

QUASI BI-SLANT SUBMANIFOLDS OF METALLIC RIEMANNIAN MANIFOLDS

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Abstract In this paper, we introduce the concept of quasi bi-slant submanifolds within the context of metallic Riemannian manifolds. We examine certain characteristics of quasi bi-slant submanifolds of metallic and golden Riemannian manifolds. We provide integrability conditions for the distributions as well as some conditions under which submanifolds in metallic and golden Riemannian manifolds are quasi bi-slant. Furthermore, we construct non-trivial examples of quasi bi-slant submanifolds of metallic and golden Riemannian manifolds.

1 Introduction

The idea of a golden structure on the Riemannian manifold was introduced by Crasmareanu and Hretcanu [1] to define a unique geometric structure based on the golden ratio and Fibonacci numbers in differential geometry. In addition, they studied the characteristics of the golden structure related to the invariant submanifold [2] and the applications of the golden ratio on Riemannian manifold [3]. Gezer et al. [4] discussed the integrability of golden Riemannian structure. Hretcanu and Crasmareanu [5] defined the metallic structure on Riemannian manifolds as a generalization of the golden structure. The metallic number $\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}$, which is the positive solution of the equation $x^2 - px - q = 0$ for $p, q \in \mathbb{N}$ serves as the model for the metallic structure ϕ , defined by Goldberg and Yano [6]. These $\sigma_{p,q}$ numbers are members of the metallic means family like golden mean, silver mean, bronze mean, copper mean, nickel mean etc., which were introduced by De Spinadel [7]. Metallic manifolds have numerous applications in mathematics, art and nature. Studying the properties of metallic structure on Riemannian manifolds allows for a deeper understanding of practical applications and advancements in different areas.

The theory of slant submanifolds of Riemannian manifolds is a significant area of study in differential geometry. It offers a comprehensive understanding of submanifold geometry within the context of Riemannian structures. The notion of slant submanifolds was introduced by Chen [8] as a generalization of holomorphic and totally real submanifolds. Many researchers studied the geometry of slant submanifolds leading to a variety of slant submanifolds including semi-slant, generic, hemi-slant, bi-slant, quasi-hemi slant submanifolds. Each type has its own properties and applications in differential geometry and in other fields. Slant submanifolds have been studied within the framework of metallic Riemannian manifolds by many researchers [9, 10, 11, 12, 13, 14, 15]. The concept of quasi bi-slant submanifolds is a generalization of the slant submanifolds. This broader structure allows for diverse geometric interactions in the ambient manifold. These submanifolds are studied for their complex geometry and potential ap-

plications in advanced mathematical contexts. It was first introduced for cosymplectic manifolds by Mehmet and Selahattin [16]. The study of quasi bi-slant submanifolds has been further developed in various contexts such as Sasakian manifolds [17], Kaehler manifolds [18] and nearly Kaehler manifolds [19]. Expanding upon existing research, we investigate the concept of quasi bi-slant submanifolds within the context of metallic Riemannian manifolds and, consequently, for Riemannian manifolds with the golden structure. This paper is organized as follows: Section 2 consists of the important preliminary notions and results required for our study. In section 3, we introduce the concept of quasi bi-slant submanifolds within the framework of metallic and golden Riemannian manifolds and provide examples of such submanifolds in these geometric settings. Furthermore, we characterize these submanifolds, investigate the integrability conditions of the distributions involved, and examine their relationship with metallic and golden Riemannian manifolds.

2 Preliminaries

Let \tilde{N} be an n -dimensional smooth manifold endowed with a $(1, 1)$ tensor field ϕ . This structure ϕ is called a *metallic structure* if it satisfies the following relation

$$\phi^2 = p\phi + qI \quad (2.1)$$

for $p, q \in \mathbb{N}$, where I is the identity operator on $\Gamma(T\tilde{N})$. Then the pair (\tilde{N}, ϕ) is called a *metallic manifold*.

In particular, if $p = q = 1$, then this manifold is called a *golden manifold*.

A Riemannian manifold (\tilde{N}, \tilde{g}) is called *metallic (or golden) Riemannian manifold* $(\tilde{N}, \tilde{g}, \phi)$ if the Riemannian metric \tilde{g} is ϕ -compatible, i.e.,

$$\tilde{g}(\phi X, Y) = \tilde{g}(X, \phi Y). \quad (2.2)$$

Then, we have

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(\phi^2 X, Y) = p\tilde{g}(\phi X, Y) + q\tilde{g}(X, Y). \quad (2.3)$$

A metallic (or golden) Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with Levi-Civita connection $\tilde{\nabla}$ on \tilde{N} is called *locally metallic (or locally golden) Riemannian manifold* if ϕ is parallel with respect to $\tilde{\nabla}$, i.e. $\tilde{\nabla}\phi = 0$.

Now we consider an m -dimensional submanifold N of the n -dimensional metallic (or golden) Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$, where $n, m \in \mathbb{N}$ and $n > m$. We denote the metric induced on the submanifold N by the same symbol \tilde{g} . For $x \in N$, we have

$$T_x\tilde{N} = T_xN \oplus (T_xN)^\perp, \quad (2.4)$$

where T_xN is the tangent space and $(T_xN)^\perp$ is the normal space of N . For $X \in \Gamma(TN)$ and $V \in \Gamma(TN)^\perp$, we have

$$\phi X = lX + mX, \quad (2.5)$$

$$\phi V = LV + MV, \quad (2.6)$$

where $lX, LV \in \Gamma(TN)$ are tangential parts and $mX, MV \in \Gamma(TN)^\perp$ are the normal parts of ϕX and ϕV , respectively. For every $X, Y \in \Gamma(TN)$ and $U, V \in \Gamma(TN)^\perp$, the above components satisfy the following properties,

$$\tilde{g}(lX, Y) = \tilde{g}(X, lY), \quad (2.7)$$

$$\tilde{g}(MU, V) = \tilde{g}(U, MV), \quad (2.8)$$

$$\tilde{g}(mX, U) = \tilde{g}(X, LU). \quad (2.9)$$

Suppose $\tilde{\nabla}, \nabla$ are the Levi-Civita connections on \tilde{N} and N , respectively. Then $\forall X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN)^\perp$, the Gauss and Weingarten formulae are given as

