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Some classification theorems and derivations in prime rings with involution

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Abstract. The focus of this article is the classification of a pair of derivations that satisfies specific differential identities in a prime rings with involution of the second kind. Several well-known results on the commutativity of prime rings have prompted further investigation and refinement.

1 Introduction

Throughout the present paper, R will denote an associative ring with center $Z(R)$. For any $x, y \in R$ the symbol $[x, y]$ denotes the commutator $xy-yx$. A ring R is called prime if $aRb = (0)$, where $a, b \in R$, implies $a = 0$ or $b = 0$; R is 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in R$. An additive mapping $* : R \longrightarrow R$ is called an involution if $*$ is an anti-automorphism of order 2; that is $(x^*)^* = x$ for all $x \in R$. An element x in a ring with involution $(R, *)$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the latter case $S(R) \cap Z(R) \neq \{0\}$. An additive mapping $d : R \to R$ is called a derivation if: $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in R$. We say that a mapping $f : R \to R$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in R$. A mapping $f : R \to R$ is said to be strong commutativity preserving (SCP) on a subset S of R if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. Over the last years, several authors have investigated the relationship between the commutativity of the ring R and certain special types of mappings on R. The first result in this direction is due to Divinsky [8], who proved that the simple artinian ring is commutative if it has a commuting nontrivial automorphism. Two years later, Posner [17] proved that the existence of nonzero centralizing derivation on prime ring forces the ring to be commutative. Over the last few decades, several authors have subsequently refined and extended these results in several directions (see [2], [5], [6], [14], [15] for references). In [13], Lanski has proved that if R is a noncommutative prime ring and d, $g \neq 0$ be two derivations of R into itself such that $[d(x), g(x)] = 0$, holds for all $x \in R$. Then $d = \lambda g$ where λ is an element from $C(R)$. Later El Mir, Mamouni and Oukhtite proved in [10], that if R is a ring, P is a prime ideal, d_1,d_2 derivations of R, then $d_1(x)d_2(y) - [x, y] \in P$ for all $x, y \in R$ implies that R/P is a commutative integral domain.

Our purpose here is to continue this line of investigation by studying commutativity criteria for rings admitting a pair of derivations satisfying certain algebraic identities.

2 Preliminary results

We start with these widely recognized Lemmas in the literature, which are simple to prove and recurrently used in our analysis.

Lemma 2.1. *Let a and b are two elements in a prime ring R with $b \in Z(R)$. If $ab \in Z(R)$, then either $a \in Z(R)$ or $b = 0$.*

Lemma 2.2. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If $H(R) \subset Z(R)$ or $S(R) \subset Z(R)$, then R is commutative.*

Lemma 2.3. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If $[h, h'] \in Z(R)$ for all $h, h' \in H(R)$ or $[s, s'] \in Z(R)$ for all $s, s' \in S(R)$, then R is commutative.*

3 Main results

In this section, we will use the previous rings theoretic results to study the commutativity of prime rings involving two derivations.

Proposition 3.1. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. There are no derivations d_1 and d_2 of R satisfying one of the following properties:*
(1) $d_1(h) \circ d_2(h) - h^2 = 0$ for all $h \in H(R)$;
(2) $d_1(h) \circ d_2(h) + h^2 = 0$ for all $h \in H(R)$.

Proof. Suppose that there exist two derivations d_1 and d_2 of R such that

$$d_1(h) \circ d_2(h) - h^2 = 0 \quad \text{for all} \ \ h \in H(R). \tag{3.1}$$

Linearizing relation (3.1), we find that

$$d_1(h) \circ d_2(h') + d_1(h') \circ d_2(h) - h \circ h' = 0 \quad \text{for all} \ \ h, h' \in H(R). \tag{3.2}$$

Replacing h' by $h'k$ in (3.2), where $k \in H(R) \cap Z(R)$, and invoking (3.2), we have

$$(d_1(h) \circ h')d_2(k) + (h' \circ d_2(h))d_1(k) = 0 \quad \text{for all} \ \ h, h' \in H(R). \tag{3.3}$$

Taking $h' \in H(R) \cap Z(R) \setminus \{0\}$ and $h = k$ in the last expression above and using the primeness of R, it is easy to verify that

$$d_1(k)d_2(k) = 0$$

As R is a prime ring, then either $d_1(k) = 0$ for all $k \in H(R) \cap Z(R)$, or $d_2(k) = 0$ for all $k \in H(R) \cap Z(R)$.
If $d_1(k) = 0$ for all $k \in H(R) \cap Z(R)$, taking $h \in H(R) \cap Z(R) \setminus \{0\}$ in (3.1), we obtain $h^2 = 0$, a contradiction.
If $d_2(k) = 0$ for all $k \in H(R) \cap Z(R)$, then using similar arguments to those above, we get a contradiction. So, there are no derivations that satisfy the first property.
We can see that (2) follows from (1) with slight modifications. □

