. Thus, $us'r = us$ for some $u \in S$. Setting $t = us$, we find that either $tx^n \in I$ or $ty^n \in I$, completing the proof.

(3) Let $x, y \in R$ such that $x^n y^n \in J \cap R$. Then there exists an element $s \in S$ such that either $sx^n \in J$ or $sy^n \in J$. Consequently, $sx^n \in J \cap R$ or $sy^n \in J \cap R$, as desired.

(4) This is straightforward. □

Let I be an ideal of a commutative ring R, and let $s \in R$. We denote $(I : s) = \{x \in R : sx \in I\}$. Then, for any $s \in R$, $(I : s)$ forms an ideal of R. Our next proposition characterizes the S-n-semiprimary ideal.

Theorem 2.4. *Let R be a commutative ring, $S \subseteq R$ be a multiplicative set, and I be an ideal of R disjoint from S. Then I is an S-n-semiprimary ideal of R if and only if $(I : s)$ is an n-semiprimary ideal of R for some $s \in S$.*

Proof. ($\Rightarrow$) Let $x^n y^n \in (I : s)$. Then $sx^n y^n \in I$. Since I is an S-n-primary ideal, either $s(sx)^n \in I$ or $sy^n \in I$. In the former case, $(s^2)^n x^n \in I$, since I is an S-n-primary ideal, either $s(s^2)^n \in I$, which contradicts to $I \cap S = \emptyset$, or $sx^n \in I$. Thus we have either $sx^n \in I$ or $sy^n \in I$, which means that $x^n \in (I : s)$ or $y^n \in (I : s)$.

($\Leftarrow$) This implication is straightforward. □

Let R and T be commutative rings with identity, let S be a multiplicative subset of R, and let $\varphi : R \longrightarrow T$ be a ring homomorphism ensuring that $\varphi(S)$ does not contain zero. It follows that $\varphi(S)$ is a multiplicative subset of $\varphi(R)$.

Proposition 2.5. *Let R and T be commutative rings with identity, and let S be a multiplicative subset of R. Suppose $\varphi : R \longrightarrow T$ is a ring homomorphism such that $\varphi(S)$ does not contain zero. Then the following statements hold:*

(1) *Let I be an ideal of R disjoint from S and satisfying Ker $(\varphi) \subseteq I$. Then I is an S-n-semiprimary ideal of R if and only if $\varphi(I)$ is a $\varphi(S)$-n-semiprimary ideal of Im (φ).*

(2) *Let J be an ideal of Im (φ) disjoint from $\varphi(S)$. Then J is a $\varphi(S)$-n-semiprimary ideal of Im (φ) if and only if $\varphi^{-1}(J)$ is an S-n-semiprimary ideal of R.*

Proof. (1) Assume I is an S-n-semiprimary ideal of R. First, we show that $\varphi(I) \cap \varphi(S) = \emptyset$. Suppose otherwise, that there exists $i \in I$ such that $\varphi(i) \in \varphi(S)$. Then $i - s \in \mathrm{Ker}\,(\varphi) \subseteq I$ for some $s \in S$, implying $s \in I$, contradicting I being disjoint from S. Hence, $\varphi(I)$ is disjoint from $\varphi(S)$. Now, let $\varphi(x), \varphi(y) \in \mathrm{Im}\,(\varphi)$ such that $\varphi(x)^n \varphi(y)^n \in \varphi(I)$. Then there exists $i \in I$ with

$x^ny^n - i \in \text{Ker}(\varphi) \subseteq I$, so $x^ny^n \in I$. Since I is S-n-semiprimary, there exists $s \in S$ such that either $sx^n \in I$ or $sy^n \in I$. Hence, either $\varphi(s)\varphi(x)^n \in \varphi(I)$ or $\varphi(s)\varphi(y)^n \in \varphi(I)$, showing $\varphi(I)$ is a $\varphi(S)$-n-semiprimary ideal of Im (φ).

Conversely, assume $\varphi(I)$ is a $\varphi(S)$-n-semiprimary ideal of Im (φ). Let $x, y \in R$ with $x^ny^n \in I$. Then $\varphi(x)^n\varphi(y)^n = \varphi(x^ny^n) \in \varphi(I)$. Assume $s \in S$ such that $\varphi(s)\varphi(x^n) \in \varphi(I)$ or $\varphi(s)\varphi(y)^n \in \varphi(I)$. This implies $sx^n - i \in \text{Ker}(\varphi) \subseteq I$ or $sy^n - i \in \text{Ker}(\varphi) \subseteq I$ for some $i \in I$, so either $sx^n \in I$ or $sy^n \in I$, confirming I is an S-n-semiprimary ideal of R.

(2) Note $J \cap \varphi(S) = \emptyset$ iff $\varphi^{-1}(J) \cap S = \emptyset$. Assume J is a $\varphi(S)$-n-semiprimary ideal of Im (φ). Let $x, y \in R$ with $x^ny^n \in \varphi^{-1}(J)$. Then $\varphi(x^n)\varphi(y^n) = \varphi(x^ny^n) \in J$, so there exists $s \in S$ with $\varphi(s)\varphi(x^n) \in J$ or $\varphi(s)\varphi(y)^n \in J$, resulting in $sx^n \in \varphi^{-1}(J)$ or $sy^n \in \varphi^{-1}(J)$, concluding $\varphi^{-1}(J)$ is an S-n-semiprimary ideal of R. The converse follows from (1). □

As an immediate consequence of Proposition 2.5, we obtain the following results:

Corollary 2.6. *Let R be a commutative ring with identity, S be a multiplicative subset of R, and $\varphi : R \longrightarrow T$ be a ring homomorphism. Then there exists a one-to-one, order-preserving correspondence between the S-n-semiprimary ideals of R containing Ker (φ) and the $\varphi(S)$-n-semiprimary ideals of $\varphi(R)$.*

Let R be a commutative ring with identity, I be an ideal of R, and S be a multiplicative subset of R. Then $S/I := \{s + I \mid s \in S\}$ is a multiplicative subset of R/I.