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad (2.10)$$

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad (2.11)$$

where σ is the second fundamental form, A is the shape operator of N and ∇^\perp is the normal connection on $\Gamma(TN)^\perp$

$$\tilde{g}(\sigma(X, Y), V) = \tilde{g}(A_V X, Y). \tag{2.12}$$

Let $\{m_i\}_{i=1}^r$ be a local orthonormal basis of $(T_x N)^\perp$ for $x \in N$, where r is the co-dimension of N in \tilde{N} . Then ϕX and ϕm_α has the following decompositions [5]

$$\phi X = lX + \sum_{\alpha=1}^r u_\alpha(X)m_\alpha, \tag{2.13}$$

$$\phi m_\alpha = \xi_\alpha + \sum_{\beta=1}^r a_{\alpha\beta}m_\beta, \tag{2.14}$$

where ξ_α are vector fields, u_α are 1-forms and $(a_{\alpha\beta})_r$ is an $r \times r$ matrix of smooth real functions on N . Using (2.5), (2.6) in (2.13), (2.14), we obtain

$$mX = \sum_{\alpha=1}^r u_\alpha(X)m_\alpha, \tag{2.15}$$

$$Lm_\alpha = \xi_\alpha, \tag{2.16}$$

$$Mm_\alpha = \sum_{\beta=1}^r a_{\alpha\beta}m_\beta. \tag{2.17}$$

Remark 2.1. [20] Let r be the co-dimension of N in \tilde{N} , $\{m_i\}_{i=1}^r$ be a local orthonormal basis of $(T_x N)^\perp$ and $A_\alpha = A_{m_\alpha}$ for any $\alpha \in \{1, \dots, r\}$, then $\forall X, Y \in \Gamma(TN)$, we have

$$\tilde{\nabla}_X m_\alpha = -A_\alpha X + \nabla_X^\perp m_\alpha, \tag{2.18}$$

$$\sigma_\alpha(X, Y) = \tilde{g}(A_\alpha X, Y). \tag{2.19}$$

Remark 2.2. [20] The normal connection $\nabla_x^\perp m_\alpha$ has the decomposition $\nabla_x^\perp m_\alpha = \sum_{\beta=1}^r t_{\alpha\beta}(X)m_\beta$ for $\alpha \in \{1, \dots, r\}$ and $\forall X \in \Gamma(TN)$, where $(t_{\alpha\beta})$ is an $r \times r$ matrix of 1-forms on N . Moreover $\tilde{g}(m_\alpha, m_\beta) = \delta_{\alpha\beta}$ implies that $\tilde{g}(\nabla_X^\perp m_\alpha, m_\beta) + \tilde{g}(m_\alpha, \nabla_X^\perp m_\beta) = 0$ which is equivalent to $t_{\alpha\beta} = -t_{\beta\alpha}$.

The covariant derivatives of l, m, L, M are defined as:

$$(\tilde{\nabla}_X l)Y = \nabla_X lY - l\nabla_X Y, \tag{2.20}$$

$$(\tilde{\nabla}_X m)Y = \nabla_X^\perp mY - m\nabla_X Y, \tag{2.21}$$

$$(\tilde{\nabla}_X L)V = \nabla_X LV - L\nabla_X^\perp V, \tag{2.22}$$

$$(\tilde{\nabla}_X M)V = \nabla_X^\perp MV - M\nabla_X^\perp V, \tag{2.23}$$

where $X, Y \in \Gamma(TN), V \in \Gamma(TN)^\perp$. From (2.2) we get

$$\tilde{g}((\tilde{\nabla}_X \phi)Y, Z) = \tilde{g}(Y, (\tilde{\nabla}_X \phi)Z) \tag{2.24}$$

$\forall X, Y, Z \in \Gamma(TN)$.

Definition 2.3. [21] A submanifold N of a metallic (or golden) Riemannian manifold \tilde{N} is called bi-slant if there exist two orthogonal differentiable distributions D_1 and D_2 on \tilde{N} such that

- (i) $TN = D_1 \oplus D_2$,
- (ii) $\phi D_1 \perp D_2$ and $\phi D_2 \perp D_1$,
- (iii) The distributions D_1 and D_2 are slant distributions with slant angles θ_1 and θ_2 , respectively.

Let m_1, m_2 be the dimensions of distributions D_1, D_2 , respectively. Then Table 1 shows the classification of slant submanifolds based on the values of m_1, m_2, θ_1 and θ_2 .

Table 1. Classification of slant submanifolds

Case	Dimension of distributions	Angle θ_1 and θ_2	Type of submanifold
(a)	$m_1 \neq 0, m_2 \neq 0$	$0 < \theta_1, \theta_2 < \frac{\pi}{2}$	proper bi-slant
(b)	$m_1 \neq 0, m_2 \neq 0$	$\theta_1 = 0, 0 < \theta_2 < \frac{\pi}{2}$	proper semi-slant
(c)	$m_1 \neq 0, m_2 \neq 0$	$\theta_1 = \frac{\pi}{2}, 0 < \theta_2 < \frac{\pi}{2}$	proper hemi-slant
(d)	$m_1 \neq 0, m_2 \neq 0$	$\theta_1 = 0, \theta_2 = \frac{\pi}{2}$	Semi-invariant
(e)	either $m_1 = 0$ or $m_2 = 0$	$0 < \theta_1 = \theta_2 < \frac{\pi}{2}$	proper slant

Proposition 2.4. [20] Suppose N is a submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$, then $\forall X, Y \in \Gamma(TN)$, we have

$$l[X, Y] = \nabla_X lY - \nabla_Y lX - A_{mY}X + A_{mX}Y, \tag{2.25}$$

$$m[X, Y] = \sigma(X, lY) - \sigma(lX, Y) + \nabla_X^\perp mY - \nabla_Y^\perp mX. \tag{2.26}$$