Theorem 3.2. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and d_1, $d_2 \neq 0$ be two nonzero derivations of R. If $[d_1(h), d_2(h)] = 0$ for all $h \in H(R)$, then $d_1 = \lambda d_2$ where λ is an element from $C(R)$.*

Proof. We are given that

$$[d_1(h), d_2(h)] = 0 \quad \text{for all} \ \ h \in H(R). \tag{3.4}$$

Then a linearization of (3.4) forces

$$[d_1(h), d_2(h')] + [d_1(h'), d_2(h)] = 0 \quad \text{for all} \ \ h, h' \in H(R). \tag{3.5}$$

Substitute $h' := h'k$ for some $k \in H(R) \cap Z(R)$ and using the last equation, we obtain

$$[d_1(h), h']d_2(k) + [h', d_2(h)]d_1(k) = 0 \quad \text{for all} \ \ h, h' \in H(R). \tag{3.6}$$

Substituting $h' := st$, where $s \in S(R)$ and $t \in S(R) \cap Z(R) \setminus \{0\}$, in (3.6), we get

$$[d_1(h), s]d_2(k) + [s, d_2(h)]d_1(k) = 0 \quad \text{for all } h \in H(R), s \in S(R). \tag{3.7}$$

Replacing k by t^2, where $t \in S(R) \cap Z(R) \setminus \{0\}$, we obtain

$$2([d_1(h), s]d_2(t) + [s, d_2(h)]d_1(t))t = 0 \quad \text{for all } h \in H(R), s \in S(R).$$

Applying Lemma 2.1, we have

$$[d_1(h), s]d_2(t) + [s, d_2(h)]d_1(t) = 0 \quad \text{for all } h \in H(R), s \in S(R). \tag{3.8}$$

Writing st instead of h' in (3.5), where $s \in S(R)$ and $t \in S(R) \cap Z(R) \setminus \{0\}$, it is obvious to see that

$$([d_1(h), d_2(s)] + [d_1(s), d_2(h)])t + [d_1(h), s]d_2(t) + [s, d_2(h)]d_1(t) = 0.$$

Invoking equation (3.8), it follows that

$$([d_1(h), d_2(s)] + [d_1(s), d_2(h)])t = 0 \quad \text{for all } h \in H(R), s \in S(R),$$

and therefore

$$[d_1(h), d_2(s)] + [d_1(s), d_2(h)] = 0 \quad \text{for all } h \in H(R), s \in S(R). \tag{3.9}$$

Replacing h by st in (3.4), where $s \in S(R)$ and $t \in S(R) \cap Z(R) \setminus \{0\}$, one can see that

$$[d_1(s), d_2(s)]t + [d_1(s), s]d_2(t) + [s, d_2(s)]d_1(t) = 0 \quad \text{for all } s \in S(R). \tag{3.10}$$

Substituting st for h in (3.8), where $s \in S(R)$ and $t \in S(R) \cap Z(R) \setminus \{0\}$, we get

$$[d_1(s), s]d_2(t) + [s, d_2(s)]d_1(t) = 0 \quad \text{for all } s \in S(R). \tag{3.11}$$

Comparing (3.10) with (3.11), we may write

$$[d_1(s), d_2(s)] = 0 \quad \text{for all } s \in S(R). \tag{3.12}$$

Taking $h = x + x^*$ and $s = x - x^*$ in (3.4) and (3.12) respectively, we have

$$[d_1(x + x^*), d_2(x + x^*)] = 0 \quad \text{for all } x \in R, \tag{3.13}$$

and

$$[d_1(x - x^*), d_2(x - x^*)] = 0 \quad \text{for all } x \in R. \tag{3.14}$$

Combining (3.13) and (3.14), we obviously get

$$[d_1(x), d_2(x)] + [d_1(x^*), d_2(x^*)] = 0 \quad \text{for all } x \in R. \tag{3.15}$$

Now in (3.9) replace h by $x + x^*$ and s by $x - x^*$, this implies

$$[d_1(x), d_2(x)] - [d_1(x^*), d_2(x^*)] = 0 \quad \text{for all } x \in R. \tag{3.16}$$