The next corollary adapts [2, Theorem 2.2(b)] to the S-setting.

Corollary 2.7. *Let R be a ring, S be a multiplicative set of R, and let I and J be two ideals of R such that $I \subseteq J$ and $J \cap S = \emptyset$. Then the following assertions are equivalent:*

(1) *J is an S-n-semiprimary ideal of R.*

(2) *J/I is an S/I-n-semiprimary ideal of R/I.*

Proof. (1) ⇔ (2) Consider the natural homomorphism $\pi : R \longrightarrow R/I$, defined by $\pi(a) = a + I$ for each $a \in R$. The equivalence of the assertions is established by applying Proposition 2.5(1). □

Corollary 2.8. *Let R be a ring, S be a multiplicative set of R, and I be a proper ideal of R. Then the following statements are equivalent:*

(1) *(I, X) is an S-n-semiprimary ideal of $R[X]$.*

(2) *I is an S-n-semiprimary ideal of R.*

Proof. (1) ⇔ (2) This equivalence follows from Corollary 2.7 and the isomorphism $(I, X)/(X) \cong I$ in $R[X]/(X) \cong R$, as required. □

We now turn our attention to the study of the localization of the S-n-semiprimary ideal property.

The following corollary presents the S-formulation of [2, Theorem 2.2(c)].

Proposition 2.9. *Let R be a ring, S be a multiplicative subset of R, and I be an ideal of R such that $I \cap S = \emptyset$. If I is an S-n-semiprimary ideal of R, then $S^{-1}I$ is an n-semiprimary ideal of $S^{-1}R$.*

Proof. Assume that I is an S-n-semiprimary ideal of R. Suppose $\left(\frac{x}{s_1}\right)^n \cdot \left(\frac{y}{s_2}\right)^n \in S^{-1}I$ for $x, y \in R$ and $s_1, s_2 \in S$. Then $\frac{x^ny^n}{(s_1)^n(s_2)^n} = \frac{a}{t}$ for some $a \in I$ and $t \in S$, leading to $s'x^ny^nt = s'a(s_1)^n(s_2)^n \in I$ for some $s' \in S$. Since I is an S-n-semiprimary ideal of R, there exists an element $s \in S$ such that either $sx^n \in I$ or $s(s'ty)^n \in I$. Consequently, either $\left(\frac{x}{s_1}\right)^n = \frac{sx^n}{s(s_1)^n} \in S^{-1}I$ or $\left(\frac{y}{s_2}\right)^n = \frac{s(s'ty^n)}{ss't(s_2)^n} \in S^{-1}I$, fulfilling the desired condition. □

The following example demonstrates that the converse of the previous proposition is not generally true.

Example 2.10. Let $E = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ be a ring, and let $S = S_0 \times \{0\}$ be a multiplicative set of $R = \mathbb{Z} \times E$, where $S_0 = \mathbb{N}^*$ is a multiplicative set of $\mathbb{N}$. Since $S^{-1}R = \mathbb{Q}$, for every $e \in E$, there exists $s \in S$ such that $se = 0$. This implies that every ideal of R is prime. On the other hand, consider $\mathcal{P} = \{p \in \mathbb{N} : p \text{ is prime}\}$, which we divide into two infinite sets $\mathcal{P}_1$ and $\mathcal{P}_2$. Define $I = 0 \times \bigoplus_{n\in\mathcal{P}_1} \mathbb{Z}/n\mathbb{Z}$. To argue by contradiction, suppose I is S-prime. Then, for some $(s, 0) \in S$ and any $(0, x), (0, y) \in R$ with $(0, x)(0, y) \in I$, it must follow that $(s, 0)(0, x) \in I$ or $(s, 0)(0, y) \in I$. Choose $p_2 \in \mathcal{P}_2$ such that $\overline{s} \neq \overline{0}$ in $\mathbb{Z}/p_2\mathbb{Z}$, and let $e \in E$ satisfy $\pi_{p_2}(e) = \overline{1}$, where $\pi_n : E \to \mathbb{Z}/n\mathbb{Z}$ is the natural projection. Since $(0, e)(0, e) = (0, 0)$, we find $(0, s)(0, e) = (0, se) \in I$, leading to $se \in \bigoplus_{n\in\mathcal{P}_2} \mathbb{Z}/n\mathbb{Z}$. Consequently, $\pi_{p_2}(se) = \overline{s} \in \pi_{p_2}\left(\bigoplus_{n\in\mathcal{P}_2} \mathbb{Z}/n\mathbb{Z}\right) = \overline{0}$, a contradiction. Thus, I is not S-prime.

The proposition below offers an S-analogue of the statement that a strongly n-semiprimary ideal is n-semiprimary [2, p. 4].

Proposition 2.11. *Let R be a ring, $S \subseteq R$ a multiplicative set, and I an ideal of R disjoint from S. Assume that there exists $s \in S$ such that for any two ideals P and Q of R, if $P^nQ^n \subseteq I$, then $sP^n \subseteq I$ or $sQ^n \subseteq I$. Then I is an S-n-semiprimary ideal.*

Proof. Suppose $x, y \in R$ such that $x^ny^n \in I$. This implies $(x^nR)(y^nR) \subseteq I$. Consequently, either $s(x^nR) \subseteq I$ or $s(y^nR) \subseteq I$, so either $sx^n \in I$ or $sy^n \in I$. □

As an immediate consequence of the previous proposition, we obtain the following:

Corollary 2.12. *Let R be a commutative ring with identity, $S \subseteq R$ a multiplicative set, and I an ideal of R disjoint from S. Assume that there exists $s \in S$ such that for any ideals $P_1, \ldots, P_k$ of R, if $P_1^n \cdots P_k^n \subseteq I$, then $sP_i^n \subseteq I$ for some $i \in \{1, \ldots, k\}$. Then I is an S-n-semiprimary ideal.*

Proposition 2.13. *Let R be a ring, S be a multiplicative set of R, and I be an ideal of R disjoint from S. Then I is an S-n-semiprimary ideal if and only if there exists $s \in S$ such that for any $x_1, \ldots, x_k \in R$, if $x_1^n \cdots x_k^n \in I$, then $sx_i^n \in I$ for some $i \in \{1, \ldots, k\}$.*

Proof. ($\Leftarrow$) Assume $k = 2$ for simplicity.