Proposition 2.5. [22] Suppose N is a submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with co-dimension r , then the Riemannian structure (\tilde{g}, ϕ) induces a structure $(l, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ on N which satisfies the following properties $\forall X, Y \in \Gamma(TN)$,

$$(\nabla_X l)Y = \sum_{\alpha=1}^r \sigma_\alpha(X, Y)\xi_\alpha + \sum_{\alpha=1}^r u_\alpha(Y)A_\alpha X, \tag{2.27}$$

$$(\nabla_X u_\alpha)Y = -\sigma_\alpha(X, lY) + \sum_{\beta=1}^r [u_\beta(Y)t_{\alpha\beta}(X) + \sigma_\beta(X, Y)a_{\beta\alpha}]. \tag{2.28}$$

Proposition 2.6. [23] Suppose N is a submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$, then

$$l[X, Y] = \nabla_X lY - \nabla_Y lX - \sum_{\alpha=1}^r [u_\alpha(Y)A_\alpha X - u_\alpha(X)A_\alpha Y], \tag{2.29}$$

$$m[X, Y] = \sum_{\alpha=1}^r [(\nabla_Y u_\alpha)X - (\nabla_X u_\alpha)Y + \sum_{\beta=1}^r u_\alpha(X)t_{\alpha\beta}(Y) - u_\alpha(Y)t_{\alpha\beta}(X)]m_\alpha, \tag{2.30}$$

where $X, Y \in \Gamma(TN)$.

3 Quasi bi-slant submanifolds of metallic (or golden) Riemannian manifolds

In this section, we introduce the definition of quasi bi-slant submanifolds of metallic (or golden) Riemannian manifolds along with supportive examples. Further, we characterize various properties of quasi bi-slant submanifolds of metallic (or golden) Riemannian manifolds and discuss the integrability conditions for the distributions involved.

Definition 3.1. A quasi bi-slant submanifold N of a metallic Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ is a submanifold that admits three orthogonal complementary distributions D, D_1, D_2 such that:

- (i) $TN = D \oplus D_1 \oplus D_2$,
- (ii) D is invariant,
- (iii) $\phi(D_1) \perp D_2, \phi(D_2) \perp D_1$,
- (iv) D_1 and D_2 are slant with angles θ_1 and θ_2 , respectively.

Table 2. Types of submanifolds

Case	Conditions	Type of Submanifold
(a)	$m_1 \neq 0, m_2 = 0, m_3 = 0$	Invariant
(b)	$m_1 = m_2 = 0, m_3 \neq 0, \theta_2 = \frac{\pi}{2}$	Anti-invariant
(c)	$m_1 \neq 0, m_2 = 0, m_3 \neq 0, \theta_2 = \frac{\pi}{2}$	Semi-invariant
(d)	$m_1 = m_2 = 0, m_3 \neq 0, 0 \leq \theta_2 \leq \frac{\pi}{2}$	Slant
(e)	$m_1 = 0, m_2 \neq 0, m_3 \neq 0, \theta_1 = 0, 0 \leq \theta_2 \leq \frac{\pi}{2}$	Semi-slant
(f)	$m_1 = 0, m_2 \neq 0, m_3 \neq 0, \theta_1 = \frac{\pi}{2}, 0 \leq \theta_2 \leq \frac{\pi}{2}$	Hemi-slant
(g)	$m_1 = 0, m_2 \neq 0, m_3 \neq 0, 0 \leq \theta_1, \theta_2 \leq \frac{\pi}{2}$	Bi-slant
(h)	$m_1 \neq 0, m_2 \neq 0, m_3 \neq 0, 0 \leq \theta_1 \leq \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}$	Quasi hemi-slant
(i)	$m_1 \neq 0, m_2 \neq 0, m_3 \neq 0, 0 < \theta_1, \theta_2 < \frac{\pi}{2}$	Proper quasi bi-slant

Let m_1, m_2 and m_3 be the dimensions of distributions D, D_1 and D_2 , respectively. Table 2 shows that the notion of quasi bi-slant submanifold generalizes the concept of various types of slant submanifolds.

Let N be a quasi bi-slant submanifold of a metallic Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$. Assume F_1, F_2 and F_3 be the projections of $X \in \Gamma(TN)$ on D, D_1 and D_2 respectively, then we have

$$X = F_1X + F_2X + F_3X, \tag{3.1}$$

Using (2.5) and (3.1), we get

$$\phi X = lF_1X + mF_1X + lF_2X + mF_2X + lF_3X + mF_3X. \tag{3.2}$$

Since the distribution D is invariant, we have

$$mF_1X = 0, \phi F_1X = lF_1X, \tag{3.3}$$

and hence, we obtain

$$\phi X = lF_1X + lF_2X + mF_2X + lF_3X + mF_3X. \tag{3.4}$$

Thus,

$$lX = lF_1X + lF_2X + lF_3X \tag{3.5}$$

and

$$mX = mF_2X + mF_3X. \tag{3.6}$$

This implies $\phi(TN) \subset D \oplus lD_1 \oplus mD_1 \oplus lD_2 \oplus mD_2$, while the normal space has the following decomposition

$$(TN)^\perp = mD_1 \oplus mD_2 \oplus \mu, \tag{3.7}$$

where μ is the orthogonal compliment of $mD_1 \oplus mD_2$ and is invariant with respect to ϕ . Moreover $\forall X \in \Gamma(D_1)$, we have

$$\cos(\theta_1)(X) = \frac{\tilde{g}(\phi F_2X, lF_2X)}{\|lF_2X\| \|\phi F_2X\|} = \frac{\|lF_2X\|}{\|\phi F_2X\|}. \tag{3.8}$$

and $\forall X \in \Gamma(D_2)$, we have

$$\cos(\theta_2)(X) = \frac{\tilde{g}(\phi F_3X, lF_3X)}{\|lF_3X\| \|\phi F_3X\|} = \frac{\|lF_3X\|}{\|\phi F_3X\|}. \tag{3.9}$$

Example 3.2. Consider the Euclidean space \mathbb{R}^8 with the usual Euclidean metric \tilde{g} . Suppose $\phi : \mathbb{R}^8 \rightarrow \mathbb{R}^8$ is the metallic structure defined by

$$\phi(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = (\sigma X_1, \sigma X_2, \bar{\sigma} X_3, \bar{\sigma} X_4, \sigma X_5, \sigma X_6, \bar{\sigma} X_7, \bar{\sigma} X_8)$$

which satisfies the property of a metallic structure i.e., $\phi^2X = p\phi X + qX$. Then, $(\mathbb{R}^8, \tilde{g}, \phi)$ is a metallic Riemannian manifold.