Comparing (3.15) with (3.16), we arrive at

$$[d_1(x), d_2(x)] = 0 \quad \text{for all } x \in R. \tag{3.17}$$

Hence, in view of ([13], Theorem 1), we get the required result. □

Theorem 3.3. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and d_1, $d_2 \neq 0$ be two nonzero derivations of R. If $[d_1(s), d_2(s)] = 0$ for all $s \in S(R)$, then $d_1 = \lambda d_2$ where λ is an element from $C(R)$.*

Proof. Suppose that
$$[d_1(s), d_2(s)] = 0 \quad \text{for all } \ s \in S(R). \tag{3.18}$$
Then a linearization of (3.18) forces
$$[d_1(s), d_2(s')] + [d_1(s'), d_2(s)] = 0 \quad \text{for all } \ s, s' \in S(R). \tag{3.19}$$
Replacing s' by $s'k$, where $k \in H(R) \cap Z(R)$ and using the last equation, we obtain
$$[d_1(s), s']d_2(k) + [s', d_2(s)]d_1(k) = 0 \quad \text{for all } \ h, h' \in H(R). \tag{3.20}$$
Substituting ht for s', where $h \in H(R)$ and $t \in S(R) \cap Z(R) \setminus \{0\}$, in (3.20), we get
$$[d_1(s), h]d_2(k) + [h, d_2(s)]d_1(k) = 0 \quad \text{for all } \ h \in H(R), s \in S(R). \tag{3.21}$$
Replacing k by t^2, where $t \in S(R) \cap Z(R) \setminus \{0\}$, we obtain
$$2([d_1(s), h]d_2(t) + [h, d_2(s)]d_1(t))t = 0 \quad \text{for all } \ h \in H(R), s \in S(R).$$
Applying Lemma 2.1, we have
$$[d_1(s), h]d_2(t) + [h, d_2(s)]d_1(t) = 0 \quad \text{for all } \ h \in H(R), s \in S(R). \tag{3.22}$$
Writing ht instead of s' in (3.19), where $h \in H(R)$ and $t \in S(R) \cap Z(R) \setminus \{0\}$, it is obvious to see that
$$([d_1(s), d_2(h)]+[d_1(h), d_2(s)])t+[d_1(s), h]d_2(t)+[h, d_2(s)]d_1(t) = 0 \quad \text{for all } \ h \in H(R), s \in S(R).$$
Invoking equation (3.22), it follows that
$$([d_1(s), d_2(h)] + [d_1(h), d_2(s)])t = 0 \quad \text{for all } \ h \in H(R), s \in S(R),$$
and therefore
$$[d_1(s), d_2(h)] + [d_1(h), d_2(s)] = 0 \quad \text{for all } \ h \in H(R), s \in S(R). \tag{3.23}$$
Replacing s by ht in (3.18), where $h \in H(R)$ and $t \in S(R) \cap Z(R) \setminus \{0\}$, one can see that
$$[d_1(h), d_2(h)]t + [d_1(h), h]d_2(t) + [h, d_2(h)]d_1(t) = 0 \quad \text{for all } \ h \in H(R). \tag{3.24}$$
Substituting ht for s in (3.22), where $h \in H(R)$ and $t \in S(R) \cap Z(R) \setminus \{0\}$, we get
$$[d_1(h), h]d_2(t) + [h, d_2(h)]d_1(t) = 0 \quad \text{for all } \ h \in H(R). \tag{3.25}$$
Comparing (3.24) with (3.25), we may write
$$[d_1(h), d_2(h)] = 0 \quad \text{for all } \ h \in H(R). \tag{3.26}$$
In view of Theorem 3.2, one can easily see that $d_1 = \lambda d_2$ for some $\lambda \in C(R)$. □

As an application of Theorem 3.2, we get the following Corollary.

Corollary 3.4. *([13], Theorem 4) Let R be a noncommutative prime ring and d, $g \neq 0$ be two derivations of R into itself such that $[d(x), g(x)] = 0$, holds for all $x \in R$. Then $d = \lambda g$, where λ is an element from $C(R)$.*

Theorem 3.5. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and d_1, d_2 be two derivations of R, then the following assertions are equivalent:*
(1) *$d_1(h)d_2(h') - [h, h'] \in Z(R)$ for all $h, h' \in H(R)$;*
(2) *$d_1(h)d_2(h') + [h, h'] \in Z(R)$ for all $h, h' \in H(R)$;*
(3) *R is a commutative integral domain.*

