Palestine Journal of Mathematics

Vol 14(3)(2025) , 749–760

$\mathtt{r}$-ideals of commutative semirings

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Communicated by Ayman Badawi

MSC 2010 Classifications: 16Y60.

Keywords and phrases: $\mathtt{r}$-ideals, $\mathtt{pr}$-ideals, semi $\mathtt{r}$-ideals, semirings.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract In this article, we introduce and study the concept of $\mathtt{r}$-ideal and $\mathtt{pr}$-ideal in commutative semirings, as a generalization of $\mathtt{r}$-ideal in commutative rings with identity. We examine various examples and characterizations of this new class of ideals and analyzing their behaviour in relation to quotient semirings, direct product of semirings and polynomial semirings with property $(\mathtt{A})$. Additionally, we demonstrate that a prime ideal J of a semiring R is an $\mathtt{r}$-ideal if and only if $J \subseteq \mathtt{zd}(\mathtt{R})$. Furthermore, we study the properties of semi $\mathtt{r}$-ideal in commutative semirings and investigate their behaviour under homomorphism, semiring of fractions and direct product of semirings.

1 Introduction

Prime ideals and semiprime ideals play a fundamental role in the structure theory of prime rings and semiprime rings, respectively. However, prime and semiprime ideals in ring theory are not necessarily equivalent to semiring theory. Golan [7] introduced the concept of prime and semiprime ideals in semirings, demonstrating with examples that some ideals are prime (or semiprime) in semiring theory but not in ring theory. In 2015, Mohamadian [13] introduced the concept of $\mathtt{r}$-ideals in commutative rings and compared them with other classical ideals, such as prime and maximal ideals. Since then, various generalizations and related classes of these ideals have been studied, including graded $\mathtt{r}$-ideals, semi $\mathtt{r}$-ideals, $\mathtt{r}$-submodules and $\mathtt{sr}$ -submodules; see [4, 11, 12]. Subsequently, Erbay [6] extended this concept to $\mathtt{r}$-ideals in commutative semigroups, examining their properties and providing examples in this context. Now, we investigate the behaviour of $\mathtt{r}$-ideals in commutative semirings. A proper ideal J of R is said to be an $\mathtt{r}$-ideal if, for any $x, y \in R$ such that $xy \in J$ and $Ann(x) = (0)$ implies that $y \in J$. In Example 3.3, we show that every $\mathtt{r}$-ideal is not always a prime ideal. However, in Theorem 3.21, we establish a condition under which an $\mathtt{r}$-ideal can be a prime ideal. In Theorem 3.8, if J is a prime ideal of a semiring R. Then J is an $\mathtt{r}$-ideal if and only if $J \subseteq \mathtt{zd}(\mathtt{R})$. Theorem 3.16 proves that if R is an entire semiring, then $J = \{0\}$ is the only $\mathtt{r}$-ideal of R. Theorem 3.30 examine the behaviour of $\mathtt{r}$-ideal in polynomial semirings with property $(\mathtt{A})$. Theorem 4.4 provides a characterization of semi $\mathtt{r}$-ideal of a semiring R and Theorem 4.15 examine the behaviour of semi $\mathtt{r}-$ ideals under direct product of semirings.

2 Preliminaries

In this section, we recall some basic definitions used throughout the article. Unless stated otherwise, R denotes a commutative semiring with an identity element.

Recall from [7] that a nonempty set R together with two binary operations addition and multiplication is said to be a semiring if (i). $(R, +)$ is a commutative monoid with identity element 0 (ii). $(R, .)$ is a monoid with identity element 1 (iii). Multiplication distributes over addition from either side (iv). $0r = 0 = r0$ for all $r \in R$ (v). $1 \neq 0$. A semiring R is said to be an entire semiring (or semi-domain) if for any $x, y \in R$ such that $xy = 0$ implies that either $x = 0$ or $y = 0$. An element x of a semiring R is called a unit if there exists an element y of R such that $xy = 1 = yx$. The element y is the inverse of x in R and if it exists, it is unique. The set of all units of R is denoted by $\mathtt{u(R)}$. A non-empty subset J of R is said to be a left (or right) ideal of R if J is a sub-semigroup of $(R, +)$ and $xy \in J$ (or $yx \in J$) for all $y \in J, x \in R$. If J is both a left and right ideal of R, then J is known to be an ideal of R. An ideal J of R is said to be a proper ideal if $J \neq R$. An ideal J of a semiring R is said to be a k-ideal (or subtractive ideal) if for all $x, y \in R, x + y \in J$ and $y \in J$ implies that $x \in J$. If every ideals of R is subtractive, then it is said to be subtractive semiring [15]. A proper ideal of R is said to be a maximal if it is not properly contained in any other ideals of R. An ideal $J \neq \{0\}$ of a semiring R is minimal if it does not contain any ideals of R other than itself and 0. The set of all minimal prime ideals of J is denoted by $Min(J)$. The radical of an ideal J is defined as $\sqrt{J} = \{x \in R : x^m \in J$ for some $m \in \mathbb{N}\}$. A proper ideal P of R is said to be prime if $x, y \in R$ such that $xy \in P$ implies that either $x \in P$ or $y \in P$. A proper ideal P of R is said to be primary if $x, y \in R$ such that $xy \in P$ implies that either $x \in P$ or $y \in \sqrt{P}$. An element x of a semiring R is said to be multiplicatively idempotent if and only if $x^2 = x$. A semiring R is said to be reduced if it contains no non-zero nilpotent elements [8]. Let R and S be semirings. A function $g : R \to S$ is said to be a homomorphism if $g(x + y) = g(x) + g(y)$ and $g(xy) = g(x)g(y)$ for all $x, y \in R$. A semiring homomorphism $g : R \to S$ is said to be an epimorphism [resp. monomorphism, isomorphism] if it is surjective [resp. injective, bijective] mapping [1]. The annihilator of an element x is defined as $Ann(x) = \{y \in R : \ yx = 0\}$. A non zero element x of a semiring R is said to be a zero divisor if there exists a non-zero element $y \in R$ such that $xy = 0$, that is $Ann(x) \neq (0)$. An element x of a semiring R is called regular if it is neither a left nor a right zero divisor [17, Definition 22]. Equivalently, an element x of a semiring R is called regular if $xy = 0$ implies that $y = 0$ for all $y \in R$ [3, Page 1] that is $Ann(x) = (0)$. The set of all zero divisors and regular elements of R are denoted by $\mathtt{zd(R)}$ and $\mathtt{r(R)}$. An ideal J of R is said to be a regular ideal if it contains atleast one regular element, that is $J \cap \mathtt{r(R)} \neq \phi$. In a semiring, every unit element is also a regular element. However, the converse is not necessarily true. For instance, in the semiring $R = \mathbb{N}$, any non-zero element of $R \setminus \{1\}$ is a regular element, while the only unit element of R is $\{1\}$. For further details on semirings, see [7].

3 r-ideals

The purpose of this section is to investigate r-ideal and pr-ideal in commutative semirings and presents several characterizations and examples to illustrate these concepts.

Definition 3.1. Let J be a proper ideal of a semiring R. Then J is said to be an r-ideal of R, if for any $x, y \in R$ such that $xy \in J$ and $Ann(x) = (0)$ implies $y \in J$.