Let $i : N \rightarrow \mathbb{R}^8$ be an immersion of N into \mathbb{R}^8 , defined by

$$i(s, u, v, w) = \left(s, \frac{\sqrt{3}}{2}v, u, \frac{1}{2}v, s, \frac{1}{2}w, u, w \right).$$

Assume $\{Z_1, Z_2, Z_3, Z_4\}$ be an orthonormal frame on the tangent space of N such that

$$Z_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5} \right), \quad Z_2 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_7} \right),$$

$$Z_3 = \frac{\sqrt{3}}{2} \frac{\partial}{\partial x_2} + \frac{1}{2} \frac{\partial}{\partial x_4}, \quad Z_4 = \frac{2}{\sqrt{5}} \left(\frac{1}{2} \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8} \right).$$

After applying the metallic structure, we obtain

$$\phi Z_1 = \frac{1}{\sqrt{2}} \left(\sigma \frac{\partial}{\partial x_1} + \sigma \frac{\partial}{\partial x_5} \right) = \sigma Z_1, \quad \phi Z_2 = \frac{1}{\sqrt{2}} \left(\bar{\sigma} \frac{\partial}{\partial x_3} + \bar{\sigma} \frac{\partial}{\partial x_7} \right) = \bar{\sigma} Z_2,$$

$$\phi Z_3 = \frac{\sqrt{3}}{2} \sigma \frac{\partial}{\partial x_2} + \frac{1}{2} \bar{\sigma} \frac{\partial}{\partial x_4}, \quad \phi Z_4 = \frac{2}{\sqrt{5}} \left(\frac{1}{2} \sigma \frac{\partial}{\partial x_6} + \bar{\sigma} \frac{\partial}{\partial x_8} \right).$$

Here, we observe that $D = span\{Z_1, Z_2\}$, $D_1 = span\{Z_3\}$ and $D_2 = span\{Z_4\}$ are distributions on TN such that

- (i) D is invariant,
- (ii) D_1 and D_2 are slant distributions with slant angles θ_1 and θ_2 , respectively,
- (iii) The slant angle θ_1 is given by

$$\theta_1 = \cos^{-1} \left(\frac{2p + \sqrt{p^2 + 4q}}{2(\sqrt{p^2 + 2p\sigma + 4q})} \right),$$

- (iv) The slant angle θ_2 is given by

$$\theta_2 = \cos^{-1} \left(\frac{5p - 3\sqrt{p^2 + 4q}}{\sqrt{5}\sqrt{4p^2 - 3p\sigma + 5q}} \right).$$

Hence, N is a quasi bi-slant submanifold of metallic Riemannian manifold $(\mathbb{R}^8, \tilde{g}, \phi)$.

For golden Riemannian manifold $p = q = 1$, therefore the slant angles θ_1 and θ_2 are reduced to

$$\theta_1 = \cos^{-1} \left(\frac{2 + \sqrt{5}}{2\sqrt{5} + 2\sigma} \right), \quad \theta_2 = \cos^{-1} \left(\frac{5 - 3\sqrt{5}}{\sqrt{5}\sqrt{9 - 3\sigma}} \right).$$

Example 3.3. Consider the Euclidean space \mathbb{R}^{11} with usual Euclidean metric \tilde{g} . Suppose $\phi : \mathbb{R}^{11} \rightarrow \mathbb{R}^{11}$ is the metallic structure defined by

$$\phi(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11}) = (\sigma X_1, \sigma X_2, \bar{\sigma} X_3, \bar{\sigma} X_4, \sigma X_5, \sigma X_6, \bar{\sigma} X_7, \bar{\sigma} X_8, \sigma X_9, \sigma X_{10}, \bar{\sigma} X_{11})$$

which satisfies the property of a metallic structure i.e., $\phi^2X = p\phi X + qX$. Then, $(\mathbb{R}^{11}, \tilde{g}, \phi)$ is a metallic Riemannian manifold.

Let $i : N \rightarrow \mathbb{R}^{11}$ be an immersion of N into \mathbb{R}^{11} , defined by

$$i(s, u, v, w, z) = (\cos v, \sin v, s, v, \cos w, \sin w, \cos z, \sin z, u, v, w).$$

Assume $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$ be an orthonormal frame on the tangent space of N such that

$$Z_1 = \frac{\partial}{\partial x_3}, \quad Z_2 = \frac{\partial}{\partial x_9},$$

$$\begin{aligned}
 Z_3 &= \frac{1}{\sqrt{3}} \left(-\sin v \frac{\partial}{\partial x_1} + \cos v \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_{10}} \right), \\
 Z_4 &= \frac{1}{\sqrt{2}} \left(-\sin w \frac{\partial}{\partial x_5} + \cos w \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_{11}} \right), \\
 Z_5 &= -\sin z \frac{\partial}{\partial x_7} + \cos z \frac{\partial}{\partial x_8}.
 \end{aligned}$$