($\Rightarrow$) Consider $x_1, \ldots, x_k \in R$ such that $x_1^n \cdots x_k^n \in I$. Thus, $(x_1^nR) \cdots (x_k^nR) \subseteq I$. By Corollary 2.12, $s(x_i^nR) \subseteq I$ for some $i \in \{1, \ldots, k\}$ and $s \in S$. Hence, $sx_i^n \in I$. □

Corollary 2.14. *Let R be a ring and I an ideal of R. Assume that for any ideals $P_1, \ldots, P_k$ of R, if $P_1^n \cdots P_k^n \subseteq I$, then $P_i^n \subseteq I$ for some $i \in \{1, \ldots, k\}$. Then I is an n-semiprimary ideal.*

Proof. Assume $S = \{1\}$ in Corollary 2.12. □

We next show that if I is an S-n-semiprimary ideal, then $\sqrt{I}$ is S-prime. This is the S-extension of [2, Theorem 2.3].

Theorem 2.15. *Let I be an S-n-semiprimary ideal of a commutative ring R. Then $\sqrt{I}$ is an S-prime ideal of R, and there exists $s \in S$ such that $sx^n \in I$ for every $x \in \sqrt{I}$.*

Proof. Assume that I is an S-n-semiprimary ideal. Consider any $x, y \in \sqrt{I}$. There exists a positive integer k such that $(x^k)^n(y^k)^n = (xy)^{kn} \in I$. Since I is an S-n-semiprimary ideal of R, there exists $s \in S$ such that $s(x^k)^n \in I$ or $s(y^k)^n \in I$. Consequently, either $sx \in \sqrt{I}$ or $sy \in \sqrt{I}$, confirming that $\sqrt{I}$ is an S-prime ideal of R.

Next, let $x \in \sqrt{I}$, and let k be the least positive integer such that $s^{k+1}x^k \in I$.

Case 1: If $k \leq 2n$, then we get the result.

Case 2: If $k > 2n$, then we have two subcases:

Subcase 1: If $k \neq mn$ for any $m \in \mathbb{N}$, then there exists $m \in \mathbb{N}$ such that $(m-1)n < k < mn$. Since $s^{mn}x^{mn} = (sx)^n(sx)^{(m-1)n} \in I$ for some $t \in R$, we obtain that $s^{n+1}x^n \in I$ or $s^{(m-1)n+1}x^{(m-1)n} \in I$, which is a contradiction, since $n < k$ and $(m-1)n < k$.

Subcase 2: If there exists $m \in \mathbb{N}$ such that $k = mn$, then $s^{mn+1}x^{mn} \in I$. Thus, $(s^2x)^n(sx)^{(m-1)n} = s^{m(n+1)}x^{mn} \in I$, and so $s(s^2x)^n \in I$ or $s(sx)^{(m-1)n} \in I$. Hence $s^{2n+1}x^n \in I$ or $s^{(m-1)n+1}x^{(m-1)n} \in I$. So $2n \geq k$ or $(m-1)n \geq k$. Since $k = mn$, we get $m \leq 2$ or $m \leq m-1$, a contradiction.

Therefore, only **Case 1** occurs, completing the proof. □

The following result extends [2, Theorem 2.5] to the S-context.

Theorem 2.16. *Let S be a multiplicative set of a ring R, and let I be a proper ideal of R such that $P = \sqrt{I}$ is an S-prime ideal of R and $P^n \subseteq I$ for a positive integer n. Then I is an S-m-semiprimary ideal of R for every integer $m \geq n$. In particular, Q^n is an S-m-semiprimary ideal of R for every S-prime ideal Q of R and for every integer $m \geq n$.*

Proof. Assume $x^n y^n = (xy)^n \in I \subseteq P$ for any $x, y \in R$. Since $P = \sqrt{I}$ is S-prime, there exists $s \in S$ such that either $sx \in P$ or $sy \in P$. Consequently, either $sx^n \in P^n \subseteq I$ or $sy^n \in P^n \subseteq I$, where $t := s^n$. Therefore, I is an S-n-semiprimary ideal of R. Moreover, $P^m \subseteq P^n \subseteq I$ for every integer $m \geq n$, thus ensuring that I is an S-m-semiprimary ideal of R for every integer $m \geq n$. The "in particular" statement is self-evident, as desired. □

Recall from [17] that a proper ideal I of a commutative ring R disjoint from S is called an S-n-absorbing ideal if there exists $s \in S$ such that for any $x_1, \ldots, x_{n+1} \in R$, if $x_1 \cdots x_{n+1} \in I$, then $s \prod_{\substack{1 \leq i \leq n+1 \\ i \neq j}} x_i \in I$ for some $1 \leq j \leq n+1$. In this case, we say that I is associated with s.

The next result provides an S-form of [2, Corollary 2.7].

Corollary 2.17. *Let I be an S-n-absorbing ideal of a commutative ring R. If $\sqrt{I}$ is an S-prime ideal of R, then I is an S-m-semiprimary ideal of R for every integer $m \geq n$.*

Proof. Assume $\sqrt{I}$, denoted as P, is an S-prime ideal of R. Since I is an S-n-absorbing ideal, it follows that $P^n = (\sqrt{I})^n \subseteq I$. Therefore, by Theorem 2.16, I is an S-m-semiprimary ideal of R for every integer $m \geq n$. □

The following finding translates [2, Corollary 2.8] into the S-framework.