Proof. (1) $\Rightarrow$ (3): Suppose that

$$d_1(h)d_2(h') - [h,h'] \in Z(R) \quad \text{for all} \;\; h,h' \in H(R). \tag{3.27}$$

If $d_1 = 0$ or $d_2 = 0$ then our assumption becomes $[h,h'] \in Z(R)$ for all $h,h' \in Z(R)$, and therefore R is commutative by Lemma 2.3.
If d_1 and d_2 are both nonzero derivations. Taking $h' = k \in H(R) \cap Z(R)$, we obtain

$$d_1(h)d_2(k) \in Z(R) \quad \text{for all} \;\; h \in H(R). \tag{3.28}$$

It follows that either $d_1(h) \in Z(R)$ for all $h \in H(R)$ or $d_2(k) = 0$ for all $k \in H(R) \cap Z(R)$.
If $d_1(h) \in Z(R)$ for all $h \in H(R)$, we replace h by $x + x^*$, then $d_1(x) + d_1(x^*) \in Z(R)$, hence $[d_1(x), d_1(x^*)] = 0$ for all $x \in R$, so R is commutative by ([7], Theorem 3.1).
If $d_2(k) = 0$ for all $k \in H(R) \cap Z(R)$, hence taking $h = k \in H(R) \cap Z(R)$, we get

$$d_1(k)d_2(h) \in Z(R) \quad \text{for all} \;\; h \in H(R). \tag{3.29}$$

As equation (3.29) is the same as equation (3.28), we conclude that R is commutative or $d_1(k) = 0$ for all $k \in H(R) \cap Z(R)$.
If $d_1(k) = 0$ for all $k \in H(R) \cap Z(R)$, now we replace h' by st, where $s \in S(R)$ and $t \in S(R) \cap Z(R) \setminus \{0\}$ in (3.27), one can easily verify that

$$d_1(h)d_2(s) - [h,s] \in Z(R) \quad \text{for all} \;\; h \in H(R), s \in S(R). \tag{3.30}$$

Taking $h = x + x^*$ and $h' = y + y^*$ in (3.27), we have

$$d_1(x+x^*)d_2(y+y^*) - [x+x^*, y+y^*] \in Z(R) \quad \text{for all} \;\; x,y \in R. \tag{3.31}$$

Now in (3.30), replace h by $x + x^*$ and s by $y - y^*$ respectively, this implies

$$d_1(x+x^*)d_2(y-y^*) - [x+x^*, y-y^*] \in Z(R) \quad \text{for all} \;\; x,y \in R. \tag{3.32}$$

Combining (3.31) and (3.32), we arrive at

$$d_1(x+x^*)d_2(y) - [x+x^*, y] \in Z(R) \quad \text{for all} \;\; x,y \in R. \tag{3.33}$$

Replacing x by xs in the last equation, where $s \in S(R) \cap Z(R) \setminus \{0\}$, we find that

$$d_1(x-x^*)d_2(y) - [x-x^*, y] \in Z(R) \quad \text{for all} \;\; x,y \in R. \tag{3.34}$$

By using (3.33) together with the above expression, we arrive at

$$d_1(x)d_2(y) - [x,y] \in Z(R) \quad \text{for all} \;\; x,y \in R. \tag{3.35}$$

Substituting yr for y in (3.35), we obtain

$$(d_1(x)d_2(y) - [x,y])r + d_1(x)yd_2(r) - y[x,r] \in Z(R) \quad \text{for all} \;\; r,x,y \in R. \tag{3.36}$$

Commuting with r, one can see that

$$[d_1(x)yd_2(r), r] - [y[x,r], r] = 0 \quad \text{for all} \;\; r,x,y \in R. \tag{3.37}$$

Replacing y by $d_1(x)y$ in (3.37), we get

$$d_1(x)[d_1(x)yd_2(r), r] + [d_1(x), r]d_1(x)yd_2(r) - d_1(x)[y[x,r], r] - [d_1(x), r]y[x,r] = 0. \tag{3.38}$$