After applying the metallic structure, we obtain

$$\begin{aligned}
 \phi Z_1 &= \bar{\sigma} \frac{\partial}{\partial x_3} = \bar{\sigma} Z_1, \quad \phi Z_2 = \sigma \frac{\partial}{\partial x_9} = \sigma Z_2, \\
 \phi Z_3 &= \frac{1}{\sqrt{3}} \left(-\sigma \sin v \frac{\partial}{\partial x_1} + \sigma \cos v \frac{\partial}{\partial x_2} + \bar{\sigma} \frac{\partial}{\partial x_4} + \sigma \frac{\partial}{\partial x_{10}} \right), \\
 \phi Z_4 &= \frac{1}{\sqrt{2}} \left(-\sigma \sin w \frac{\partial}{\partial x_5} + \sigma \cos w \frac{\partial}{\partial x_6} + \bar{\sigma} \frac{\partial}{\partial x_{11}} \right), \\
 \phi Z_5 &= -\bar{\sigma} \sin z \frac{\partial}{\partial x_7} + \bar{\sigma} \cos z \frac{\partial}{\partial x_8} = \bar{\sigma} Z_5.
 \end{aligned}$$

Here, we observe that $D = span\{Z_1, Z_2, Z_5\}$, $D_1 = span\{Z_3\}$ and $D_2 = span\{Z_4\}$ are distributions on TN such that

- (i) D is invariant,
- (ii) D_1 and D_2 are slant distributions with slant angles θ_1 and θ_2 , respectively,
- (iii) The slant angle θ_1 is given by

$$\theta_1 = \cos^{-1} \left(\frac{p + \sigma}{\sqrt{3}\sqrt{p^2 + p\sigma + 3q}} \right),$$

- (iv) The slant angle θ_2 is given by

$$\theta_2 = \cos^{-1} \left(\frac{p}{\sqrt{2}\sqrt{p^2 + 2q}} \right).$$

Hence, N is a quasi bi-slant submanifold of metallic Riemannian manifold $(\mathbb{R}^{11}, \tilde{g}, \phi)$. For golden Riemannian manifold $p = q = 1$, therefore, the slant angles θ_1 and θ_2 are reduced to

$$\theta_1 = \cos^{-1} \left(\frac{1 + \sigma}{\sqrt{3}\sqrt{4 + \sigma}} \right), \quad \theta_2 = \cos^{-1} \left(\frac{1}{\sqrt{6}} \right).$$

Example 3.4. Consider the Euclidean space \mathbb{R}^7 with the usual Euclidean metric \tilde{g} . Suppose $\phi : \mathbb{R}^7 \rightarrow \mathbb{R}^7$ is the metallic structure defined by

$$\phi(X_1, X_2, X_3, X_4, X_5, X_6, X_7) = (\bar{\sigma}X_1, \bar{\sigma}X_2, \sigma X_3, \bar{\sigma}X_4, \bar{\sigma}X_5, \sigma X_6, \sigma X_7)$$

which satisfies the property of a metallic structure i.e., $\phi^2 X = p\phi X + qX$. Then, $(\mathbb{R}^7, \tilde{g}, \phi)$ is a metallic Riemannian manifold.

Let $i : N \rightarrow \mathbb{R}^7$ be an immersion of N into \mathbb{R}^7 , defined by

$$i(u, v, w) = \left(\frac{1}{2} \cos u, \frac{1}{2} \sin u, \frac{\sqrt{3}}{2} w, \frac{1}{2} w, v, \sin v, \cos v \right).$$

Assume $\{Z_1, Z_2, Z_3\}$ be an orthonormal frame on the tangent space of N such that

$$Z_1 = \frac{1}{\sqrt{2}} \left(-\frac{1}{2} \sin u \frac{\partial}{\partial x_1} + \frac{1}{2} \cos u \frac{\partial}{\partial x_2} \right), \quad Z_2 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_5} + \cos v \frac{\partial}{\partial x_6} - \sin v \frac{\partial}{\partial x_7} \right),$$

$$Z_3 = \frac{\sqrt{3}}{2} \frac{\partial}{\partial x_3} + \frac{1}{2} \frac{\partial}{\partial x_4}.$$

After applying the metallic structure, we obtain

$$\begin{aligned} \phi Z_1 &= \frac{1}{\sqrt{2}} \left(-\bar{\sigma} \frac{1}{2} \sin u \frac{\partial}{\partial x_1} + \bar{\sigma} \frac{1}{2} \cos u \frac{\partial}{\partial x_2} \right) = \bar{\sigma} Z_1, \\ \phi Z_2 &= \frac{1}{\sqrt{2}} \left(\bar{\sigma} \frac{\partial}{\partial x_5} + \sigma \cos v \frac{\partial}{\partial x_6} - \sigma \sin v \frac{\partial}{\partial x_7} \right), \\ \phi Z_3 &= \sigma \frac{\sqrt{3}}{2} \frac{\partial}{\partial x_3} + \bar{\sigma} \frac{1}{2} \frac{\partial}{\partial x_4}. \end{aligned}$$

Here, we observe that $D = span\{Z_1\}$, $D_1 = span\{Z_2\}$ and $D_2 = span\{Z_3\}$ are distributions on TN such that

- (i) D is invariant,
- (ii) D_1 and D_2 are slant distributions with slant angles θ_1 and θ_2 , respectively,
- (iii) The slant angle θ_1 is given by

$$\theta_1 = \cos^{-1} \left(\frac{p}{\sqrt{2p^2 + 4q}} \right),$$

- (iv) The slant angle θ_2 is given by

$$\theta_2 = \cos^{-1} \left(\frac{p + 2\sigma}{2\sqrt{p^2 + 2p\sigma + 4q}} \right).$$

Hence, N is a quasi bi-slant submanifold of metallic Riemannian manifold $(\mathbb{R}^7, \tilde{g}, \phi)$.