Corollary 2.18. *Let S be a multiplicative subset of a ring R, and let $P_1 \subseteq \cdots \subseteq P_i$ be S-prime ideals of R with positive integers $n_1, \ldots, n_i$. Then $I = P_1^{n_1} \cdots P_i^{n_i}$ is an S-m-semiprimary ideal of R for every integer $m \geq n_1 + \cdots + n_i$.*

Proof. Note that $\sqrt{I} = P_1$ is an S-prime ideal of R, and $P_1^n \subseteq P_1^{n_1} \subseteq \cdots \subseteq P_i^{n_i} = I$, where $n = n_1 + \cdots + n_i$. So, by Theorem 2.16, I is an S-m-semiprimary ideal of R for every integer $m \geq n$, as needed. □

We next collect some results of S-n-semiprimary ideals.

The subsequent result corresponds to the S-version of [2, Theorem 2.14].

Theorem 2.19. *Let I be an S-n-semiprimary ideal of R. Then, the following assertions hold:*

(1) *If $x^m y^k \in I$ for $x, y \in R$ and positive integers m and k, then either $sx^n \in I$ or $sy^n \in I$. In particular, if $x^m \in I$ for $x \in R$ and m a positive integer, then $sx^n \in I$.*

(2) *I is an S-m-semiprimary ideal of R for every positive integer $m \geq n$.*

Proof. (1) Consider $x^m y^k \in I$ for $x, y \in R$. We may assume that $m \geq k$. Therefore, $x^m y^m = (x^m y^k) y^{m-k} \in I$, implying that $xy \in \sqrt{I}$. Consequently, there exists $t \in S$ such that $tx^n y^n = t(xy)^n \in I$, by Theorem 2.15. Since I is an S-n-semiprimary ideal of R, there exists an element $s \in S$ such that either $sx^n \in I$ or $sy^n \in I$. The "in particular" statement is straightforward.

(2) Let $x^m y^k \in I$ for $x, y \in R$, with $m \geq n$. By part (1), either $sx^n \in I$ or $sy^n \in I$. Therefore, $sx^m = sx^{m-n} x^n \in I$ or $sy^m = sy^{m-n} y^n \in I$, since $m \geq n$. This ensures that I operates as an S-m-semiprimary ideal of R. □

The next proposition studies the S-n-semiprimary ideal under a decomposable ring.

Proposition 2.20. *Let R_1 and R_2 be commutative rings, and let S_1 and S_2 be multiplicative subsets of R_1 and R_2, respectively. The following statements are equivalent:*

(1) *$I_1 \times I_2$ is an $(S_1 \times S_2)$-n-semiprimary ideal of $R_1 \times R_2$.*

(2) *I_1 is an S_1-n-semiprimary ideal of R_1 and $S_2 \cap I_2 \neq \emptyset$, or I_2 is an S_2-n-semiprimary ideal of R_2 and $S_1 \cap I_1 \neq \emptyset$.*

Proof. (1) $\Rightarrow$ (2) First, we show that either $S_1 \cap I_1 \neq \emptyset$ or $S_2 \cap I_2 \neq \emptyset$. Given that $(0,0) = (1,0)^n(0,1)^n \in I_1 \times I_2$, either $(s_1,s_2)(1,0) \in I_1 \times I_2$ or $(s_1,s_2)(0,1) \in I_1 \times I_2$ must hold, implying $s_1 \in I_1$ or $s_2 \in I_2$. Thus, $S_1 \cap I_1 \neq \emptyset$ or $S_2 \cap I_2 \neq \emptyset$.

Assume now that $I_1 \times I_2$ is an $(S_1 \times S_2)$-n-semiprimary ideal of $R_1 \times R_2$. For $(a,b),(c,d) \in R_1 \times R_2$ such that $(a,b)^n(c,d)^n \in I_1 \times I_2$, since $(a,0)^n(c,0)^n \in I_1 \times I_2$, there exists $(s_1,s_2) \in S_1 \times S_2$ such that either $(s_1,s_2)(a,0)^n \in I_1 \times I_2$ or $(s_1,s_2)(c,0)^n \in I_1 \times I_2$, i.e., $s_1a^n \in I_1$ or $s_1c^n \in I_1$. Therefore, I_1 is an S_1-n-semiprimary ideal of R_1. Similarly, if $S_2 \cap I_2 \neq \emptyset$, then I_2 is an S_2-n-semiprimary ideal of R_2.

(2) $\Rightarrow$ (1) Assume $S_2 \cap I_2 \neq \emptyset$ and I_1 is an S_1-n-semiprimary ideal of R_1. For any $(a,b),(c,d) \in R_1 \times R_2$ with $(a,b)^n(c,d)^n \in I_1 \times I_2$, since $(a,0)^n(c,0)^n \in I_1 \times I_2$, there exists $(s_1,s_2) \in S_1 \times S_2$ such that either $(s_1,s_2)(a,0)^n \in I_1 \times I_2$ or $(s_1,s_2)(c,0)^n \in I_1 \times I_2$, i.e., $s_1a^n \in I_1$ or $s_1c^n \in I_1$. Let $s_2 \in S_2 \cap I_2$. Thus, $(s_1,s_2)(a,b)^n = (s_1a^n, s_2b^n) \in I_1 \times I_2$ or $(s_1,s_2)(c,d)^n = (s_1c^n, s_2d^n) \in I_1 \times I_2$. Hence, $I_1 \times I_2$ is an $(S_1 \times S_2)$-n-semiprimary ideal of $R_1 \times R_2$, as required. □

Corollary 2.21. *$I_1 \times I_2$ is an n-semiprimary ideal of $R_1 \times R_2$ if and only if $I_1 = R_1$ and I_2 is an n-semiprimary ideal of R_2, or $I_2 = R_2$ and I_1 is an n-semiprimary ideal of R_1.*

3 S-n-Powerful Semiprimary Ideals

In this section, we study S-n-powerful semiprimary ideals in an integral domain.

Recall from [2] that a proper ideal I of an integral domain R with quotient field K and n a positive integer is an n-powerful ideal of R if whenever $x^ny^n \in I$ for $x,y \in K$, then $x^n \in R$ or $y^n \in R$. An ideal I is an n-powerful semiprimary ideal of R if whenever $x^ny^n \in I$ for $x,y \in K$, then $x^n \in I$ or $y^n \in I$. One of the main topics is the S-version of this concept.