Example 3.2. (i) The set of all non-negative integers $\mathbb{N}$ in [7, Example 1.3] forms a commutative and an entire semiring under usual addition and multiplication. However, $\mathbb{N}$ is not a ring. Consider $R = \mathbb{N}$ and let $J = (0)$ be an ideal of R. Then J is an r-ideal of R. Since $x, y \in R$ such that $xy \in J$ and $Ann(x) = (0)$ implies that $y \in J$.

(ii) Consider the semiring $\mathbb{B} = \{0, 1\}$ with addition and multiplication is defined as follows:

+	0	1
0	0	1
1	1	1

and

.	0	1
0	0	0
1	0	1

If $J = \{0\}$, then J is an r-ideal of semiring $\mathbb{B}$. While $\mathbb{B}$ is not a ring.

(iii) Consider the semiring $R = \mathbb{Z}_m$ of integer modulo $m \geq 2$. Then every ideal of $\mathbb{Z}_m$ is an r-ideal. If J is a non-zero proper ideal of $\mathbb{Z}_m$ and $xy \in J$ with $Ann(x) = (0)$, then x and m are relatively prime and so x has an inverse in $\mathbb{Z}_m$. Therefore, $x^{-1}(xy) \in J$, which implies $y \in J$.

The notions of prime ideals and r-ideals are distinct as shown by the following example:

Example 3.3. (i) In the semiring $R = \mathbb{N}$ of non-negative integers, the ideal $5\mathbb{N}$ is a prime ideal but not an r-ideal. Since $5 \cdot 3 \in 5\mathbb{N}$ and $Ann(5) = (0)$ but $3 \notin 5\mathbb{N}$.

(ii) Consider the semiring $R = \mathbb{Z}_8$ and let $J =< 4 >$ be a proper ideal of R. By Example 3.2 (iii), J is an r-ideal. However, J is not a prime ideal since $2 \cdot 2 \in J$ but $2 \notin J$.

Definition 3.4. Let J be a proper ideal of a semiring R. Then J is said to be a pr-ideal of R, if for any $x, y \in R$ such that $xy \in J$ and $Ann(x) = (0)$ implies that $y^m \in J$.

Every r-ideal is a pr-ideal, but the converse is not always true, as demonstrated by the following example:

Example 3.5. Consider the semiring $R = \mathbb{N}[x, x^2, x^3]$, where $\mathbb{N}$ is a semifield and let $J =< x^3 >$ be an ideal of R. Then J is a pr-ideal, while J is not an r-ideal, since $x \cdot x^2 \in J$ and $Ann(x) = (0)$, but $x^2 \notin J$.

Additionally, the concepts of pr-ideal and primary ideal are distinct.

Example 3.6. (i) Consider the semiring $R = \mathbb{Z}_{12}$ and let $J =< 6 >$ be a proper ideal of R. Assume $x, y \in R$ such that $xy \in J$ and $Ann(x) = (0)$. Since R is a finite semiring, then the set of all units in R and all regular elements in R are equal so that x has an inverse in R. Thus, $x^{-1}(xy) = y \in J$. Hence, J is a pr-ideal. However, it is not a primary ideal since $2 \cdot 3 \in J$ but neither $2 \in J$ nor $3^n \in J$ and similarly, neither $3 \in J$ nor $2^n \in J$ for any $n \in \mathbb{N}$.

(ii) Consider $R = \mathbb{N}$, the semiring of the set of non-negative integers. The ideal $9\mathbb{N}$ is a primary ideal of R, but it is not a pr-ideal. Since $9 \cdot 2 \in 9\mathbb{N}$ and $Ann(9) = (0)$ but $2^n \notin 9\mathbb{N}$ for any $n \in \mathbb{N}$.

Theorem 3.7. *Let J be an ideal of a semiring R. Then J is a* pr*-ideal if and only if $\sqrt{J}$ is an* r*-ideal of R.*

Proof. Let J be a pr-ideal of R. Assume $x, y \in R$ such that $xy \in \sqrt{J}$ with $Ann(x) = (0)$. Then $(xy)^m = x^m y^m \in J$ for some $m \in \mathbb{N}$. It is obvious that, $Ann(x^m) = (0)$. Since J is a pr-ideal of R, there exists some $k \in \mathbb{N}$ such that $y^{mk} \in J$ implies that $y \in \sqrt{J}$. Conversely, suppose $\sqrt{J}$ is an r-ideal of R. Assume $x, y \in R$ such that $xy \in \sqrt{J}$ with $Ann(x) = (0)$. Since $xy \in \sqrt{J}$, it follows that $y \in \sqrt{J}$, which implies that there exists some $m \in \mathbb{N}$ such that $y^m \in J$. □

Theorem 3.8. *Let J be a prime ideal of a semiring R. Then J is an* r*-ideal if and only if $J \subseteq$* zd(R).

Proof. Suppose $J \nsubseteq$ zd(R). Then there exists a regular element $j \in J$ such that $j \notin$ zd(R). Thus $Ann(j) = (0)$. Now, let $j \in R$ and $1 \in R$ such that $j.1 \in J$. If $j \notin J$, then $1 \in J$, which contradicts the fact that J is an r-ideal. Hence, $J \subseteq$ zd(R). Conversely, Assume J is a prime ideal of R and $J \subseteq$ zd(R). Let $x, y \in R$ such that $xy \in J$ and $Ann(x) = (0)$. Sine J is a prime ideal, either $x \in J$ or $y \in J$. As $J \subseteq$ zd(R) and $Ann(x) = (0)$, we have $x \notin J$ and so, $y \in J$. Therefore, J is an r-ideal of R. □

If the condition that J is prime is dropped, then $J \subseteq$ zd(R), does not necessarily imply that J is an r− ideal.

Example 3.9. Consider the semiring $R = \mathbb{N} \times \mathbb{N}$ and let $J = 0\mathbb{N} \times 0\mathbb{N}$ be an ideal of R. Note that $J \subseteq$ zd(R). Since $(2, 0), (0, 3) \in R$ such that $(2, 0)(0, 3) \in J$ and $Ann(2, 0) = (0)$ while $(0, 3) \notin J$. Hence, J is not an r− ideal.

Theorem 3.10. *Let R be a semiring and P be a primary ideal of R. Then P is a* pr*-ideal if and only if $P \subseteq$* zd(R).

Proof. Assume P is a primary ideal of R and $P \subseteq$ zd(R). Let $x, y \in R$ such that $xy \in P$ and $Ann(x) = (0)$. Since $P \subseteq$ zd(R) and $Ann(x) = (0)$, it follows that $x \notin P$, which implies $y^n \in P$ for some $n \in \mathbb{N}$, since P is a primary ideal of R. Thus, P is a pr-ideal. Conversely, assume P is a pr-ideal. By Theorem 3.7, $\sqrt{P}$ is an r-ideal and by Theorem 3.8, $P \subseteq \sqrt{P} \subseteq$ zd(R). □

Theorem 3.11. *Let R be a semiring and J be an ideal of R. Then J is an* $\mathtt{r}$*-ideal if and only if for every* $x \in \mathtt{r(R)}$ *such that* $xJ = xR \cap J$ *and* $Ann(x) = (0)$.