Multiplying (3.37) by $d_1(x)$ on the left side, we find that

$$d_1(x)[d_1(x)yd_2(r), r] - d_1(x)[y[x,r], r] = 0 \tag{3.39}$$

Comparing this relation with (3.38), then

$$[d_1(x), r]d_1(x)yd_2(r) - [d_1(x), r]y[x,r] = 0. \tag{3.40}$$

Now we replace y by $yd_2(r)$ in the last equation, we obtain

$$[d_1(x), r]d_1(x)y(d_2(r))^2 - [d_1(x), r]yd_2(r)[x, r] = 0. \tag{3.41}$$

We multiply (3.40) from the right by $d_2(r)$, we get

$$[d_1(x), r]d_1(x)y(d_2(r))^2 - [d_1(x), r]y[x, r]d_2(r) = 0. \tag{3.42}$$

Combining (3.41) and (3.42), we arrive at

$$[d_1(x), r]y[d_2(r), [x, r]] = 0 \quad \text{for all} \ \ r, x, y \in R. \tag{3.43}$$

Hence, for any $r \in R$ we have either $[d_1(x), r] = 0$ for all $x \in R$ or $[d_2(r), [x, r]] = 0$ for all $x \in R$. Let $r \in R$, if $[d_1(x), r] = 0$ for all $x \in R$, then $r \in Z(R)$, and therefore $[d_2(r), [x, r]] = 0$ for all $x \in R$, we conclude that

$$[d_2(r), [x, r]] = 0 \quad \text{for all} \ \ r, x \in R. \tag{3.44}$$

Replacing x by xy in (3.44), we find that

$$x[d_2(r), [y, r]]+[d_2(r), x][y, r]+[d_2(r), [x, r]]y+[x, r][d_2(r), y] = 0 \quad \text{for all} \ \ r, x, y \in R. \tag{3.45}$$

Invoking (3.44), we get

$$[d_2(r), x][y, r] + [x, r][d_2(r), y] = 0 \quad \text{for all} \ \ r, x, y \in R. \tag{3.46}$$

Taking $r = x$, then

$$[d_2(x), x][y, x] = 0 \quad \text{for all} \ \ x, y \in R. \tag{3.47}$$

Substituting yr for y, we obviously get

$$[d_2(x), x]y[r, x] = 0 \quad \text{for all} \ \ r, x, y \in R. \tag{3.48}$$

Since R is prime, the last equation assures that $[r, x] = 0$ for all $r \in R$ or $[d_2(x), x] = 0$, for all $x \in R$. Let $x \in R$, if $[r, x] = 0$ for all $r \in R$, then $x \in Z(R)$, it follows that $[d_2(x), x] = 0$, we conclude that $[d_2(x), x] = 0$ for all $x \in R$. Consequently, R is commutative by ([17], Lemma 3).
(2) $\Rightarrow$ (3) follows from the first implication with slight modifications. □

Following the the same arguments as in Theorem 3.5, we can obtain the following result.

Theorem 3.6. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and d_1, d_2 be two derivations of R, then the following assertions are equivalent:*
(1) $d_1(s)d_2(s') - [s, s'] \in Z(R)$ *for all* $s, s' \in S(R)$;
(2) $d_1(s)d_2(s') + [s, s'] \in Z(R)$ *for all* $s, s' \in S(R)$;
(3) $d_1(h)d_2(s) - [h, s] \in Z(R)$ *for all* $h \in H(R)$, $s \in S(R)$;
(4) $d_1(h)d_2(s) + [h, s] \in Z(R)$ *for all* $h \in H(R)$, $s \in S(R)$;
(5) *R is a commutative integral domain.*

As Corollaries we find the following results

Corollary 3.7. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and d_1, d_2 be two derivations of R, then the following assertions are equivalent:*
(1) $d_1(x)d_2(y) - [x, y] \in Z(R)$ *for all* $x, y \in R$;
(2) $d_1(x)d_2(y) + [x, y] \in Z(R)$ *for all* $x, y \in R$;
(3) *R is a commutative integral domain.*

Corollary 3.8. *([10], Corollary 1) Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind, d_1 and d_2 are derivations of R. Then $d_1(x)d_2(y) - [x, y] = 0$ for all $x, y \in R$ if and only if R is commutative.*

Theorem 3.9. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and d_1, d_2 be two derivations of R, then the following assertions are equivalent:*
(1) $d_1(h) \circ d_2(h) - h^2 \in Z(R)$ *for all* $h \in H(R)$;
(2) $d_1(h) \circ d_2(h) + h^2 \in Z(R)$ *for all* $h \in H(R)$;
(3) *R is a commutative integral domain.*

Proof. It is obvious that (3) implies (1) and (2). So we need to prove that each of (1) and (2) implies (3).
(1) $\Rightarrow$ (3) Assume that

$$d_1(h) \circ d_2(h) - h^2 \in Z(R) \quad \text{for all } \ h \in H(R). \tag{3.49}$$