Definition 3.1. Let R be an integral domain with quotient field K, n a positive integer, and S a multiplicative set of R. An ideal I of R, disjoint from S, is called:

(1) An S-n-powerful ideal of R if there exists an $s \in S$ such that for all $x,y \in K$ with $x^ny^n \in I$, either $sx^n \in R$ or $sy^n \in R$.

(2) An S-n-powerful semiprimary ideal of R if there exists an $s \in S$ such that for all $x,y \in K$ with $x^ny^n \in I$, either $sx^n \in I$ or $sy^n \in I$.

We next collect some immediate classes of S-n-powerful semiprimary ideals.

Remark 3.2. Let R be a ring and S a multiplicative set of R. Then the following statements hold:

(1) Every n-powerful (respectively, n-powerful semiprimary) ideal of R is an S-n-powerful (respectively, S-n-powerful semiprimary) ideal of R.

(2) Every S-n-powerful semiprimary ideal is an S-n-powerful ideal for every multiplicative set S of R.

(3) If I is an S-n-powerful ideal of R, then I is an S-mn-powerful ideal of R for every positive integer m.

In the following proposition, we demonstrate that S-n-powerful ideals exhibit similar properties to those of S-n-semiprimary ideals.

The result that follows is the S-analogue of [2, Theorem 4.2(b) and (c)].

Proposition 3.3. *Let R be an integral domain with quotient field K, and let S be a multiplicative set of R. The following assertions hold:*

(1) *Let $I \subseteq J$ be proper ideals of R. If J is an S-n-powerful ideal of R, then I is also an S-n-powerful ideal of R.*

(2) *If I is an S-n-powerful ideal of R with $I \cap S = \emptyset$, then $S^{-1}I$ is an n-powerful ideal of $S^{-1}R$.*

(3) *If I is an S-n-powerful semiprimary ideal of R with $I \cap S = \emptyset$, then $S^{-1}I$ is an n-powerful semiprimary ideal of $S^{-1}R$.*

Proof. (1) Assume $x^n y^n \in I \subseteq J$ for $x, y \in K$. Since J is an S-n-powerful ideal of R, there exists $s \in S$ such that either $sx^n \in R$ or $sy^n \in R$. Thus, I is an S-n-powerful ideal of R.

(2) Omitted proof (similar to Proposition 2.9).

(3) Omitted proof (similar to Proposition 2.9). □

The next theorem represents the S-generalization of [2, Theorem 4.25].

Proposition 3.4. *Let S be a multiplicative set of R and I be an S-prime ideal of an integral domain R with quotient field K. If I is an S-n-powerful semiprimary ideal of R, then I is an S-mn-powerful semiprimary ideal of R for every positive integer m. Additionally, if $x^m \in I$ for a positive integer m and $x \in K$, then $sx^n \in I$.*

Proof. Suppose $x^{mn} y^{mn} \in I$ for $x, y \in K$ and let m be a positive integer. Consequently, $(x^m)^n (y^m)^n \in I$. Since I is an S-n-powerful semiprimary ideal of R, there exists $s \in S$ such that $s(x^m)^n = sx^{mn} \in I$ or $s(y^m)^n = sy^{mn} \in I$. Thus, I is an S-mn-powerful semiprimary ideal of R.

Next, assume $x^m \in I$ for $x \in K$ and some positive integer m. Hence $(tx)^{mn} = t^{mn}(x^m)^n \in I$ for any $t \in S$. Let d be the smallest positive integer such that $(tx)^{dn} \in I$. Since $(tx)^{(d-1)n}(tx)^n = (tx)^{dn} \in I$ and I is an S-n-powerful semiprimary ideal of R, there exists $u \in S$ such that either $u\left((tx)^{d-1}\right)^n \in I$ or $u(tx)^n \in I$. By Proposition 2.4(3), $d = 1$, and thus $sx^n \in I$. □

In the subsequent discussion, which is the S-counterpart of [2, Theorem 4.3], we explore how the concepts of S-n-powerful semiprimary ideals and S-n-powerful ideals coincide for S-prime ideals.

Theorem 3.5. *Let S be a multiplicative set of R and I be an S-prime ideal of an integral domain R with quotient field K. The following statements are equivalent:*

(1) *I is an S-n-powerful semiprimary ideal of R.*

(2) *I is an S-n-powerful ideal of R.*

Proof. (1) ⇒ (2) This direction is straightforward.

(2) ⇒ (1) Assume I is an S-n-powerful ideal of R. Then there exists $s_1 \in S$ such that for all $x, y \in K$ with $x^n y^n \in I$, we have $s_1 x^n \in R$ or $s_1 y^n \in R$.

Since I is S-prime, there exists $s_2 \in S$ such that for all $x, y \in R$ with $xy \in I$, we have $s_2 x \in I$ or $s_2 y \in I$. Set $s = s_1 s_2$. Thus, I is an S-n-powerful (resp., S-prime) ideal of R with respect to s.

Now, let $x^n y^n \in I$ for $x, y \in K$, and suppose $s^2 x^n, s^2 y^n \in R$. Since I is an S-prime ideal of R with $s^2 x^n s^2 y^n \in I$, we get $s^3 x^n \in I$ or $s^3 y^n \in I$. Thus, we may assume $s^2 x^n \notin R$, and hence $s^2 y^n \in R$ since I is an S-n-powerful ideal of R.

Since $s^2 x^n \notin R$ and I is an S-n-powerful ideal of R, we have $s^2 x^{2n} = (sx^n)(sx^n) \notin I$. Assume $s^2 x^{2n} \in R$, so $s^2 x^{2n}, y^{2n} \in R$. Since $sx^{2n} sy^{2n} = s^2 (x^n y^n)^2 \in I$ and $s^2 x^{2n} \notin I$, we have $s^2 y^{2n} \in I$. Hence, $(s^2 y^n)(s^2 y^n) \in I$ and $s^2 y^n \in R$. Therefore, we get $s^3 y^n \in I$ since I is S-prime with respect to s.