Proof. First, suppose J is an $\mathtt{r}$-ideal of R . Clearly, $xJ \subseteq xR$ and $xJ \subseteq J$ which implies $xJ \subseteq xR \cap J$. Now, if $y \in xR \cap J$, then there exists some $r \in R$ such that $y = xr$. Since J is an $\mathtt{r}$-ideal, we have $r \in J$ and so $y = xr \in xJ$. Hence, $xJ = xR \cap J$. Conversely, assume that for all $x \in \mathtt{r(R)}$, $xJ = xR \cap J$ and $Ann(x) = (0)$. Now, we have to show that J is an $\mathtt{r}$-ideal. Let $xy \in J$ and $Ann(x) = (0)$ for some $x, y \in \mathtt{r(R)}$ and so $xy \in xR \cap J = xJ$. Thus, there exists some $z \in J$ such that $xy = xz$ which implies $x(y - z) = 0$. Since $Ann(x) = (0)$, we conclude that, $y - z = 0$ implies that $y = z \in J$. Therefore, J is an $\mathtt{r}$-ideal. □

Theorem 3.12. *Let J be an ideal of a semiring R. Then J is an* $\mathtt{r}$*-ideal if and only if for each* $x \in \mathtt{r(R)}$ *and* $Ann(x) = (0)$ *such that* $(J : x) = J$.

Proof. Assume J is an $\mathtt{r}$-ideal of R, and let $x \in \mathtt{r(R)}$ such that $Ann(x) = (0)$. Clearly, $J \subseteq (J : x)$. If $y \in (J : x)$, then $xy \in J$. Since J is an $\mathtt{r}$-ideal and $Ann(x) = (0)$, it follows that $y \in J$. Therefore, $(J : x) = J$. Conversely, suppose that $(J : x) = J$. Let $xy \in J$ and $Ann(x) = (0)$ for some $x, y \in \mathtt{r(R)}$. Then $y \in (J : x) = J$. Hence, J is an $\mathtt{r}$-ideal of R. □

Theorem 3.13. *Let J be an* $\mathtt{r}$*-ideal and K be a non-empty subset of R such that* $K \nsubseteq J$*. If K contains regular element, then* $(J : K)$ *is an* $\mathtt{r}$*-ideal. In particular,* $Ann(K)$ *is an* $\mathtt{r}$*-ideal.*

Proof. Assume $xy \in (J : K)$ and $Ann(x) = (0)$. Then there exists some $k \in K$ such that $xyk \in J$. If k is a regular element of K, then xk is also a regular element. Since J is an $\mathtt{r}$-ideal and $xyk \in J$, it follows that $y \in J$. Therefore, $y \in J \subseteq (J : K)$. □

Theorem 3.14. *Let J be an ideal of a semiring R. Then J is an* $\mathtt{r}$*-ideal of R if and only if for any ideals K and L of R, whenever* $KL \subseteq J$ *and* $K \cap \mathtt{r(R)} \neq \phi$*, then* $L \subseteq J$.

Proof. First, assume J is an $\mathtt{r}$-ideal of R and consider any two ideals K and L of R such that $KL \subseteq J$ and $K \cap \mathtt{r(R)} \neq \phi$. Since $K \cap \mathtt{r(R)} \neq \phi$, there exists $x \in K \cap \mathtt{r(R)}$. If $y \in L$, then $xy \in J$. Since J is an $\mathtt{r}$-ideal, we have $y \in J$. Therefore, $L \subseteq J$. Conversely, assume $L \subseteq J$. Let $xy \in J$ and $Ann(x) = (0)$ for some $x, y \in \mathtt{r(R)}$. Consider the ideals $K =< x >$ and $L =< y >$ of R with $x \in K \cap \mathtt{r(R)}$ and $KL \subseteq J$. Since $L \subseteq J$, it follows that $y \in J$. Therefore, J is an $\mathtt{r}$-ideal of R. □

Corollary 3.15. *Let J and* J_1 *be ideals of R with* $J \cap \mathtt{r(R)} \neq \phi$*. Then the following statements hold:*

(i) If K and L are $\mathtt{r}$*-ideals of R such that* $JK = JL$*, then* $K = L$.

(ii) If JJ_1 *is an* $\mathtt{r}$*-ideal of R, then* $J = JJ_1$*. Especially, J is an* $\mathtt{r}-$ *ideal.*

Proof. (i) Since $JK \subseteq L$ and L is an $\mathtt{r}$-ideal, by Theorem 3.14, we have $K \subseteq L$. Similarly, $L \subseteq K$. Thus, $K = L$.

(ii) Since JJ_1 is an $\mathtt{r}$-ideal of R and $JJ_1 \subseteq JJ_1$, by Theorem 3.14, it follows that $J_1 \subseteq JJ_1 \subseteq J_1$. Hence, $J = JJ_1$.

□

Recall from [2, Corollary 2.7], if R is a semiring. Then R is a semi-domain if and only if $J = (0)$ is the only prime ideal of R. An interesting result can be derived by replacing the prime ideal with the $\mathtt{r}-$ ideals.

Theorem 3.16. *Let R be a semiring. Then the following statements are equivalent:*

(i) R is a an entire semiring.

(ii) The zero ideal is the only $\mathtt{r}$*-ideal of R.*

(iii) $Ann(xy) = Ann(x) \cup Ann(y)$ *for all* $x, y \in R$.

Proof. **(i) ⇒ (ii)**. Let R be an entire semiring and J be a nonzero proper ideal of R. Then there exists $0 \neq x \in J$. Since R is an entire semiring, we have $Ann(x) = (0)$ and $x.1 \in J$. If J is an $\mathtt{r}$-ideal, then $1 \in J$, which leads to a contradiction. Therefore, the zero ideal is the only $\mathtt{r}$-ideal of R.
(ii) ⇒ (iii). Let $J = (0)$ be an $\mathtt{r}$-ideal of R. By Theorem 3.13, for any non-zero element $x \in R$, $Ann(x)$ is an $\mathtt{r}$-ideal. Thus, by assumption, we have $Ann(x) = (0)$. Therefore, for any $x, y \in R$, we have $Ann(xy) = Ann(x) \cup Ann(y)$.
(iii) ⇒ (i). Let $x, y \in R$ such that $xy = 0$. Then $Ann(xy) = (0)$. Since $Ann(xy) = Ann(x) \cup Ann(y)$, we have $0 \in Ann(x) \cup Ann(y)$, which implies that $x = 0$ or $y = 0$. Therefore, R is an entire semiring.

□

Theorem 3.17. *Let $\{z_j : j \in K\}$ be a set of idempotent elements in a semiring R. Then $J = \sum_{j\in K} z_j R$ is an $\mathtt{r}$-ideal.*

Proof. Let $x, y \in R$ such that $xy \in J$ and $Ann(x) = (0)$. Since $J = \sum_{j\in K} z_j R$, which implies that $xy = \sum_{k=1}^{n} z_{j_k} r_{j_k}$ for some $j_1, ..., j_n \in K$ and $r_{j_1}, ..., r_{j_n} \in R$. Let $u = \prod_{k=1}^{n}(1 - z_{j_k})$. Then $xyu = 0$ and since $Ann(x) = (0)$, this implies $yu = 0$. Moreover, there exists $v \in J$ such that $u = 1 - v$. Then $y(1 - v) = 0$, which implies $y = yv \in J$. Therefore, J is an $\mathtt{r}$-ideal of R. □

Example 3.18. Let $R = \mathbb{Z}_6$ be a semiring of integer modulo 6. By Example 3.2(iii), $< 2 >$ and $< 3 >$ are $\mathtt{r}$-ideal, but their sum $< 2 > + < 3 >= \mathbb{Z}_6$ is not necessarily an $\mathtt{r}$-ideal.

The sum of two $\mathtt{r}$-ideals is not always an $\mathtt{r}$-ideal. In Theorem 3.19, we shows that the sum of two annihilator ideals of a semiring R is an $\mathtt{r}$-ideal. Furthermore, in Theorem 3.20, we prove that the sum of a minimal prime ideal and an annihilator ideal in a reduced semiring is also an $\mathtt{r}$-ideal.