A linearization of relation (3.49) gives

$$d_1(h) \circ d_2(h') + d_1(h') \circ d_2(h) - h \circ h' \in Z(R) \quad \text{for all } \ h, h' \in H(R). \tag{3.50}$$

Substituting $h'k$ for h' in (3.50), where $k \in H(R) \cap Z(R)$, we find that

$$(d_1(h) \circ d_2(h'))k + (d_1(h) \circ h')d_2(k) + (d_1(h') \circ d_2(h))k + (h' \circ d_2(h))d_1(k) - (h \circ h')k \in Z(R).$$

In light of (3.50), the above expression yields

$$(d_1(h) \circ h')d_2(k) + (h' \circ d_2(h))d_1(k) \in Z(R) \quad \text{for all } \ h, h' \in H(R). \tag{3.51}$$

Taking $h' \in H(R) \cap Z(R) \setminus \{0\}$ in (3.51) and using Lemma 2.1, one can see that

$$d_1(h)d_2(k) + d_2(h)d_1(k) \in Z(R) \quad \text{for all } \ h \in H(R). \tag{3.52}$$

Now in (3.52) replace h by hk, and use relation (3.52), so we obtain

$$hd_1(k)d_2(k) \in Z(R) \quad \text{for all } \ h \in H(R). \tag{3.53}$$

In view of Lemma 2.1, hence either $h \in Z(R)$ for all $h \in H(R)$, in which case R is commutative by Lemma 2.2, or $d_1(k)d_2(k) = 0$ for all $k \in H(R) \cap Z(R)$. In the latter case, we have $d_1(k) = 0$ for all $k \in H(R) \cap Z(R)$, or $d_2(k) = 0$ for all $k \in H(R) \cap Z(R)$.
If $d_1(k) = 0$ for all $k \in H(R) \cap Z(R)$, so (3.51) becomes

$$(d_1(h) \circ h')d_2(k) \in Z(R) \quad \text{for all } \ h, h' \in H(R). \tag{3.54}$$

Then either $d_1(h) \circ h' \in Z(R)$ for all $h, h' \in H(R)$, or $d_2(k) = 0$ for all $k \in H(R) \cap Z(R)$.
Assume $d_1(h) \circ h' \in Z(R)$ for all $h, h' \in H(R)$ and using ([9], Theorem 2.7), we conclude that R is commutative.
If $d_2(k) = 0$ for all $k \in H(R) \cap Z(R)$, so by taking $h' \in H(R) \cap Z(R) \setminus \{0\}$ in (3.50), and using Lemma 2.1, one can easily see that

$$h \in Z(R) \quad \text{for all } \ h \in H(R).$$

By Lemma 2.2, we conclude that R is commutative.
If $d_2(k) = 0$ for all $k \in H(R) \cap Z(R)$, then reasoning as above we obtain that R is commutative.
(2) $\Rightarrow$ (3) follows from the first implication with slight modifications. □

Using the same arguments as in Theorem 3.9, we can obtain the following result.

Theorem 3.10. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and d_1, d_2 be two derivations of R, then the following assertions are equivalent:*
(1) $d_1(h) \circ d_2(h) - h \circ h \in Z(R)$ *for all* $h \in H(R)$;
(2) $d_1(h) \circ d_2(h) + h \circ h \in Z(R)$ *for all* $h \in H(R)$;
(3) *R is a commutative integral domain.*

Arguing as in the Theorem 3.9, we can find the following result.

Theorem 3.11. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and d_1, d_2 be two derivations of R, then the following assertions are equivalent:*
(1) $d_1(s) \circ d_2(s) - s^2 \in Z(R)$ *for all* $s \in S(R)$;
(2) $d_1(s) \circ d_2(s) + s^2 \in Z(R)$ *for all* $s \in S(R)$;
(3) *R is a commutative integral domain.*

As an application of Theorem 3.10, we get the following Corollaries.

Corollary 3.12. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and d_1, d_2 be two derivations of R, then the following assertions are equivalent:*
(1) $d_1(h) \circ d_2(h') - h \circ h' \in Z(R)$ for all $h, h' \in H(R)$;
(2) $d_1(h) \circ d_2(h') + h \circ h' \in Z(R)$ for all $h, h' \in H(R)$;
(3) R is a commutative integral domain.