Now, in the first case, assume $s^2 x^{2n} \in R$ (and we also have $s^2 y^n \in R$), then $(sx^{2n})(s^2 x^n) = s^3 x^n (x^n y^n) \in I$. Hence, using the fact that I is an S-prime ideal of R (with respect to s), we conclude that either $ssx^{2n} = (sx)^{2n} \in I$ (and thus $s^2 x^n \in I$, which is impossible since $s^2 x^n \notin R$) or $ss^2 y^n \in I$, and this is our conclusion.

In the second case, if $s^2 x^{2n} \notin R$, then $x^n y^n = \left(\frac{y^2}{xy}\right)^n x^{2n} \in I$, and since I is an S-n-powerful ideal of R with respect to s, we have $sy^{2n}/x^n y^n \in R$. Thus, $sy^{2n} = sx^n y^n \left(\frac{y^{2n}}{x^n y^n}\right) \in I$. Therefore, $(sy^n)^2 \in I$, and since I is S-prime with respect to s, we have $s^2 y^n \in I$. This implies that $s^3 y^n \in I$.

Consequently, in all cases, we have $s^3 x^n \in I$ or $s^3 y^n \in I$. Thus, I is an S-n-powerful semiprimary ideal of R with respect to s^3, as required. □

Proposition 3.6. *Let R be a ring, S a multiplicative set of R, and I an ideal of R disjoint from S. Let J be an ideal of R such that $J \cap S \neq \emptyset$. If I is an S-n-powerful semiprimary ideal of R, then JI is also an S-n-powerful semiprimary ideal of R.*

Proof. Let $s \in S \cap J$. Assume $x^n y^n \in JI \subseteq I$ for $x, y \in K$. Since I is an S-n-powerful semiprimary ideal, there exists $s_1 \in S$ such that either $s_1 x^n \in I$ or $s_1 y^n \in I$. Consequently, either $ss_1 x^n \in JI$ or $ss_1 y^n \in JI$, ensuring that JI is an S-n-powerful semiprimary ideal of R. This completes the proof. □

We conclude this section with the following result, extending [2, Theorem 4.4] to the S-context.

Theorem 3.7. *Let $I \subseteq J$ be S-prime ideals of an integral domain R. If J is an S-n-powerful semiprimary ideal of R, then I is also an S-n-powerful semiprimary ideal of R.*

Proof. Assuming J is an S-n-powerful ideal of R, it follows from Proposition 3.3(1) that I is an S-n-powerful ideal of R. Therefore, by Theorem 3.5, I is an S-n-powerful semiprimary ideal of R, as required. □

4 Idealization of S-n-semiprimary ideals

We start this section by discussing the transfer of S-n-semiprimary ideal property to trivial ring extensions. First, note that if S is a multiplicative set of R, then $S \ltimes 0$ is a multiplicative set of $R \ltimes M$. Our goal is to derive an S-generalization of [2, Theorem 3.8].

Theorem 4.1. *Let R be a ring, $S \subseteq R$ a multiplicative subset, I a proper ideal of R, M an R-module, and N a submodule of M with $IM \subseteq N$. The following statements are true:*

(1) *If $I \ltimes N$ is an $(S \ltimes 0)$-n-semiprimary ideal of $R \ltimes M$, then I is an S-n-semiprimary ideal of R.*

(2) *If there exists $s \in S$ such that $sM \subseteq N$, then $I \ltimes N$ is an $(S \ltimes 0)$-n-semiprimary ideal of $R \ltimes M$ if and only if I is an S-n-semiprimary ideal of R. Specifically, $I \ltimes M$ is an $(S \ltimes 0)$-n-semiprimary ideal if and only if I is an S-n-semiprimary ideal.*

(3) *If I is an S-n-semiprimary ideal of R, then $I \ltimes N$ is an $(S \ltimes 0)$-$(n+1)$-semiprimary ideal of $R \ltimes M$.*

Proof. (1) Let $a, b \in R$ such that $a^n b^n \in I$. Then $(a,0)^n (b,0)^n = (a^n b^n, 0) \in I \ltimes N$. According to the hypothesis, there exists $(s,0) \in (S \ltimes 0)$ such that either $(s,0)(a,0)^n = (sa^n, 0) \in I \ltimes N$ or $(s,0)(b,0)^n = (sb^n, 0) \in I \ltimes N$. This implies $sa^n \in I$ or $sb^n \in I$, respectively. Therefore, I is an S-n-semiprimary ideal of R.

(2) The direct implication follows from (1). Conversely, assume that I is an S-n-semiprimary ideal and let $(a,b), (c,d) \in R \ltimes M$ such that $(a,b)^n (c,d)^n \in I \ltimes N$. This implies $a^n c^n \in I$. By hypothesis, there exists $s_1 \in S$ such that $s_1 a^n \in I$ or $s_1 c^n \in I$. Hence, $(s_1, 0)(a,b)^n \in I \ltimes N$ or $(s_1, 0)(c,d)^n \in I \ltimes N$, confirming that $I \ltimes N$ is an $(S \ltimes 0)$-n-semiprimary ideal of $R \ltimes M$.

(3) Suppose that I is an S-n-semiprimary ideal of R, and let $(a,m), (b,h) \in R \ltimes M$. Then $(a,m)^{n+1}(b,h)^{n+1} = (a^{n+1}b^{n+1}, z) \in I \ltimes N$ for some z. By applying Theorem 2.19(1), since I is an S-n-semiprimary ideal of R, either $sa^{n+1} \in I$ or $sb^{n+1} \in I$. Assuming $sa^{n+1} \in I$, it follows that $s(n+1)a^n m \in IM \subseteq N$. Thus, $(s,0)(a,m)^{n+1} = (sa^{n+1}, s(n+1)a^n m) \in I \ltimes N$, confirming that $I \ltimes N$ is an $(S \ltimes 0)$-$(n+1)$-semiprimary ideal of $R \ltimes M$. □

Remark 4.2. Let I be an S-n-semiprimary ideal that is not an S-$(n-1)$-semiprimary ideal. Then $I \ltimes N$ is an $(S \ltimes 0)$-$(n+1)$-semiprimary ideal but not an $(S \ltimes 0)$-$(n-1)$-semiprimary ideal. If there exists $s \in S$ such that $sM \subseteq N$, $I \ltimes N$ is an $(S \ltimes 0)$-n-semiprimary ideal. However, in general, $I \ltimes N$ is not necessarily an $(S \ltimes 0)$-n-semiprimary ideal, as demonstrated in the following example.