Theorem 3.19. *Let J be an ideal of a semiring R and let $x + y = 1$ for some $x, y \in R$. Then $J = Ann(x) + Ann(y)$ is an $\mathtt{r}$-ideal of R.*

Proof. Let $u, v \in R$ such that $uv \in J$ and $Ann(u) = (0)$. Then there exist $x_1 \in Ann(x)$ and $y_1 \in Ann(y)$ such that $uv = x_1 + y_1$. It is clear that $uvxy = 0$, which implies $vxy = 0$, since $Ann(u) = 0$. As a result, $vx \in Ann(y)$ and $vy \in Ann(x)$. Thus, $v = v(x + y) = vx + vy \in J$. Therefore, J is an $\mathtt{r}$-ideal of R. □

Theorem 3.20. *Let z be an idempotent element of a reduced semiring R and $K \in Min(R)$. Then $J = K + Ann(z)$ is an $\mathtt{r}$-ideal.*

Proof. Let $x, y \in R$ such that $xy \in J$ and $Ann(x) = (0)$. Then there exist $u \in K$ and $vz = 0$ such that $xy = u + v$. It is obvious that, $uw = 0$ for some $w \notin K$. This implies $zwxy = 0$, and so $zwy = 0$. Hence, $zy \in K$. Since $y = zy + (1 - z)y \in K + Ann(z) = J$. Therefore, J is an $\mathtt{r}$-ideal. □

In Example 3.3, we show that every $\mathtt{r}$-ideal is not always a prime ideal. The following result establishes a condition under which an $\mathtt{r}$-ideal can be a prime ideal.

Theorem 3.21. *Every maximal $\mathtt{r}$-ideal of a semiring R is a prime ideal.*

Proof. Assume J is a maximal $\mathtt{r}$-ideal. Let $x, y \in R$ such that $xy \in J$ and $x \notin J$. By Theorem 3.12, $(J : x)$ is an $\mathtt{r}$-ideal of R. Clearly, $J \subseteq (J : x)$ and $y \in (J : x)$. By the maximality of J, we have $J = (J : x)$, so $y \in J$. Therefore, J is a prime ideal of R. □

Recall from [5, Definition 3.7], let R be a semiring and J be a non-zero ideal. If for every nonzero ideal K of R, $J \cap K \neq (0)$, then J is said to be an essential ideal.

Theorem 3.22. *Let R be a reduced semiring and J be a non-zero $\mathtt{r}$-ideal that is not essential. Then there exists a minimal prime ideal $J \subseteq M$, which is a maximal $\mathtt{r}$-ideal.*

Proof. Let K be a nonzero ideal of R such that $J \cap K = (0)$, as J is not an essential. Since $(0) \neq K$ and R is reduced semiring, then there exists $M \in Min(R)$ such that $K \nsubseteq M$. Thus, there exists some $z \in K \setminus M$. By Zorn's Lemma, there exists a maximal $\mathtt{r}$-ideal J_1 containing J such that $J_1 \cap K = (0)$. As a result, $KJ_1 = (0)$ or $zJ_1 = (0) \subseteq M$. Hence $J_1 \subseteq M$. Therefore, $J \subseteq J_1 = M$. □

In the following results, we examine the behaviour of $\mathtt{r}$-ideal and $\mathtt{pr}$-ideal with quotient semirings.

Theorem 3.23. *Let K be an* $\mathtt{r}$*-ideal in R that is contained in J. If* J/K *is an* $\mathtt{r}$*-ideal in* R/K*, then J is also an* $\mathtt{r}$*-ideal in R.*

Proof. Let $x, y \in R$ such that $xy \in J$ and $Ann(x) = (0)$. If $xy \in K$. Since K is an $\mathtt{r}$-ideal, we have $y \in K \subseteq J$. If $xy \in J \setminus K$, this implies $xy + K = (x + K)(y + K) \in J/K$. Since $Ann(x) = (0)$, then $Ann(x + K) = (0)$. If $Ann(x + K) \neq (0)$, then there exists a non-zero element $z + K \in R/K$ such that $(x + K)(z + K) = 0 + K$ implies that $xz \in K$. Since K is an $\mathtt{r}$-ideal and $Ann(x) = (0)$, it follows that $z \in K$, which is a contradiction. Hence, $Ann(x + K) = (0)$. Furthermore, since $Ann(x + K) = (0)$ and J/K is an $\mathtt{r}$-ideal in R/K, we conclude that $y + K \in J/K$, which implies $y \in J$. Therefore, J is an $\mathtt{r}$-ideal in R. □

Theorem 3.24. *Let J be an* $\mathtt{r}$*-ideal and K be an ideal of R such that* $J \subseteq K$*. If* K/J *is a* $\mathtt{pr}$*-ideal of* R/J*, then K is a* $\mathtt{pr}$*-ideal of R.*

Proof. Let $x, y \in R$ such that $xy \in J$ and $Ann(x) = (0)$. If $xy \in J$ and J is an $\mathtt{r}$-ideal, then $y \in J$ and so $y \in K$. Now assume $xy \notin J$. We will prove that $Ann(x + J) = (0)_{R/J}$. Let $z + J \in R/J$ such that $(x + J)(z + J) = 0 + J$. This implies $xz + J = 0 + J$ and so $xz \in J$. Since $Ann(x) = (0)$ and J is an $\mathtt{r}$-ideal, we have $z \in J$. Thus, $x + J = 0_{R/J}$. Hence, $Ann(x + J) = (0)_{R/J}$. Since K/J is a $\mathtt{pr}$-ideal of R/J and $(x + J)(y + J) = xy + J \in K/J$, there exists some $n \in \mathbb{N}$ such that $(y + J)^n = y^n + J \in K/J$, which implies $y^n \in K$. Therefore, K is a $\mathtt{pr}$-ideal of R. □

In this result, we investigate the behaviour of $\mathtt{pr}$-ideal with direct product of semirings. Let $R = R_1 \times R_2$, where R_1 and R_2 are two commutative semirings with unity and $(x, y)(z, w) = (xz, yw)$ for all $x, z \in R_1$ and $y, w \in R_2$.

Theorem 3.25. *Let* $R = R_1 \times R_2$ *where* R_1 *and* R_2 *are two commutative semirings and let* J_1 *be an ideal of* R_1 *and* J_2 *be an ideal of* R_2 *such that* $J = J_1 \times J_2$*. Then the following statements are equivalent:*

(i) J is a $\mathtt{pr}$*-ideal of R.*

(ii) $J_1 = R_1$ *and* J_2 *is a* $\mathtt{pr}$*-ideal of* R_2 *or* $J_2 = R_2$ *and* J_1 *is a* $\mathtt{pr}$*-ideal of* R_1 *or both* J_1 *and* J_2 *are* $\mathtt{pr}$*-ideal of* R_1 *and* R_2 *respectively.*