Corollary 3.13. *([16], Theorem 2.11) Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. Let d_1 and d_2 be nonzero derivations of R, such that $d_1(x) \circ d_2(x^*) = \pm x \circ x^*$ for all $x \in R$. Then R is commutative.*

Theorem 3.14. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and d_1, d_2 be two derivations of R. Then $d_1(h)h - hd_2(h) \in Z(R)$ for all $h \in H(R)$ if and only if ($d_1 = d_2 = 0$ or R is a commutative integral domain).*

Proof. Assume that

$$d_1(h)h - hd_2(h) \in Z(R) \quad \text{for all } \ h \in H(R). \tag{3.55}$$

Linearizing (3.55), we find that

$$d_1(h)h' + d_1(h')h - hd_2(h') - h'd_2(h) \in Z(R) \quad \text{for all } \ h, h' \in H(R). \tag{3.56}$$

Replacing h' by $h'k$ in (3.56), where $k \in H(R) \cap Z(R)$, we obtain

$$d_1(h)h'k + d_1(h')hk + h'hd_1(k) - hd_2(h')k - h'd_2(h)k - hh'd_2(k) \in Z(R) \tag{3.57}$$

By invoking (3.56), the last equation becomes

$$h'hd_1(k) - hh'd_2(k) \in Z(R) \quad \text{for all } \ h, h' \in H(R). \tag{3.58}$$

Taking $h' \in H(R) \cap Z(R) \setminus \{0\}$, we obviously get

$$h(d_1(k) - d_2(k)) \in Z(R) \quad \text{for all } \ h, h' \in H(R). \tag{3.59}$$

Using Lemma 2.1, it follows that either $h \in Z(R)$ for all $h \in H(R)$ or $d_1(k) = d_2(k)$ for all $k \in H(R) \cap Z(R)$.
If $h \in Z(R)$ for all $h \in H(R)$, hence by Lemma 2.2, R is commutative.
If $d_1(k) = d_2(k)$ for all $k \in H(R) \cap Z(R)$, (3.58) becomes

$$[h, h']d_1(k) \in Z(R) \quad \text{for all } \ h, h' \in H(R). \tag{3.60}$$

The last equation implies that either $[h, h'] \in Z(R)$ for all $h, h' \in H(R)$ or $d_1(k) = 0$ for all $k \in H(R) \cap Z(R)$.
If $[h, h'] \in Z(R)$ for all $h, h' \in H(R)$, so R is commutative by Lemma 2.3.
If $d_1(k) = 0$ for all $k \in H(R) \cap Z(R)$, then $d_2(k) = 0$ for all $k \in H(R) \cap Z(R)$, and thus $d_1(Z(R)) = d_2(Z(R)) = \{0\}$. Now we replace h' by st in (3.56), where $s \in S(R)$ and $t \in S(R) \cap Z(R) \setminus \{0\}$, one can easily see that

$$d_1(h)s + d_1(s)h - hd_2(s) - sd_2(h) \in Z(R) \quad \text{for all } \ h \in H(R), s \in S(R). \tag{3.61}$$

Substituting st instead of h in (3.55), where $s \in S(R)$ and $t \in S(R) \cap Z(R) \setminus \{0\}$, we get

$$d_1(s)s - sd_2(s) \in Z(R) \quad \text{for all } \ s \in S(R). \tag{3.62}$$

Taking $h = x + x^*$ and $s = x - x^*$ in (3.55) and (3.62) respectively, we obtain

$$d_1(x + x^*)(x + x^*) - (x + x^*)d_2(x + x^*) \in Z(R) \quad \text{for all } \ x \in R, \tag{3.63}$$

and

$$d_1(x - x^*)(x - x^*) - (x - x^*)d_2(x - x^*) \in Z(R) \quad \text{for all } \ x \in R. \tag{3.64}$$

Replacing h by $x + x^*$ and s by $x - x^*$ in (3.61), we find that

$$d_1(x+x^*)(x-x^*)+d_1(x-x^*)(x+x^*)-(x+x^*)d_2(x-x^*)-(x-x^*)d_2(x+x^*) \in Z(R). \tag{3.65}$$

Comparing (3.63), (3.64) and (3.65), we obviously get

$$d_1(x)x - xd_2(x) \in Z(R) \quad \text{for all } \; x \in R. \tag{3.66}$$

Now replacing x by $x+t$ where $t \in S(R) \cap Z(R) \setminus \{0\}$, one can easily see that

$$(d_1 - d_2)(x) \in Z(R) \quad \text{for all } \; x \in R. \tag{3.67}$$

Using ([17], Lemma 3), then $d_1 = d_2$ or R is commutative.
If $d_1 = d_2$, our assumption becomes $[d_1(x), x] \in Z(R)$ for all $x \in R$, then by ([17], Theorem 2), R is commutative or $d_1 = d_2 = 0$. □

Corollary 3.15. *([1], Theorem 2.1) Let $(R,*)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$. Let d_1 and d_2 be nonzero derivations of R. Then the following statements are equivalent:*
(1) $d_1(x)x^* - x^*d_2(x) = 0$ for all $x \in R$.
(2) R is a commutative integral domain and $d_1 = d_2$.