For golden Riemannian manifold $p = q = 1$, therefore, the slant angles θ_1 and θ_2 are reduced to

$$\theta_1 = \cos^{-1} \left(\frac{1}{\sqrt{6}} \right), \quad \theta_2 = \cos^{-1} \left(\frac{1 + 2\sigma}{2\sqrt{5 + 2\sigma}} \right).$$

Theorem 3.5. *Let N be a quasi bi-slant submanifold of a metallic Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with slant angles θ_1 and θ_2 , then $\forall X, Y \in \Gamma(TN)$, we have*

$$\tilde{g}(lF_2X, lF_2Y) = \cos^2(\theta_1)[p\tilde{g}(\phi F_2X, F_2Y) + q\tilde{g}(F_2X, F_2Y)], \tag{3.10}$$

$$\tilde{g}(lF_3X, lF_3Y) = \cos^2(\theta_2)[p\tilde{g}(\phi F_3X, F_3Y) + q\tilde{g}(F_3X, F_3Y)], \tag{3.11}$$

$$\begin{aligned} \tilde{g}(mX, mY) &= \sin^2(\theta_1)[p\tilde{g}(\phi F_2X, F_2Y) + q\tilde{g}(F_2X, F_2Y)] + \\ &\quad \sin^2(\theta_2)[p\tilde{g}(\phi F_3X, F_3Y) + q\tilde{g}(F_3X, F_3Y)]. \end{aligned} \tag{3.12}$$

Proof. Using (3.8), we get

$$\cos^2 \theta_1 = \frac{\tilde{g}(\phi F_2X, lF_2X)}{\tilde{g}(\phi F_2X, \phi F_2X)}.$$

Replacing X by $X + Y$ we obtain,

$$\tilde{g}(\phi F_2X, lF_2Y) = \cos^2 \theta_1 \tilde{g}(\phi F_2X, \phi F_2Y),$$

which implies

$$\tilde{g}(lF_2X, lF_2Y) = \cos^2 \theta_1 \tilde{g}(\phi F_2X, \phi F_2Y), \tag{3.13}$$

which implies (3.10).

Similarly, for the distribution D_2 , we get (3.11).

Again from (3.4) and (3.6), we get

$$lF_2X = \phi X - lF_1X - lF_3x - mX.$$

Now using (3.1) and (3.4), we obtain

$$lF_2X = \phi F_2X + \phi F_3X - lF_3X - mX.$$

Substituting above values in (3.10), we get

$$\begin{aligned} \tilde{g}(\phi F_2X + \phi F_3X - lF_3X - mx, \phi F_2Y + \phi F_3Y - lF_3Y - mY) &= \cos^2 \theta_1 [p\tilde{g}(\phi F_2X, F_2Y) \\ &+ q\tilde{g}(F_2X, F_2Y)]. \end{aligned} \tag{3.14}$$

On solving (3.14), we get

$$\begin{aligned} \tilde{g}(\phi F_2X, \phi F_2Y) + \tilde{g}(\phi F_3X, \phi F_3Y) - \tilde{g}(lF_3X, lF_3Y) - \tilde{g}(mX, mY) &= \cos^2 \theta_1 [p\tilde{g}(\phi F_2X, F_2Y) \\ &+ q\tilde{g}(F_2X, F_2Y)]. \end{aligned}$$

Using (3.10) and (3.11) in above equation, we get the required result (3.12). □

Corollary 3.6. *Let N be a quasi bi-slant submanifold of a golden Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with slant angles θ_1 and θ_2 , then $\forall X, Y \in \Gamma(TN)$, we have*

$$\tilde{g}(lF_2X, lF_2Y) = \cos^2(\theta_1)[\tilde{g}(\phi F_2X, F_2Y) + \tilde{g}(F_2X, F_2Y)], \tag{3.15}$$

$$\tilde{g}(lF_3X, lF_3Y) = \cos^2(\theta_2)[\tilde{g}(\phi F_3X, F_3Y) + \tilde{g}(F_3X, F_3Y)], \tag{3.16}$$

$$\begin{aligned} \tilde{g}(mX, mY) &= \sin^2(\theta_1)[\tilde{g}(\phi F_2X, F_2Y) + \tilde{g}(F_2X, F_2Y)] + \\ &\sin^2(\theta_2)[\tilde{g}(\phi F_3X, F_3Y) + \tilde{g}(F_3X, F_3Y)]. \end{aligned} \tag{3.17}$$

Lemma 3.7. *Let N be a quasi bi-slant submanifold of a metallic Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with slant angles θ_1 and θ_2 , then $\forall X, Y \in \Gamma(TN)$, we have*

$$l^2 F_2 = \cos^2 \theta_1 [p\phi F_2 + qF_2], \tag{3.18}$$

$$l^2 F_3 = \cos^2 \theta_2 [p\phi F_3 + qF_3]. \tag{3.19}$$

Proof. Using (2.3) in (3.10), we obtain

$$\tilde{g}(l^2 F_2X, F_2Y) = \cos^2 \theta_1 [p\tilde{g}(\phi F_2X, F_2Y) + q\tilde{g}(F_2X, F_2Y)],$$

which implies (3.18).

Similarly, we get (3.19).

For golden Riemannian manifold $p = q = 1$, therefore (3.18) and (3.19) are reduced to

$$l^2 F_2 = \cos^2 \theta_1 [\phi F_2 + F_2],$$

$$l^2 F_3 = \cos^2 \theta_2 [\phi F_3 + F_3].$$

□

Theorem 3.8. *Let N be a quasi bi-slant submanifold of a metallic Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with slant angles θ_1 and θ_2 , then*

$$l^2 = (plF_1 + qF_1) + \lambda_1 [plF_2 + qF_2] + \lambda_2 [plF_3 + qF_3], \tag{3.20}$$

where $\lambda_i = \cos^2 \theta_i, i = 1, 2$.