Proof. (i)⇒ (ii). Assume J is a $\mathtt{pr}$-ideal and $J_2 = R_2$. Let $x, y \in R_1$ such that $xy \in J_1$ and $Ann(x) = (0)_{R_1}$. Then $(x, 1_{R_2}), (y, 0_{R_2}) \in R$ such that $(x, 1_{R_2})(y, 0_{R_2}) = (xy, 0_{R_2}) \in J$ and $Ann(x, 1_{R_2}) = (0_{R_1}, 0_{R_2})$. Since J is a $\mathtt{pr}$-ideal, it follows that $(y, 0_{R_2})^n = (y^n, 0_{R_2}) \in J_1 \times J_2$. Thus, $y^n \in J_1$. Therefore, J_1 is a $\mathtt{pr}$-ideal of R_1. Similarly, if $J_1 = R_1$, then J_2 is a $\mathtt{pr}$-ideal of R_2. Similarly, if J_1 and J_2 are proper ideals of R_1 and R_2, then J_1 and J_2 are $\mathtt{pr}$-ideal of R_1 and R_2.
(ii)⇒ (i). Assume J_1 and J_2 are $\mathtt{pr}$-ideal of R_1 and R_2. Let $(x, z), (y, w) \in R_1 \times R_2$ such that $(x, z)(y, w) \in J_1 \times J_2$ and $Ann(x, z) = (0_{R_1}, 0_{R_2})$. Then $x, y \in R_1$ such that $xy \in J_1$ with $Ann(x) = (0)_{R_1}$ and $z, w \in R_2$ such that $zw \in J_2$ with $Ann(z) = 0_{R_2}$. Since J_1 and J_2 are $\mathtt{pr}$-ideal of R_1 and R_2, we have $y^n \in J_1$ and $w^n \in J_2$, which implies $(y, w)^n \in J_1 \times J_2$. Therefore, $J = J_1 \times J_2$ is a $\mathtt{pr}$-ideal of $R = R_1 \times R_2$. In other cases, one can similarly prove that J is a $\mathtt{pr}$-ideal of R. □

Corollary 3.26. *Let* $R = R_1 \times R_2 \times ... \times R_n$ *where* $R_1, R_2, ..., R_n$ *are commutative semirings and let* $J_j^{\prime}s$ *be ideals of* R_j *such that* $J = J_1 \times J_2 \times ... \times J_n$*. Then the following statements are equivalent:*

(i) J is a $\mathtt{pr}$*-ideal of R.*

(ii) There exist $i_1, ..., i_j \in \{1, 2, ..., n\}$ such that $J_i = R_i$ for each $i \in \{i_1, ..., i_j\}$ and J_i is a pr*-ideal of R_i for each $i \in \{1, 2, ..., n\} - \{i_1, ..., i_j\}$.*

Finally, in this section, we study the behaviour of r-ideal and pr-ideal in polynomial semirings with property (A). Recall from [15] that $R[y]$ denote the polynomial semiring over R with indeterminates y. For a polynomial $g(y) = b_0 + b_1 y + ... + b_m y^m$, where $b_m \neq 0$, then $deg(g) = m$. The content of g is defined as $c(g) = (b_0, b_1, ..., b_m)$. For more details on polynomial semirings, refer to [9, 15].

Definition 3.27. [15, Definition 52] A semiring R is said to have Property (A), if every finitely generated ideal $J \subseteq$ zd(R) has a nonzero annihilator.

Theorem 3.28. *[15, Theorem 3] Let R be a semiring. Then the following conditions are equivalent:*

(i) R is a subtractive semiring.

(ii) If $g, h \in R[y]$ and $deg(h) = n$, then $c(g)^{n+1}c(h) = c(g)^n c(gh)$. For an ideal J of R, the ideal $J[y] = \{g(y) \in R[y] : c(g) \subseteq J\}$.

Lemma 3.29. *Let R be a semiring. If $g(y) \in$ zd(R[y]), then $c(g) \subseteq$ zd(R). The converse is true if R satisfies property (A).*

Proof. Let G be a finite set and $< G > \subseteq$ zd(R). Clearly, $g(y) \in R[y]$ with $c(g) = G$. Since $g(y) \in$ zd(R[y]), there exists some nonzero element $f \in R$ such that $fg = 0$. Thus, $fG = \{0\}$ and so $Ann(g) \neq (0)$. Hence, $c(g) \subseteq$ zd(R). Conversely, assume R satisfies property (A) and $c(g) \subseteq$ zd(R). Since $c(g)$ is finitely generated ideal, it follows that $Ann(c(g)) \neq (0)$. Hence, $g(y) \in$ zd(R[y]). □

Theorem 3.30. *Let R be a subtractive reduced semiring and let J be an* r*-ideal in R. Then R satisfies property* (A) *if and only if $J[y]$ is an* r*-ideal in $R[y]$.*

Proof. Assume J is an r-ideal of R and let $g, h \in R[y]$ such that $gh \in J[y]$ and $Ann_{R[y]}(g) = (0)$. By lemma 3.29, we have $c(g) \nsubseteq$ zd(R). Thus, there exists $k \in c(g)$ with $Ann_R(k) = (0)$. It is obvious that $c(gh) \subseteq J$. Now, by Theorem 3.28, we have $c(g)^{m+1}c(h) = c(g)^m c(gh)$, where m is a degree of g. Therefore, $c(g)^{m+1}c(h) \subseteq J$. Since $k^{m+1} \in c(g)^{m+1}$, it follows that $k^{m+1}c(h) \subseteq J$. Moreover, $Ann_R(k^{m+1}) = (0)$. Since J is an r-ideal of R, we have $c(h) \subseteq J$ implies that $h \in J[y]$. Therefore, $J[y]$ is an r-ideal in $R[y]$. Conversely, assume $J[y]$ is an r-ideal in $R[y]$. We will prove this by contradiction. Suppose that R does not satisfies property (A). By Lemma 3.29, there exists $k \in R[y]$ with $Ann_{R[y]}(k) = (0)$ and $J = c(k) \subseteq$ zd(R). Thus by Theorem 3.8, there exists a prime ideal P of R such that $J \subseteq P$, which implies $c(k) \subseteq P$. Thus, $k \in P[y]$, but k is regular element. Hence, $P[y]$ is not an r-ideal, which is a contradiction. Therefore, R satisfies property (A). □

Definition 3.31. [9, p. 82] A semiring R is said to be Armendariz if $f = \sum_{j=0}^{n} a_j y^j$ and $g = \sum_{k=0}^{m} b_k y^k \in R[y]$ such that $fg = 0$ then $a_j b_k = 0$ for all j and k.

Theorem 3.32. *Let R be a subtractive Armendariz semiring. Then J is a* pr*-ideal of R if and only if $J[y]$ is a* pr*-ideal of $R[y]$.*

Proof. Assume $J[y]$ is a pr-ideal ideal of $R[y]$ and let $x, y \in R$ such that $xy \in J$ with $Ann(x) = (0)$. Choose $g(y) = x$ and $h(y) = u$ such that $g(y)h(y) = xu \in J[y]$ with $Ann_{R[y]}(g(y)) = (0)$. Since $J[y]$ is a pr-ideal of $R[y]$, there exists $n \in \mathbb{N}$ such that $(h(y))^n = u^n \in J[y]$ and so $u^n \in J$. Hence, J is a pr-ideal of R. Conversely, suppose J is a pr-ideal of R and let $g(y), h(y) \in R[y]$ such that $g(y)h(y) \in J[y]$ with $Ann_{R[y]}(g(y)) = (0)$. Since R is an Armendariz semiring and $Ann_{R[y]}(g(y)) = Ann(c(g))[y] = (0)$, we have $c(g) \nsubseteq$ zd(R), there exists $z \in c(g)$ such that $Ann(z) = (0)$. Also $c(gh) \subseteq J$. By Theorem 3.28, $c(g)^{n+1}c(h) = c(g)^n c(gh)$, where n is a degree of g. Since $z^{n+1} \in c(g)^{n+1}$ and $Ann(z^{n+1}) = (0)$, we have $z^{n+1}c(h) \subseteq J$. Now assume there exist some $r_j \in R$ such that $c(h) = (r_0, r_1, ..., r_k)$. Thus, for any $r_j \in c(h)$, we have $r_j^n \in J$. It is clear that $c(h)^{(k+1)n} \subseteq J$ and so $c(h^{(k+1)n}) \subseteq c(h)^{(k+1)n} \subseteq J$. Thus, $h(y)^{(k+1)n} \in J[y]$. Therefore, $J[y]$ is an pr-ideal in $R[y]$. □

4 Semi $\mathtt{r}$-ideals

The purpose of this section is to investigate various characterizations and examples of semi $\mathtt{r}$-ideals in commutative semirings.