Theorem 3.16. *Let $(R,*)$ be a 2-torsion free prime ring with involution of the second kind and d_1, d_2 be two derivations of R. Then $d_1(s)s - sd_2(s) \in Z(R)$ for all $s \in S(R)$ if and only if $d_1 = d_2 = 0$ or R is a commutative integral domain.*

Proof. Assume that

$$d_1(s)s - sd_2(s) \in Z(R) \quad \text{for all } \; s \in S(R). \tag{3.68}$$

Linearizing (3.68), we find that

$$d_1(s)s' + d_1(s')s - sd_2(s') - s'd_2(s) \in Z(R) \quad \text{for all } \; s, s' \in S(R). \tag{3.69}$$

Replacing s' by $s'k$ in (3.69), where $k \in H(R) \cap Z(R)$, we obtain

$$d_1(s)s'k + d_1(s')sk + s'sd_1(k) - sd_2(s')k - s'd_2(s)k - ss'd_2(k) \in Z(R). \tag{3.70}$$

In light of (3.69), the last relation becomes

$$s'sd_1(k) - ss'd_2(k) \in Z(R) \quad \text{for all } \; s, s' \in S(R). \tag{3.71}$$

Taking $s' \in S(R) \cap Z(R) \setminus \{0\}$, one can see that

$$s(d_1(k) - d_2(k)) \in Z(R) \quad \text{for all } \; s \in H(R). \tag{3.72}$$

Using Lemma 2.1, it follows that either $s \in Z(R)$ for all $s \in S(R)$ or $d_1(k) = d_2(k)$ for all $k \in H(R) \cap Z(R)$.
If $s \in Z(R)$ for all $s \in S(R)$, hence by Lemma 2.2, R is commutative.
If $d_1(k) = d_2(k)$ for all $k \in H(R) \cap Z(R)$, (3.71) becomes

$$[s, s']d_1(k) \in Z(R) \quad \text{for all } \; s, s' \in H(R). \tag{3.73}$$

The last equation implies that either $[s, s'] \in Z(R)$ for all $s, s' \in S(R)$ or $d_1(k) = 0$ for all $k \in H(R) \cap Z(R)$.
If $[s, s'] \in Z(R)$ for all $s, s' \in S(R)$, so R is commutative by Lemma 2.3.
If $d_1(k) = 0$ for all $k \in H(R) \cap Z(R)$, then $d_2(k) = 0$ for all $k \in H(R) \cap Z(R)$ which implies that $d_1(Z(R)) = d_2(Z(R)) = \{0\}$. Substituting ht instead of s in (3.68), where $h \in H(R)$ and $t \in S(R) \cap Z(R) \setminus \{0\}$, we get

$$d_1(h)h - hd_2(h) \in Z(R) \quad \text{for all } \; h \in H(R). \tag{3.74}$$

By using Theorem 3.14, one can easily see that $d_1 = d_2 = 0$ or R is commutative. □

4 Examples

The following example proves that the condition "$*$ is of the second kind" is necessary in Theorems 3.5, 3.9 and 3.14.

Example 4.1.
Let us consider $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
It is straightforward to check that $(R, *)$ is a prime ring with involution of the first kind. We set $d_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$, and $d_2 = d_1$. We can easily see that d_1 and d_2 satisfy the conditions of Theorems 3.5, 3.9 and 3.14 but R is not commutative.

The following example proves that the primeness hypothesis of R is necessary in Theorems 3.5, 3.9 and 3.14.

Example 4.2. Let $(R, *)$ and d_1, d_2 be as in the Example 4.1, and consider $\mathcal{R} = R \times \mathbb{C}$, it is obvious to verify that $(\mathcal{R}, \sigma)$ is a semi prime ring with involution of the second kind defined by: $\sigma(X, z) = (X^*, \overline{z})$. We set the derivations D_1 and D_2 of $\mathcal{R}$ defined as $D_1(X, z) = D_2(X, z) = (d_1(X), 0)$. It is obvious that D_1 and D_2 satisfy the conditions of Theorems 3.5, 3.9 and 3.14 but R is not commutative.

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