Proof. For the distribution D , we have

$$\tilde{g}(l^2 F_1X, F_1Y) = \tilde{g}(lF_1X, lF_1Y) = \tilde{g}(\phi F_1X, \phi F_1Y) = \tilde{g}(plF_1X + qF_1X, F_1Y),$$

which implies

$$l^2 F_1 = plF_1 + qF_1. \tag{3.21}$$

Also, for the distribution D_1 , we have

$$\tilde{g}(l^2 F_2 X, F_2 Y) = \cos^2 \theta_1 [p\tilde{g}(\phi F_2 X, F_2 Y) + q\tilde{g}(F_2 X, F_2 Y)],$$

which implies

$$l^2 F_2 = \cos^2 \theta_1 (p\phi F_2 + qF_2). \quad (3.22)$$

Similarly for the distribution D_2 , we obtain

$$l^2 F_3 = \cos^2 \theta_2 [p\phi F_3 + qF_3]. \quad (3.23)$$

Using (3.21), (3.22) and (3.23) in (3.5), we get (3.20). \square

Corollary 3.9. Let N be a quasi bi-slant submanifold of a golden Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with slant angles θ_1 and θ_2 , then we obtain

$$l^2 = (lF_1 + F_1) + \lambda_1 [lF_2 + F_2] + \lambda_2 [lF_3 + F_3], \quad (3.24)$$

where $\lambda_i = \cos^2 \theta_i$, $i = 1, 2$.

Theorem 3.10. Let N be a quasi bi-slant submanifold of a metallic Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with slant angles θ_1 and θ_2 , then there exist a constant $\lambda \in [0, 1)$ such that $\mathcal{D} = \{X \in \Gamma(TN) : l^2 X = \lambda(p\phi X + qX)\}$ is a distribution and $\mathcal{D}^\perp = D$.

Proof. Let N be a quasi bi-slant submanifold, then taking $\lambda_i = \cos^2 \theta_i$, where $i = 1, 2$ and $\lambda_i \in [0, 1)$ in (3.18) and (3.19), we get

$$l^2 F_2 X = \lambda_1 [p\phi F_2 X + qF_2 X],$$

$$l^2 F_3 X = \lambda_2 [p\phi F_3 X + qF_3 X],$$

which means that $F_2 X, F_3 X \in \Gamma(\mathcal{D})$, i.e.,

$$D_1 \oplus D_2 \subset \mathcal{D}. \quad (3.25)$$

Now, let $X \in \Gamma(\mathcal{D})$ be a non-zero vector field, then by using (3.1), we get

$$X = F_1 X + F_2 X + F_3 X.$$

Using (2.1) and (3.3), we get

$$p l F_1 X + q F_1 X = \phi(\phi F_1 X) = \phi(l F_1 X) = l^2 F_1 X.$$

Also, as $X \in \Gamma(\mathcal{D})$, from the above equations, we get

$$p l F_1 X + q F_1 X = \lambda(p l F_1 X + q F_1 X)$$

$$\implies (\lambda - 1)(p l F_1 X + q F_1 X) = 0$$

$$\implies p l F_1 X + q F_1 X = 0, (\lambda \neq 1)$$

$$\implies F_1 X = 0.$$

Hence, we get

$$\mathcal{D} \subset D_1 \oplus D_2. \quad (3.26)$$

From (3.25) and (3.26), we obtain

$$\mathcal{D} = D_1 \oplus D_2.$$

This completes the proof. \square

Corollary 3.11. Let N be a quasi bi-slant submanifold of a golden Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with slant angles θ_1 and θ_2 , then there exist a constant $\lambda \in [0, 1)$ such that $\mathcal{D} = \{X \in \Gamma(TN) : l^2 X = \lambda(\phi X + X)\}$ is a distribution and $\mathcal{D}^\perp = D$.

Theorem 3.12. *Let N be a quasi bi-slant submanifold of a metallic Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$. Let D_1 and D_2 be the slant distributions with slant angles θ_1 and θ_2 such that for any $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$, $\theta_1 = \theta_2 = \theta$ with $\tilde{g}(\phi X, Y) = 0$, then N is a semi-slant submanifold of the metallic Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with slant angle θ .*

Proof. For any $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$, we have $\tilde{g}(\phi X, Y) = 0$. Using (2.2), we get

$$\tilde{g}(X, lY) = 0.$$

Similarly, we have

$$\tilde{g}(lX, Y) = 0$$

Using $\theta_1 = \theta_2 = \theta$ in (3.8) and (3.9), we obtain

$$\cos^2 \theta = \frac{\tilde{g}(lX, lX)}{\tilde{g}(\phi X, \phi X)}$$

Thus, N is a semi-slant submanifold with slant angle θ . □

Corollary 3.13. *Let N be a quasi bi-slant submanifold of a golden Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$. If D_1 and D_2 are the slant distributions with slant angles θ_1 and θ_2 such that for any $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$, $\theta_1 = \theta_2 = \theta$ with $\tilde{g}(\phi X, Y) = 0$, then N is a semi-slant submanifold of golden Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with slant angle θ .*

Theorem 3.14. *Let N be a quasi bi-slant submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with codimension r , then the distribution D is integrable if and only if $\forall X, Y \in \Gamma(D)$, we have*

$$(\nabla_Y u_\alpha)X = (\nabla_X u_\alpha)Y \quad \forall \alpha \in \{1, \dots, r\}, \tag{3.27}$$

$$\sigma(X, lY) = \sigma(lX, Y), \tag{3.28}$$

$$\phi A_V X = A_V \phi X. \tag{3.29}$$

Proof. We know that D is integrable if and only if $m[X, Y] = 0 \forall X, Y \in \Gamma(D)$. Now $\phi D \subset D$, $mX = mY = 0$. Hence, from (2.15), we obtain

$$u_\alpha(X)t_{\alpha\beta}(Y) = u_\alpha(Y)t_{\alpha\beta}(X) = 0.$$

Therefore (2.30) reduces to (3.27).