Definition 4.1. [7] An ideal J of a semiring R is said to be a semiprime ideal if for any ideal I of R such that $I^2 \subseteq J$ implies $I \subseteq J$.

Definition 4.2. Let J be a proper ideal of a semiring R. Then J is said to be a semi $\mathtt{r}$-ideal if for any $x \in R$ such that $x^2 \in J$ with $Ann(x) = (0)$, then $x \in J$.

The following example demonstrates that semiprime ideals, $\mathtt{r}$-ideals and semi $\mathtt{r}$-ideals are distinct.

Example 4.3. (i) By Example 3.2(i), $R = \mathbb{N}$ is an entire semiring. Any non-zero semiprime ideal in R is a semi $\mathtt{r}$-ideal, while it is not an $\mathtt{r}$-ideal.

(ii) In any semiring R, the zero ideal is always a semi $\mathtt{r}$-ideal, but it is not a semiprime ideal unless R is a semiprime semiring.

In the following result, we provide a characterization of semi $\mathtt{r}$-ideal of R.

Theorem 4.4. *Let R be a semiring and J be a proper ideal of R. The following statements are equivalent:*

(i) J is a semi $\mathtt{r}$-ideal of R.

(ii) If $x \in R$ such that $0 \neq x^2 \in J$ and $Ann(x) = (0)$, then $x \in J$.

(iii) If $x \in R$ such that $x^m \in J$ and $Ann(x) = (0)$, then $x \in J$ for some $m \in \mathbb{N}$.

(iv) $\sqrt{J} \subseteq \mathtt{zd(R)} \cup J$.

Proof. **(i)⇔(ii).** Assume (ii) hold and let $x \in R$ such that $x^2 \in J$ and $Ann(x) = (0)$. Now, we have to show that J is a semi $\mathtt{r}$-ideal of R. If $x^2 = 0$, then $x = 0$, which implies J is a semi $\mathtt{r}$-ideal of R because zero ideal is a semi $\mathtt{r}$-ideal of R. If $x^2 \neq 0$, then by assumption, $x \in J$. The converse part is clear.
(i)⇒(iii). Let J be a semi $\mathtt{r}$-ideal of R. Assume $x \in R$ and for some $m \in \mathbb{N}$ such that $x^m \in J$ and $Ann(x) = (0)$. We prove this by induction on m. If $m \leq 2$, the result holds. Assume it is true for $2 < n < m$. Now, we show that the result is true for m. If m is even such that $m = 2k$ for some positive integer k. Since J is a semi $\mathtt{r}$-ideal, $x^m = (x^{2k}) = (x^k)^2 \in J$ and $Ann(x^k) = (0)$, then $x^k \in J$. Thus, by induction, $x \in J$. If m is odd with $m + 1 = 2n$ for some positive integer n. Since J is a semi $\mathtt{r}$-ideal and $x^{m+1} = (x^{2n}) = (x^n)^2 \in J$ and $Ann(x^n) = (0)$, then $x^n \in J$. Thus by induction, $x \in J$.
(iii)⇒(iv). Assume $x \in \sqrt{J}$. Then for some $m \geq 1$ such that $x^m \in J$. By assumption $x \in \mathtt{zd(R)}$ or $x \in J$. Thus, $\sqrt{J} \subseteq \mathtt{zd(R)} \cup J$.
(iv)⇒(i). This implication is straightforward from the definitions. □

Corollary 4.5. *Let R be a semiring. Then the following statements hold:*

(i) If J is a semi $\mathtt{r}$-ideal of R, K is an ideal of R and m is a positive integer such that $K^m \subseteq J$ and $K \cap \mathtt{zd(R)} = \{0\}$, then $K \subseteq J$.

(ii) Whenever J and K are proper ideals of R such that $J \cap \mathtt{zd(R)} = K \cap \mathtt{zd(R)} = \{0\}$. If J and K are semi $\mathtt{r}$-ideals of R and $J^2 = K^2$, then $J = K$.

Proof. (i) Assume K is an ideal of R and there exists a positive integer m such that $K^m \subseteq J$ and $K \cap \mathtt{zd(R)} = \{0\}$. Let $0 \neq x \in K$. Since $K \cap \mathtt{zd(R)} = \{0\}$, it follows that $Ann(x) = (0)$. Thus, $x^m \in J$. By Theorem 4.4 (iii), we conclude that $x \in J$. Therefore, $K \subseteq J$.

(ii) By (i), if $J^2 \subseteq K$ and $K \cap \mathtt{zd(R)} = \{0\}$, then $J \subseteq K$. Similarly, if $K^2 \subseteq J$ and $J \cap \mathtt{zd(R)} = \{0\}$, then $K \subseteq J$. Hence, $J = K$.

□

Lemma 4.6. *Let R be a semiring and T be a non-empty subset of R such that $T \cap \mathtt{zd(R)} = \phi$. If J is a semi $\mathtt{r}$-ideal and $T \nsubseteq J$, then $(J : T)$ is a semi $\mathtt{r}$-ideal of R.*

Proof. Let $x \in R$ such that $x^2 \in (J : T)$ and $Ann(x) = (0)$. Then there exists an element $t \in T$ such that $(xt)^2 \in J$. Since J is a semi $\mathtt{r}$-ideal, either $xt \in \mathtt{zd(R)}$ or $xt \in J$. If $xt \in \mathtt{zd(R)}$, then $T \cap \mathtt{zd(R)} = \phi$, which implies $x \in \mathtt{zd(R)}$, which contradicts $Ann(x) = (0)$. Therefore, $xt \in J$ implies $x \in (J : T)$. Thus, $(J : T)$ is a semi $\mathtt{r}$-ideal of R. □

In Example 4.3 (i), a semi $\mathtt{r}$-ideal is not necessarily an $\mathtt{r}$-ideal. The following result provides a condition under which a semi $\mathtt{r}$-ideal becomes an $\mathtt{r}$-ideal.

Theorem 4.7. *Let R be a semiring and J be a maximal semi $\mathtt{r}$-ideal of R contained in $\mathtt{zd(R)}$. Then J is an $\mathtt{r}$-ideal.*

Proof. Let J be a maximal semi $\mathtt{r}$-ideal of R contained in $\mathtt{zd(R)}$. Assume $x, y \in R$ such that $xy \in J$ and $Ann(x) = (0)$. This implies that $x \notin J \cup \mathtt{zd(R)}$. By Lemma 4.6, $(J : x)$ is a semi $\mathtt{r}$-ideal of R. Since $(J : x) \subseteq \mathtt{zd(R)}$ and $J \subseteq (J : x)$, the maximality of J implies that $J = (J : x)$. Thus, $y \in J$. Hence, J is an $\mathtt{r}$-ideal. □

Definition 4.8. Let S be an over semiring of a semiring R with $S \subseteq R$. Then S is said to be an essential in R if for every non-zero ideal J of R, $S \cap J \neq (0)$.