Example 4.3. Let $R = \mathbb{Z} \ltimes \mathbb{Q}$ be a ring, where $S_0 = \mathbb{Z}^*$ and $S = S_0 \ltimes 0$ form a multiplicative set in R. Consider the ideal $0 \ltimes \mathbb{Z}$, which is an S-2-semiprimary but not an S-1-semiprimary ideal. Since 0 is a prime ideal, it is an S_0-1-semiprimary ideal, and thus $0 \ltimes \mathbb{Z}$ is also S-2-semiprimary. However, $0 \ltimes \mathbb{Z}$ is not S-prime. To illustrate, suppose $(a,b)(c,d) \in 0 \ltimes \mathbb{Z}$ implies for some $(s,0) \in S$ that either $(s,0)(a,b) \in 0 \ltimes \mathbb{Z}$ or $(s,0)(c,d) \in 0 \ltimes \mathbb{Z}$. If $(s,0)$ is a unit, then absurdly, $0 \ltimes \mathbb{Z}$ would be prime. Conversely, if $(s,0)$ is nonunit, $\frac{1}{s}$ belongs to $\mathbb{Q} \setminus \mathbb{Z}$. As such, $(0, \frac{1}{s^2})^2 = (0,0) \in 0 \ltimes \mathbb{Z}$, leading to the contradiction that $(0, \frac{1}{s}) = s(0, \frac{1}{s^2}) \in 0 \ltimes \mathbb{Z}$.

Example 4.4. Let $R = \mathbb{Z} \ltimes \mathbb{Z}$ be a ring, and let $S_0 = \{2^n : n \in \mathbb{N}\}$ with $S = S_0 \ltimes 0$ forming a multiplicative set of R. Consequently, $0 \ltimes 2\mathbb{Z}$ is an S-1-semiprimary ideal, which is distinct from being a 1-semiprimary (prime) ideal.

To conclude this section, we derive the S-version of [2, Theorem 3.9].

Theorem 4.5. *Let R be a ring with characteristic $n \geq 2$, and let S be a multiplicative set of R. Suppose I is a proper ideal of R, M is an R-module, and N is a submodule of M such that $IM \subseteq N$. Then $I \ltimes N$ is an $(S \ltimes 0)$-n-semiprimary ideal of $R \ltimes M$ if and only if I is an S-n-semiprimary ideal of R.*

Proof. If $J = I \ltimes N$ is an $(S \ltimes 0)$-n-semiprimary ideal of $A = R \ltimes M$, then I must be an S-n-semiprimary ideal of R. Conversely, assume I is an S-n-semiprimary ideal of R. Consider $(a,m), (b,h) \in A$ such that $(a,m)^n(b,h)^n = (a^n b^n, z) \in J$. Then $sa^n \in I$ or $sb^n \in I$, following that I is an S-n-semiprimary ideal of R. Assuming $sa^n \in I$, and noting that the characteristic of R being $n \geq 2$ implies $na^{n-1}m = 0 \in N$, it follows that $(s,0)(a,m)^n = (sa^n, sna^{n-1}m) = (sa^n, 0) \in J$. Thus, J is confirmed as an $(S \ltimes 0)$-n-semiprimary ideal of A. □

5 Amalgamation of S-n-semiprimary ideals

Let A and B be commutative rings, let J be an ideal of B, and let $f : A \to B$ be a ring homomorphism. We can then define the amalgamation subring $A \bowtie^f J$ of $A \times B$ as follows:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

This subring is known as the amalgamation of A with B along J with respect to f. Readers can refer to [6, 5, 18] for more on amalgamated algebras. The weakly nil-clean property, coherent property, and arithmetical property in the amalgamated algebra were studied in [3], [8], and [13], respectively.

Furthermore, $A \bowtie^f J$ always possesses a prime ideal (see [5, Proposition 2.6(3)]), ensuring the existence of S-n-semiprimary ideals in $A \bowtie^f J$. In this section, we employ some of the results from Section 2, specifically Proposition 2.5(1) and Corollary 2.7.

This section examines the conditions under which well-known ideals of $A \bowtie^f J$ become S-n-semiprimary ideals.

Let I be an ideal of A, S a multiplicative set of A, and K an ideal of $f(A) + J$. Set:

$$S^{\bowtie^f} := \{(s, f(s)) \mid s \in S\},$$
$$I \bowtie^f J := \{(i, f(i) + j) \mid i \in I, j \in J\},$$
$$\overline{K}^f := \{(a, f(a) + j) \mid a \in A, j \in J, f(a) + j \in K\}.$$

Clearly, $S^{\bowtie^f}$ is a multiplicative subset of $A \bowtie^f J$. Moreover, both $I \bowtie^f J$ and $\overline{K}^f$ are ideals of $A \bowtie^f J$. Therefore, we have the following theorem:

Theorem 5.1. *Let $f : A \to B$ be a ring homomorphism and let S be a multiplicative set in A such that $0 \notin f(S)$. Suppose I is an ideal of A disjoint from S, J is an ideal of B, and K is an ideal of $f(A) + J$ disjoint from $f(S)$. Then the following assertions hold:*

(1) *I is an S-n-semiprimary ideal if and only if $I \bowtie^f J$ is an $S^{\bowtie^f}$-n-semiprimary ideal of $A \bowtie^f J$.*

(2) *K is an $f(S)$-n-semiprimary ideal of $f(A) + J$ if and only if $\overline{K}^f$ is an $S^{\bowtie^f}$-n-semiprimary ideal of $A \bowtie^f J$.*

Proof. (1) Consider the ring homomorphism $\psi : A \to (A \bowtie^f J)/(0 \bowtie^f J)$ defined by $\psi(a) = (a, f(a) + (0 \bowtie^f J))$. This map ψ is an isomorphism, and it maps S to $S^{\bowtie^f}/(0 \bowtie^f J)$. By Proposition 2.5 (1), I is an S-n-semiprimary ideal of A if and only if $(I \bowtie^f J)/(0 \bowtie^f J)$ is an $\psi(S)$-n-semiprimary ideal of $(A \bowtie^f J)/(0 \bowtie^f J)$. Further, by Corollary 2.7, $(I \bowtie^f J)/(0 \bowtie^f J)$ is an $\psi(S)$-n-semiprimary ideal of $(A \bowtie^f J)/(0 \bowtie^f J)$ if and only if $I \bowtie^f J$ is an $S^{\bowtie^f}$-n-semiprimary ideal of $A \bowtie^f J$.