Similarly, from (2.26), we obtain

$$\sigma(X, lY) = \sigma(lX, Y).$$

From above equation, we get

$$\sigma(X, lY) - \sigma(lX, Y) = 0, \tag{3.30}$$

which proves (3.28). Also, using (3.30), we obtain

$$\tilde{g}(\sigma(X, lY) - \sigma(lX, Y), V) = 0,$$

which implies

$$\tilde{g}(\sigma(X, \phi Y) - \sigma(\phi X, Y), V) = 0.$$

Using (2.12), we get

$$\tilde{g}(A_V X, \phi Y) - \tilde{g}(A_V \phi X, Y) = 0.$$

Now using (2.2), we have

$$\tilde{g}(\phi A_V X, Y) - \tilde{g}(A_V \phi X, Y) = 0,$$

which proves (3.29). □

Theorem 3.15. *Let N be a quasi bi-slant submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with codimension r , then the distribution D_1 is integrable if and only if for all $X, Y \in \Gamma(D_1)$, either of the following holds*

$$\begin{aligned} (F_1 + F_3)(\nabla_X lY - \nabla_Y lX) &= \sum_{\alpha=1}^r [u_\alpha(Y)(F_1 + F_3)A_\alpha X - u_\alpha(X)(F_1 + F_3)A_\alpha Y] \\ &\quad - \sum_{\alpha=1}^r [(\nabla_Y u_\alpha)X - (\nabla_X u_\alpha)Y + \sum_{\beta=1}^r u_\alpha(X)t_{\alpha\beta}(Y) \\ &\quad - u_\alpha(Y)t_{\alpha\beta}(X)]F_3 m_\alpha, \end{aligned} \tag{3.31}$$

$$\begin{aligned} (F_1 + F_3)(\nabla_X lY - \nabla_Y lX) &= (F_1 + F_3)(A_{mY}X - A_{mX}Y) - F_3[\sigma(X, lY) \\ &\quad - \sigma(lX, Y) + \nabla_X^\perp mY - \nabla_Y^\perp mX]. \end{aligned} \tag{3.32}$$

Proof. Since D_1 is integrable if and only if $F_1[X, Y] + F_3[X, Y] = 0$, therefore, from (2.29) and (2.30), we get

$$\begin{aligned} (F_1 + F_3)(\nabla_X lY - \nabla_Y lX) &= \sum_{\alpha=1}^r [u_\alpha(Y)(F_1 + F_3)A_\alpha X - u_\alpha(X)(F_1 + F_3)A_\alpha Y] \\ &\quad - \sum_{\alpha=1}^r [(\nabla_Y u_\alpha)X - (\nabla_X u_\alpha)Y + \sum_{\beta=1}^r u_\alpha(X)t_{\alpha\beta}(Y) \\ &\quad - u_\alpha(Y)t_{\alpha\beta}(X)]F_3 m_\alpha. \end{aligned}$$

Again using (2.25) and (2.26), we obtain

$$\begin{aligned} (F_1 + F_3)(\nabla_X lY - \nabla_Y lX) &= (F_1 + F_3)(A_{mY}X - A_{mX}Y) - F_3[\sigma(X, lY) \\ &\quad - \sigma(lX, Y) + \nabla_X^\perp mY - \nabla_Y^\perp mX], \end{aligned}$$

which completes the proof. □

Theorem 3.16. *Let N be a quasi bi-slant submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{N}, \tilde{g}, \phi)$ with codimension r , then the distribution D_2 is integrable if and only if for all $X, Y \in \Gamma(D_2)$, either of the following holds*

$$\begin{aligned} (F_1 + F_2)(\nabla_X lY - \nabla_Y lX) &= \sum_{\alpha=1}^r [u_\alpha(Y)(F_1 + F_2)A_\alpha X - u_\alpha(X)(F_1 + F_2)A_\alpha Y] \\ &\quad - \sum_{\alpha=1}^r [(\nabla_Y u_\alpha)X - (\nabla_X u_\alpha)Y + \sum_{\beta=1}^r u_\alpha(X)t_{\alpha\beta}(Y) \\ &\quad - u_\alpha(Y)t_{\alpha\beta}(X)]F_2 m_\alpha, \end{aligned} \tag{3.33}$$

$$\begin{aligned} (F_1 + F_2)(\nabla_X lY - \nabla_Y lX) &= (F_1 + F_2)(A_{mY}X - A_{mX}Y) - F_2[\sigma(X, lY) \\ &\quad - \sigma(lX, Y) + \nabla_X^\perp mY - \nabla_Y^\perp mX]. \end{aligned} \tag{3.34}$$

Proof. Since D_2 is integrable if and only if $F_1[X, Y] + F_2[X, Y] = 0$, therefore, from (2.29) and (2.30), we get

$$\begin{aligned} (F_1 + F_2)(\nabla_X lY - \nabla_Y lX) &= \sum_{\alpha=1}^r [u_\alpha(Y)(F_1 + F_2)A_\alpha X - u_\alpha(X)(F_1 + F_2)A_\alpha Y] \\ &\quad - \sum_{\alpha=1}^r [(\nabla_Y u_\alpha)X - (\nabla_X u_\alpha)Y + \sum_{\beta=1}^r u_\alpha(X)t_{\alpha\beta}(Y) \\ &\quad - u_\alpha(Y)t_{\alpha\beta}(X)]F_2 m_\alpha. \end{aligned}$$

Again using (2.25) and (2.26), we obtain

$$(F_1 + F_2)(\nabla_X lY - \nabla_Y lX) = (F_1 + F_2)(A_{mY}X - A_{mX}Y) - F_2[\sigma(X, lY) - \sigma(lX, Y) + \nabla_X^\perp mY - \nabla_Y^\perp mX],$$

which completes the proof. \square

4 Conclusion remarks

In this paper, we introduce the concept of quasi bi-slant submanifolds in metallic and golden Riemannian manifolds and analyze their fundamental properties. We derive integrability conditions for the associated distributions and establish necessary conditions for a submanifold to be quasi bi-slant. Additionally, we provide non-trivial examples to illustrate our theoretical findings. These findings enrich the study of submanifolds and provide a crucial new understanding of their characteristics. Furthermore, this concept can be extended to indefinite metallic manifolds and other generalized structures, which will help to analyze their geometric and topological properties in a more effective manner.

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