Theorem 4.9. *Let S be an over semiring of a semiring R such that $S \subseteq R$ and S is essential in R. If K is a semi $\mathtt{r}$-ideal of R, then $K \cap S$ is a semi $\mathtt{r}$-ideal in R.*

Proof. Let $r^2 \in K \cap S$ and $Ann_S(r) = (0)$ for all $r \in S$. Then $r \in R$ such that $r^2 \in K$ and $Ann_R(r) = (0)$. If $Ann_S(r) \neq (0)$, then $Ann_S(r) \cap S \neq \{0\}$, since S is essential in R. Then there exists a non-zero element $s \in S$ such that $s \in Ann_R(r)$. Thus, $s \in Ann_S(r)$, which leads to a contradiction. Since K is a semi $\mathtt{r}$-ideal of R, we conclude that $r \in K \cap S$. Therefore, $K \cap S$ is a semi $\mathtt{r}$-ideal in R. □

In the following result if the condition S is essential is removed, the result does not generally hold:

Example 4.10. Consider $R = \mathbb{N}$ and $S = \mathbb{N} \times \mathbb{N}$ as semirings. Define a mapping $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by $f(x) = (x, 0)$. Thus, f is a monomorphism and so $R = f(\mathbb{N})$ is an entire. Also $K = Ann_S((0, 1))$ is a semi $\mathtt{r}$-ideal of R. However, $R \subseteq K$. Therefore, $R = K \cap R$ is not a semi $\mathtt{r}$-ideal in R.

In the following theorem, we examine the behavior of semi $\mathtt{r}$-ideal ideal under homomorphism.

Theorem 4.11. *Let R and S be semirings and let $g : R \to S$ be a homomorphism. The following statements hold:*

(i) If g is an epimorphism and J is a semi $\mathtt{r}$-ideal of R containing Ker(g) such that $J \cap \mathtt{zd(R)} = \{0\}$, then $g(J)$ is a semi $\mathtt{r}$-ideal of S.

(ii) If g is an isomorphism and K is a semi $\mathtt{r}$-ideal of S, then $g^{-1}(K)$ is a semi $\mathtt{r}$-ideal of R.

Proof. (i) Let $s \in S$ such that $s^2 \in g(J)$ and $s \notin g(J)$. Then there exists some element $r \in R \setminus J$ such that $s = g(r)$. Since $s^2 = g(r^2) \in g(J)$ and $Ker(g) \subseteq J$, we have $r^2 \in J$. Since J is a semi $\mathtt{r}$-ideal of R, it follows that $r \in \mathtt{zd(R)}$. If $r = 0$, then $s = g(r) \in \mathtt{zd(S)}$. If $r \neq 0$, then there exists a non-zero element $t \in R$ such that $rt = 0$. Thus, $g(t) \neq 0$, but $t \in J \cap \mathtt{Zd(R)}$, which is a contradiction. Therefore, $s = g(r) \in \mathtt{zd(S)}$ and so $g(J)$ is a semi $\mathtt{r}$-ideal of S.

(ii) Assume K is a semi $\mathtt{r}$-ideal of S. Let $r \in R$ such that $r^2 \in g^{-1}(K)$ and $r \notin g^{-1}(K)$. Then $g(r^2) = g(r)^2 \in K$ and $g(r) \notin K$, which implies that $g(r) \in \mathtt{zd(S)}$. Since g is an isomorphism, $r \in \mathtt{Zd(R)}$. Therefore, $g^{-1}(K)$ is a semi $\mathtt{r}$-ideal of R.

□

Recall from [14], consider R as a semiring and let T be the set of all multiplicatively cancellable elements of R (note that $1 \in T$). Clearly, T is a multiplicatively closed set. Define a relation $\sim$ on $R \times T$ as follows: for $(r, s), (t, u) \in R \times T$, then $(r, s) \sim (t, u)$ if and only if $ru = st$. This relation $\sim$ is an equivalence relation on $R \times T$. For any $(r, s) \in R \times T$, denote the equivalence class containing (r, s) by r/s, and R_T denote the set of all such equivalence classes. The set R_T can be given the structure of a commutative semiring with the following operations: $r/s + t/u = (ru + st)/su$, $(r/s)(t/u) = rt/su$ for all $r, t \in R$ and $t, u \in T$. This new semiring R_T is called the semiring of fractions of R with respect to T. The zero element in R_T is $0/1$, the multiplicative identity is $1/1$ and every element of T has a multiplicative inverse in R_T.

In the following result, we give the relationship between semi $\mathtt{r}$-ideals of a semiring and those of its semiring of fraction by using the notation $z_J(R)$ which denotes the set $\{s \in R \,|st \in J$ for some $t \in R \setminus J\}$.

Theorem 4.12. *Let R be a semiring and T be a multiplicatively closed subset of R such that $T \cap \mathtt{zd(R)} = \phi$. Then the following statements hold:*

(i) If J is a semi $\mathtt{r}$-ideal of R and $J \cap T = \phi$, then J_T is a semi $\mathtt{r}$-ideal of R_T.

(ii) If J_T is a semi $\mathtt{r}$-ideal of R_T and $T \cap z_J(R) = \phi$, then J is a semi $\mathtt{r}$-ideal of R.

Proof. (i) Let $\frac{r}{t} \in R_T$ such that $\left(\frac{r}{t}\right)^2 \in J_T$ and $\frac{r}{t} \notin J_T$. Then there exists an element $s \in T$ such that $sr^2 \in J$. This implies that $(sr)^2 \in J$. Since J is a semi $\mathtt{r}$-ideal and $sr \notin J$, it follows that $sr \in \mathtt{zd(R)}$. If $sr \in \mathtt{zd(R)}$, then there exists some non-zero element $u \in R$ such that $(sr)u = 0$. Since $T \cap \mathtt{zd(R)} = \phi$, we conclude that $\frac{r}{t}.\frac{u}{1} = \frac{sru}{st} = 0_{R_T}$ and so $\frac{u}{1} \neq 0_{R_T}$. Therefore, $\frac{r}{t} \in \mathtt{zd(R_T)}$. Thus, J_T is a semi $\mathtt{r}$-ideal of R_T.

(ii) Let $r \in R$ such that $r^2 \in J$. Since J_T is a semi $\mathtt{r}$-ideal of R_T and $\left(\frac{r}{1}\right)^2 \in J_T$, either $\frac{r}{1} \in R_T$ or $\frac{r}{1} \in \mathtt{zd(R_T)}$. If $\frac{r}{1} \in J_T$, then there exists $t \in T$ such that $tr \in J$. Thus, $r \in J$ as $T \cap z_J(R) = \phi$. If $\frac{r}{1} \in \mathtt{zd(R_T)}$, then there exists some non-zero element $\frac{s}{u} \in R_T$ such that $\frac{r}{1}.\frac{s}{u} = \frac{rs}{u} = 0_{R_T}$. Hence, there exists $t_1 \in T$ such that $t_1 rs = 0$. Since $T \cap \mathtt{zd(R)} = \phi$, which implies $rs = 0$. Thus, $r \in z_J(R)$ as $s \neq 0$. Therefore, J is a semi $\mathtt{r}$-ideal of R.