(2) Define the epimorphism $\psi : A \bowtie^f J \to f(A) + J$ by $\psi((a, f(a) + j)) = f(a) + j$. The kernel of ψ is $f^{-1}(J) \times \{0\}$, and ψ maps $\overline{K}^f$ to K and $S^{\bowtie^f}$ to $f(S)$. Since $f^{-1}(J) \times \{0\} \subseteq \overline{K}^f$, it follows that K is an $f(S)$-n-semiprimary ideal of $f(A) + J$ if and only if $\overline{K}^f$ is an $S^{\bowtie^f}$-n-semiprimary ideal of $A \bowtie^f J$, in accordance with Proposition 2.5 (1). □

By Theorem 5.1, we have the following example:

Example 5.2. Let A and B be commutative rings such that $A \subseteq B$, and let S be a multiplicative subset of A. Define $I_1 = XB[X]$ and $I_2 = XB[[X]]$.

(1) Consider the natural embedding $\iota_1 : A \to B[X]$. It is straightforward to verify that $A \bowtie^{\iota_1} I_1$ is isomorphic to $A + XB[X]$. Therefore, by Theorem 5.1 (1), an ideal I is an S-n-semiprimary ideal of A if and only if $I + XB[X]$ is an S-n-semiprimary ideal of $A + XB[X]$.

(2) Consider the natural embedding $\iota_2 : A \to B[[X]]$. Verification shows that $A \bowtie^{\iota_2} I_2$ is isomorphic to $A + XB[[X]]$. Hence, by Theorem 5.1 (1), an ideal I is an S-n-semiprimary ideal of A if and only if $I + XB[[X]]$ is an S-n-semiprimary ideal of $A + XB[[X]]$.

Assuming I is an ideal of A and $id_A : A \to A$ is the identity homomorphism on A, then

$$A \bowtie I = A \bowtie^{id_A} I := \{(a, a + i) \mid a \in A \text{ and } i \in I\}$$

is the amalgamated duplication of A along I, as introduced and studied by D'Anna and Fontana in [6]. The next corollary is an immediate consequence of assertion (1) of Theorem 5.1 concerning the transfer of the S-n-semiprimary property to duplications.

Corollary 5.3. *Let A be a ring, I an ideal of A, and S a multiplicative set of A. Let K be an ideal of A disjoint from S. Then $K \bowtie I$ is an $S^{\bowtie^f}$-n-semiprimary ideal of $A \bowtie I$ if and only if K is an S-n-semiprimary ideal of A.*

Let L be an ideal of $A \bowtie^f J$ disjoint from $S^{\bowtie^f}$. Define $\pi_1 : A \bowtie^f J \to A$ by $\pi_1(a, f(a) + j) = a$ and $\pi_2 : A \bowtie^f J \to f(A) + J$ by $\pi_2(a, f(a) + j) = f(a) + j$. Both π_1 and π_2 are epimorphisms. As a consequence of Theorem 5.1, we have the following corollary:

Corollary 5.4. *Let $f : A \to B$ be a ring homomorphism, and let S be a multiplicative set of A such that $f(S)$ does not contain zero. Let I be an ideal of A disjoint from S, J an ideal of B, and K an ideal of $f(A) + J$ disjoint from $f(S)$. Then the following assertions hold:*

(1) *Every $S^{\bowtie^f}$-n-semiprimary ideal of $A \bowtie^f J$ containing $0 \times J$ is of the form $I \bowtie^f J$, where I is an S-n-semiprimary ideal of A.*

(2) *Every $S^{\bowtie^f}$-n-semiprimary ideal of $A \bowtie^f J$ containing $f^{-1}(J) \times \{0\}$ is of the form $\overline{K}^f$, where K is an $f(S)$-n-semiprimary ideal of $f(A) + J$.*

Proof. (1) By Theorem 5.1(1), $I \bowtie^f J$ is an $S^{\bowtie^f}$-n-semiprimary ideal of $A \bowtie^f J$ for any S-n-semiprimary ideal I of A. Suppose that L is an $S^{\bowtie^f}$-n-semiprimary ideal of $A \bowtie^f J$ containing $0 \times J$. It is straightforward to show that $L = \pi_1(L) \bowtie^f J$. Since $\ker(\pi_1) = 0 \times J$, we deduce that $\pi_1(L)$ is an S-n-semiprimary ideal of A, as required by Proposition 2.5(1).

(2) By Theorem 5.1 (2), $\overline{K}^f$ is an $S^{\bowtie^f}$-n-semiprimary ideal of $A \bowtie^f J$ for any $f(S)$-n-semiprimary ideal K of $f(A) + J$. Suppose that L is an $S^{\bowtie^f}$-n-semiprimary ideal of $A \bowtie^f J$ containing $f^{-1}(J) \times \{0\}$. It is straightforward to show that $L = \overline{\pi_2(L)}^f$. Since $\ker(\pi_2) = f^{-1}(J) \times \{0\}$, we conclude that $\pi_2(L)$ is an $f(S)$-n-semiprimary ideal of $f(A) + J$, as supported by Proposition 2.5(1). □

Open Question.

What rings satisfy the condition that every S-n-semiprimary ideal is also an n-semiprimary ideal?

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