□

We now examine the notion of content semirings, which is analogous to Property $(*)$ for rings (see [11, Definition 2]).

Definition 4.13. A semiring R is said to satisfy property $(*)$ if for every $g \in \mathtt{r(R[y])}$, then $c(g) \setminus \{0\} \subseteq \mathtt{r(R)}$.

Theorem 4.14. *Let R be a subtractive semiring and J be an ideal of R. Then the following statements hold:*

(i) If $J[y]$ is a semi $\mathtt{r}$-ideal of $R[y]$, then J is a semi $\mathtt{r}$-ideal of R.

(ii) If J is a semi $\mathtt{r}$-ideal of R and R satisfies the property $(*)$, then $J[y]$ is a semi $\mathtt{r}$-ideal of $R[y]$.

Proof. (i) Assume $J[y]$ is a semi $\mathtt{r}$-ideal of $R[y]$. Let $r \in R$ such that $r^2 \in J$ and $Ann(r) = (0)$. Thus, clearly $r^2 \in J[y]$ and $Ann_{R[y]}(r) = (0)$. Since $J[y]$ is a semi $\mathtt{r}$-ideal of $R[y]$, $r \in J[y]$, which implies $r \in J$. Therefore, J is a semi $\mathtt{r}$-ideal of R.

(ii) Assume J is a semi $\mathtt{r}$-ideal of R and R satisfies property $(*)$. Let $g(y) \in R[y]$ such that $(g(y))^2 \in J[y]$ and $Ann_{R[y]}(g(y)) = (0)$. By the content formula, $c(g)^2 = c(g^2) \subseteq J$. Since R satisfies property $(*)$ and $c(g) \cap \mathtt{zd(R)} = \{0\}$, by Corollary 4.5(i), we have $c(g) \subseteq J$. Thus, $g[y] \in J[y]$, and so $J[y]$ is a semi $\mathtt{r}$-ideal of $R[y]$.

□

In the following results, we examine the behaviour of semi $\mathtt{r}-$ ideals under direct product of semirings.

Theorem 4.15. *Let R_1 and R_2 be two semirings such that $R = R_1 \times R_2$ and let J_1 be a proper ideal of R_1 and J_2 be a proper ideal of R_2. Then*

(i) J_1 is a semi $\mathtt{r}$-ideal of R_1 if and only if $J_1 \times R_2$ is a semi $\mathtt{r}$-ideal of R.

(ii) J_2 is a semi $\mathtt{r}$-ideal of R_2 if and only if $R_1 \times J_2$ is a semi $\mathtt{r}$-ideal of R.

Proof. (i) Assume J_1 is a semi $\mathtt{r}$-ideal of R_1. Consider $(r,s)^2 \in J_1 \times R_2$ and $Ann_R(r,s) = (0_{R_1}, 0_{R_2})$. Thus, $r^2 \in J_1$ and $Ann_{R_1}(r) = (0)$, which implies $r \in J_1$. Consequently, $(r,s) \in J_1 \times R_2$. Hence, $J_1 \times R_2$ is a semi $\mathtt{r}$-ideal of R. Conversely, assume $J_1 \times R_2$ is a semi $\mathtt{r}$-ideal of R and let $r \in R_1$ such that $r^2 \in J_1$ and $Ann_{R_1}(r) = (0)$. Then $(r, 1_{R_2})^2 \in J_1 \times R_2$ and $Ann_R(r, 1_{R_2}) = (0_{R_1}, 0_{R_2})$ implies that $(r, 1_{R_2}) \in J_1 \times R_2$, and so $r \in J_1$. Thus, J_1 is a semi $\mathtt{r}$-ideal of R_1.

(ii) The proof follows similarly to part (i).

□

Theorem 4.16. *Let $R = R_1 \times R_2$ where R_1 and R_2 are two semirings and J_1 is a proper ideal of R_1 and J_2 is a proper ideal of R_2. Then the following statements hold:*

(i) If J_1 and J_2 are semi $\mathtt{r}$-ideals of R_1 and R_2, then $J = J_1 \times J_2$ is a semi $\mathtt{r}$-ideal of R.

(ii) If $J = J_1 \times J_2$ is a semi $\mathtt{r}$-ideal of R, then either J_1 is a semi $\mathtt{r}$-ideal of R_1 or J_2 is a semi $\mathtt{r}$-ideal of R_2.

(iii) If $J = J_1 \times J_2$ is a semi $\mathtt{r}$-ideal of R and $J_1 \nsubseteq \mathtt{zd}(\mathtt{R_1})$, then J_2 is a semi $\mathtt{r}$-ideal of R_2.

(iv) If $J = J_1 \times J_2$ is a semi $\mathtt{r}$-ideal of R and $J_2 \nsubseteq \mathtt{zd}(\mathtt{R_2})$, then J_1 is a semi $\mathtt{r}$-ideal of R_1.

Proof. (i) Assume that J_1 and J_2 are semi $\mathtt{r}$-ideals of R_1 and R_2. Let $(r,s) \in R$ such that $(r,s)^2 = (r^2, s^2) \in J$ and $Ann_R(r,s) = (0_{R_1}, 0_{R_2})$. Then $r^2 \in J_1$, $s^2 \in J_2$ and $Ann_{R_1}(r) = (0)_{R_1}$, $Ann_{R_2}(s) = (0)_{R_2}$. Since J_1 and J_2 are semi $\mathtt{r}$-ideals of R_1 and R_2, it follows that $r \in J_1$, $s \in J_2$ and so, $(r,s) \in J$. Thus, $J = J_1 \times J_2$ is a semi $\mathtt{r}$-ideal of R.

(ii) Let $J = J_1 \times J_2$ be a semi $\mathtt{r}$-ideal of R. Assume that J_1 and J_2 are not semi $\mathtt{r}$-ideal of R_1 and R_2, then there exist elements $x \in R_1$ and $y \in R_2$ such that $x^2 \in J_1$, $Ann_{R_1}(x) = (0)_{R_1}$ and $y^2 \in J_2$, $Ann_{R_2}(y) = (0)_{R_2}$ while $x \notin J_1$ and $y \notin J_2$. Thus, $(x^2, y^2) = (x,y)^2 \in J$ and $Ann_R(x,y) = (0_{R_1}, 0_{R_2})$. Since $J = J_1 \times J_2$ is a semi $\mathtt{r}$-ideal of R, it follows that $(r,s) \in J$, which contradicts our assumption. Thus, either J_1 is a semi $\mathtt{r}$-ideal of R_1 or J_2 is a semi$\mathtt{r}$-ideal of R_2.

(iii) Assume $J = J_1 \times J_2$ is a semi $\mathtt{r}$-ideal of R and $J_1 \nsubseteq \mathtt{zd}(\mathtt{R_1})$. Let $s \in R_2$ such that $s^2 \in R_2$ and $Ann_{R_2}(s) = (0)_{R_2}$. Since $J_1 \nsubseteq \mathtt{zd}(\mathtt{R_1})$, there exists some $r \in J_1 \cap \mathtt{r}(\mathtt{R_1})$ such that $(r^2, s^2) = (r,s)^2 \in J$ and $Ann_R(r,s) = (0_{R_1}, 0_{R_2})$. Since $J = J_1 \times J_2$ is a semi $\mathtt{r}$-ideal of R, then $(r,s) \in J$, so $s \in J_2$. Hence, J_2 is a semi $\mathtt{r}$-ideal of R_2.

(iv) The proof follows similarly to (iii).

□

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Received: 2024-10-11
Accepted: 2025-05-15